Exploring Warped Compactifications of Extra Dimensions

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Abstract of the Dissertation

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In 1920s, the concept of extra dimensions was considered for the first time to unify gravity and electromagnetism. Since then there have been many developments to understand the unification of fundamental forces using extra dimensions. In this thesis, we study this idea of extra dimensions in higher dimensional gravity theories such as String Theory or Supergravity to make connections with cosmology. We construct a family of non-singular time-dependent solutions of a six-dimensional gravity with a warped geometry. The warp factor is time-dependent and breaks the translation invariance along one of the extra directions. Our solutions have the desired property of homogeneity and isotropy along the non-compact space. These geometries are supported by matter that does not violate the null energy condition. These 6D solutions do not have a closed trapped surface and hence the Hawking-Penrose singularity theorems do not apply to these solutions. These solutions are constructed from 7D locally flat solution by performing Kaluza-Klein reduction. We also study warped compactifications of string/M theory with the help of effective potentials for the construction of de Sitter vacua. The dynamics of the conformal factor of the internal metric is explored to investigate instabilities. The results works the best mainly in the case of a slowly varying warp factor. We also present interesting ideas to find AdS vacua of N=1 flux compactifications using smooth, compact toric manifolds as internal space.

To Family and Friends!

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Chapter 1

Review

1.1 Introduction

In 20th century, to study fundamental forces in nature, two different theories, Standard Model and General Relativity were constructed. The Standard Model of particle physics explains physics at small length scales and very accurately predicts interactions of elementary particles such as quarks, electrons and neutrinos. These interactions come from the strong force, the weak force and the electromagnetic force. The fourth fundamental force, Gravity is described by Einstein's theory of General Relativity. Predictions of these theories are tested with great accuracies by various experiments. But at small length scales, General Relativity description of gravity breaks down and hence, the construction of a renormalizable quantum field theory of gravity is a challenge for theoretical physicists.

Superstring theory is the main candidate for a quantum theory of gravity and a unified theory of the fundamental forces of nature right now. To achieve that it has to make connections with the observations of Standard Model of particle physics and cosmology. If string theory describes the universe, then there exist six or seven(for M-Theory) extra dimensions of space, not yet verified by experiments. Since 1984, many phenomenologically relevant features were studied from compactification on Calabi-Yau manifolds. One of the main problems of such compactifications was the presence of possibly a large number of moduli fields. The concept of warped flux compactifications has given us the way to fix moduli. In particle phenomenology, warping can be used to generate the exponentially small ratio of M_{weak}/M_{Planck}.

Observational evidences of late-time cosmology indicate that our universe has a small cosmological constant which is positive. It has been a great challenge to obtain such a positive cosmological constant in pure supergravity backgrounds because of standard No-Go theorems[1]. In string compactifications, it is possible to construct de Sitter vacua by evading these No-Go theorems[2, 3]. This happens because of extra stringy sources such as D-branes and O-planes. Presence of fluxes and stringy sources naturally lead to non-trivial warping. Cosmological inflation plays a crucial role in understanding the isotropy and homogeneity of our universe. Constructing inflation in supergravity theories is very difficult. On the other hand, warped flux compactifications are used to build inflationary models[4].

Given the implications arising from the study of flux vacua and warped compactifications, it is extremely important to understand such compactifications and their dynamics. Presently effects of warping are not as well studied as standard Kaluza-Klein models. With String theory models addressing important aspects for the theory of inflation, such effects and time-dependent properties have to be studied carefully. One major focus should be to understand the effects of warping on the 4D effective theories. We also establish a procedure to understand Non-singular time-dependent solutions of higher dimensional gravity using warping.

1.2 Kaluza-Klein Reduction

In 1920s, Kaluza and Klein considered the idea of unifying Einstein's gravity and Electromagnetism by using compactified extra dimensions, actually a circle [5]. Such compactifications of gravitational theory in an arbitrary D-dimensional space-time to four dimensions, lead to the four-dimensional metric and vector bosons related to a gauge group, which is the isometry group of the internal manifold. The higher dimensional theories such as String theory are usually theories of gravity coupled to matter fields. In this section, to understand Kaluza-Klein ideas, we follow the conventions of C. Pope's lecture notes[6]and we will mainly focus on a circle as compactifying manifold. Let us start with Einstein gravity in (D+1) dimensions. The lagragian can be expressed in usual Einstein-Hilbert form as follows:

$$\mathcal{L} = \sqrt{-\hat{g}}\hat{R} \tag{1.2.1}$$

Here fields with hats are defined in (D+1) dimensions. We would like to study the dimensional reduction of this theory by compactifying it on a circle (y-coordinate) of radius L. Using the properties of periodic functions, one can expand metric components using Fourier series.

$$\hat{g}_{MN}(x, y) = \sum_{n} g^{(n)}_{MN}(x)e^{\frac{iny}{L}} \tag{1.2.2}$$

The modes with non-zero n are massive modes such that masses are proportional to n/L. Usually in Kaluza-Klein approach, it is argued that the radius of the compactifying circle (L) is very small such that these non-zero modes are extremely massive and they are usually neglected. In technical terms, such restriction to study massless modes are called truncation.

Let's write the (D+1) dimensional metric in the following form

$$d\hat{s}^2 = e^{2\alpha\phi} g_{\mu\nu}(x) dx^\mu dx^\nu + e^{2\beta\phi}(dy + \mathcal{A}_\mu dx^\mu)^2 \tag{1.2.3}$$

This way the (D+1)-dimensional metric is written in terms of D-dimensional fields, metric $g_{\mu\nu}$, vector boson $\mathcal{A}$ and scalar field ϕ. The various (D+1)-metric components are

$$\begin{aligned} \hat{g}_{\mu\nu} &= e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} \mathcal{A}_\mu \mathcal{A}_\nu \\ \hat{g}_{\mu y} &= e^{2\beta\phi} \mathcal{A}_\mu \\ \hat{g}_{yy} &= e^{2\beta\phi} \end{aligned} \tag{1.2.4}$$

At this stage, α and β are free parameters. In order to obtain the reduced action in standard Einstein-Hilbert form, β is set to $\beta = -(D-2)\alpha$ and to get kinetic term of the scalar field in standard form, $\alpha^2 = \frac{1}{2(D-1)(D-2)}$.

The reduced Lagrangian with a field strength ($\mathcal{F} = d\mathcal{A}$) looks like

$$L = \sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} e^{-2(D-1)\alpha\phi} \mathcal{F}^2\right] \tag{1.2.5}$$

One important point to understand in this setup is that the diffeomorphism invariance in the y-dimension becomes the D-dimensional U(1) gauge invariance associated with the vector boson. Thus, dimensionally reducing pure gravity over a circle gives rise to lower dimensional Einstein-Maxwell-Dilaton theory.

1.3 Supersymmetric Compactifications

1.3.1 Type II Supergravity/String Theory

In this section, we give the basic idea about 10 dimensional Type II Superstring theories with their field contents.

The bosonic part of the massless spectrum contains the metric g_{MN}, the antisymmetric 2-form B_{MN} and the dilaton ϕ coming from NS-NS sector and R-R sector contains p-form potentials C_p such that $p = 1, 3, 5, 7, 9$ for Type IIA and $p = 0, 2, 4, 6, 8$ for Type IIB. The important point for Type IIB theory is that 4-form RR potential C_4 has a self-duality constraint.

The fermionic part consists of two Majorana-Weyl gravitinos, ψ_M^A, $A = 1, 2$. These gravitinos are or opposite chirality in IIA

$$\gamma_{11}\psi_M^1 = +\psi_M^1, \ \gamma_{11}\psi_M^2 = -\psi_M^2 \tag{1.3.1}$$

and the same chirality in IIB

$$\gamma_{11}\psi_M^1 = +\psi_M^1, \ \gamma_{11}\psi_M^2 = +\psi_M^2. \tag{1.3.2}$$

The NS-NS B field and R-R potentials have field strengths, given by

$$H = dB, \ F_p = dC_{p-1} - H \wedge C_{p-3} \tag{1.3.3}$$

The RR fields have a constraint coming from Hodge duality.

$$F_p = (-1)^{[p/2]} \star F_{10-p} \tag{1.3.4}$$

The Bianchi identities assosiated with NS flux and RR fluxes are

$$dH = 0, \ dF_p - H \wedge F_{p-2} = 0 \tag{1.3.5}$$

So far we haven't considered any sources such as D-branes and Orientifold planes. When sources are present, one cannot have the globally well-defined potentials and Bianchi identities get modified accordingly.

1.3.2 Compactifications

For this discussion, we will restrict to supersymmetric compactification of String Theory ($d = 10$) to 4-dimensional spacetime. A vacuum of type II supergravity is a solution of its equations of motion and Bianchi identities, such that M_{10} is fibered over a spacetime $\mathcal{M}_4$, and such that the whole solution has maximal symmetry in four dimensions (that is, for example, Poincaré for $M_4 = Mink_4$). Usual approach is to consider 10-dimensional spacetime as a product of 4-dimensional non-compact spacetime and 6 dimensional compact internal manifold such that maximal symmetry is preserved in four dimensions.

$$M_{10} = \mathcal{M}_4 \times X_6 \tag{1.3.6}$$

In last 10-15 years, with phenomenological implications (Randall-Sundrum models), 10D spacetime is considered as a warped product of $\mathcal{M}_4$ and X_6.

$$ds_{10}^2 = e^{2A(y)} ds_4^2 + g_{mn}(y) dy^m dy^n$$

In general, X_6 can be strongly curved (i.e. $R_{(6)} \sim \frac{1}{\alpha'}$) or can break SUSY at the compactification scale $M_{SUSY} \sim \frac{1}{R}$ or can be of string size. Even today, these conditions are extremely hard to analyze quantitatively. We will stick with compactifications where geometric treatment is valid, manifold is weakly curved and large. We would also like to understand how to preserve some amount of supersymmetry after compactification.

1.3.3 $\mathcal{N} = 1$ Supersymmteric Flux Compactifications

$\mathcal{N} = 1$ supersymmetry is a solution of the hierarchy problem between the weak scale (a TeV) and the Planck scale. Supersymmetry provides the answer to this large hierarchy. It is largely believed that $\mathcal{N} = 1$ supersymmetry will survive down to the TeV-scale which might be tested in next couple of years at LHC. Hence, it is natural to consider 4D compactifications with $\mathcal{N} = 1$ supersymmetry.
In 10 dimensions, string theories are equipped with $\mathcal{N} = 1$ or 2 supersymmetry. All string compactifications studied in the 80s mainly had one major drawback, known as moduli problem. A large number of massless scalar fields, string moduli arise from small deformations of the String background and variations of the size of internal manifolds. If such moduli were present, then we would have tested their long range interactions.
While partially breaking $\mathcal{N} = 2$ supersymmetry obtained from Calabi-Yau compactifications down to $\mathcal{N} = 1$, fluxes are used which give vacuum expectation values to some of the moduli arising from compactifications[2]. With the help of some non-perturbative corrections, all moduli could get vacuum expectation values [3] in Type IIB theories. Later it was shown that fluxes alone can stabilize all moduli classically within the valid supergravity approximation for massive Type IIA theories. The study of moduli stabilization plays a key role in making connections to real world physics from string backgrounds.
In this section, we review the main features of flux compactifications of Type II theories. The metric has a warp factor but it maintains maximal symmetry in 4D spacetime (i.e AdS, Mink or dS). This puts constraints on choices of NS or RR fluxes one can have. Consider 3-form flux H, all indices should be internal because presence of one or more spacetime indices would break maximal symmetry in 4 dimensions. This logic holds for RR fluxes F_p when $p < 4$. For higher fluxes, one can consider $F_{0123a_1..a_{6-p}}$. In more compact way, one can say $F = f + vol_{4d} \wedge (-1)^{[\frac{p}{2}]}(\star_6 f)$.

We start with ten dimensional Majorana-Weyl spinors. As said before, these spinors are of opposite(same) chirality in Type IIA (B). The maximal symmetry in 4D requires the vacuum expectation value of the fermionic fields to be zero. To obtain supersymmetric vacuum, we impose $< \delta_\epsilon \chi >= 0$, where ϵ is the supersymmetry parameter and χ is any fermionic field. Using democratic formulation, in string frame, supersymmetric variations are given by [70],

$$\delta\psi_M = \nabla_M \epsilon + \frac{1}{4}\ \not{H}_M \mathcal{P}\epsilon + \frac{1}{16} e^{\phi} \sum_n\ \not{F}_n^{(10)} \Gamma_M \mathcal{P}_n \epsilon,$$

$$\delta\lambda = \left(\not{\partial}\phi + \frac{1}{2}\ \not{H}\mathcal{P} \right) \epsilon + \frac{1}{8} e^{\phi} \sum_n (-1)^n (5-n)\ \not{F}_n^{(10)} \mathcal{P}_n \epsilon.$$

Here M takes values from 0 to 9 and $\psi_M = \begin{pmatrix} \psi_M^1 \\ \psi_M^2 \end{pmatrix}$. H_M stands for $\frac{1}{2} H_{MNP}\Gamma^{NP}$ and $\not{H}$ stands for $\frac{1}{2} H_{MNP}\Gamma^{MNP}$. In type IIA, $\mathcal{P} = \Gamma_{11}$ and $\mathcal{P}_n = \Gamma_{11}^{(n/2)}\sigma^1$ and in case of Type IIB, $\mathcal{P} = -\sigma^3$ and $\mathcal{P}_n = \sigma^1$ for even $\frac{n+1}{2}$ and $\mathcal{P}_n = i\sigma^2$ for odd $\frac{n+1}{2}$.

To understand four dimensional supersymmetic compactifications, 10D spinors are split into 4D spinors($\xi^{1,2}$) and 6D spinor(η) with appropriate chiralities such that $(\xi_+^{1,2})^* = \xi_-^{1,2}$ and $(\eta_+)^* = \eta_-$. For Type IIA, we get

$$\begin{aligned} \epsilon^1 &= \xi_+^1 \otimes \eta_+ + \xi_-^1 \otimes \eta_-, \\ \epsilon^2 &= \xi_+^2 \otimes \eta_- + \xi_-^2 \otimes \eta_+, \end{aligned} \tag{1.3.7}$$

and in case of Type IIB,

$$\epsilon^{1,2} = \xi_+^{1,2} \otimes \eta_+ + \xi_-^{1,2} \otimes \eta_-. \tag{1.3.8}$$

These Eq. (1.3.7) and (1.3.8) can be used in gravitino variation to understand the conditions for supersymmetric vacua. The presence of one such internal spinor leads to $\mathcal{N} = 2$ supersymmetry in four dimensions which is the usual case for Calabi-Yau compactifications with one covariantly constant spinor and no flux.
Once fluxes are present, one cannot work with Calabi-Yau manifolds and the internal geometry should admit a globally well defined non-vanishing spinor to preserve supersymmetry. Such geometries are known and studied extensively as $SU(3)$-structure manifolds. These geometries are more explained in chapter 4. To obtain $\mathcal{N} = 1$ supersymmetry from 10D spinors, one can start with four-dimensional spinors ξ_+ and ξ_-, Majorana conjugates.

10D spinors take following forms, in the case of Type IIA,

$$\begin{aligned} \epsilon^1 &= \xi_+ \otimes \eta_+^1 + \xi_- \otimes \eta_-^1, \\ \epsilon^2 &= \xi_+ \otimes \eta_-^2 + \xi_- \otimes \eta_+^2, \end{aligned} \tag{1.3.9}$$

and in case of Type IIB,

$$\epsilon^{1,2} = \xi_+ \otimes \eta_+^{1,2} + \xi_- \otimes \eta_-^{1,2}. \tag{1.3.10}$$

$SU(3)$-structure manifolds and $\mathcal{N} = 1$ supersymmetry are obtained when we impose the proportionality between η_1 and η_2 with proper choice of fluxes. If η_1 and η_2 are independent, then one gets more supersymmetry with $SU(2)$-structure geometry which has additional topological constraints.

1.4 Warped compactifications

1.4.1 Type IIB review

In this section, we will review the warped Type IIB string compactification. This is a solution at leading order in α' which are Type IIB supergravity solutions with D-branes(D3/D7) and Orientifolds[2].

In Einstein frame, the supergravity action of Type IIB string theory is given by

$$\begin{aligned} S_{IIB} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(R - \frac{\partial_M\tau\partial^M\bar{\tau}}{2(Im\tau)^2} - \frac{G_3\cdot\bar{G}_3}{12Im\tau} - \frac{\tilde{F}_5^2}{4\cdot 5!}\right) \\ &\quad + \frac{1}{8i\kappa_{10}^2}\int\frac{C_4\wedge G_3\wedge\bar{G}_3}{Im\tau} + S_{sources} \end{aligned} \tag{1.4.1}$$

where axio-dilaton $\tau = C_0 + ie^{-\phi}$ and $G_3 = F_3 - \tau H_3$ and 5-form flux is such that $\tilde{F}_5 = *\tilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3$.

To obtain solutions with maximal symmetry (Poincare) in 4 dimensions, the 10D metric takes following form

$$ds_{10}^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)}\tilde{g}_{mn}dy^m dy^n \tag{1.4.2}$$

To obtain the maximal symmetry, axio-dilaton is $\tau = \tau(y)$, 3-form flux G_3 has all components in the internal directions and $\tilde{F}_5 = (1+*)[d\alpha(y)\wedge dx^0\wedge dx^2\wedge dx^2\wedge dx^3]$. The sources obey following condition, $\frac{1}{4}(T_m^m - T_\mu^\mu) \geq T_3\rho_3^{source}$, where ρ_3 is the D3 charge density of the localized sources. These sources allow us to evade the standard Supergravity No-Go theorems.

The general solution at the leading order in α' under above conditions has following features:

1) Internal manifold is a conformal Calabi-Yau, i.e. $\tilde{g}_{mn} = g_{mn}^{CY}$ together

with the orientifold projection. Thus, this geometry comes with Complex structure and Kahler moduli.

2) Closed 3-form fluxes F_3, H_3 obey quantization conditions, $\frac{1}{2\pi\alpha'}\int F_3 \in 2\pi\mathbb{Z}$ and $\frac{1}{2\pi\alpha'}\int H_3 \in 2\pi\mathbb{Z}$.

3) D-brane charges follow Gauss-law condition [integrated Bianchi identity], $\int_M H_3 \wedge F_3 + (2\kappa_{10}^2 T_3)Q_3^{source} = 0$.

4) Two important features of this solution are $*_6 G_3 = iG_3$ and $\alpha = e^{4A}$. The imaginary self-dual primitive 3-form fixes Complex structure moduli and axio-dilaton.

Now, it is important to discuss the 4D effective description of these Type IIB orientifold models with RR and NS fluxes. Calabi-Yau compactifications lead to complex structure (z^α) and Kăhler moduli(t^a). Kăhler potential for axio-dilaton and complex structure moduli is given by

$$\mathcal{K}_C = -\log\left(i\int_M \Omega\wedge\bar{\Omega}\right) - \log(-i(\tau-\bar{\tau})). \tag{1.4.3}$$

For Kăhler moduli, Volume is described using Kăhler form (J) by $V = \int_M J\wedge J\wedge J = \frac{1}{6}S_{abc}t^a t^b t^c$. With this information, Kăhler potential is schematically given by

$$\mathcal{K}_K = -2\log(V).$$

With NS and RR fluxes, superpotential is generated for axio-dilaton and complex moduli, which is given by

$$W = \int_M G_3\wedge\Omega. \tag{1.4.4}$$

Now, we have all ingradients to write down the potential in $\mathcal{N}=1$ supergravity.

$$V = e^{\mathcal{K}_C+\mathcal{K}_K}\left(G^{i\bar{j}}D_iW\overline{D_jW} - 3|W|^2\right) \tag{1.4.5}$$

It is important to notice that at tree level, W does not depend on Kăhler moduli, thus $D_aW = \mathcal{K}_{,a}W$. Thus, potential simplifies to

$$V = e^{\mathcal{K}_{total}}\left(G^{\alpha\bar{\beta}}D_\alpha W\overline{D_\beta W} + G^{a\bar{b}}\mathcal{K}_{,a}\overline{\mathcal{K}_{,a}}|W|^2 - 3|W|^2\right). \tag{1.4.6}$$

Using the form of $\mathcal{K}$ we have, the potential term further simplifies to

$$V = e^{\mathcal{K}_{total}} \left(G^{\alpha\bar{\beta}} D_\alpha W \overline{D_\beta W} \right). \tag{1.4.7}$$

One can notice that in this setup, it is possible to get supersymmetric vacua with $D_\phi W = D_a W = D_\alpha W = 0$ while non-supersymmetric vacua with $D_\alpha W \neq 0$ for some Kähler modulus. Thus, one can construct Supersymmetric vacua with $V = 0$ with Calabi-Yau orientifolds. The challenge left here is to generate a potential for Kähler moduli.

Superpotentials in No-scale models receive no corrections at all orders in perturbations. Non-pertuebatively there can be corrections from instantons which are usually Kähler modulus dependent. After adding such contributions, superpotential takes following form, $W = W_{pert} + Ae^{ia\rho}$ with Kähler modulus ρ.

In [3], it was shown that after addition of non-perturbative effects, all moduli can be stabilized for small W_{pert}. Having negative cosmological constant with moduli fixed, these solutions are not good to describe our universe. KKLT therefore uplifted the AdS minima to positive minima by adding anti-D3-branes. This uplifting term adds the following term to the moduli potential $V_{uplifting} = \frac{D}{(\rho+\bar{\rho})^2}$. Such de Sitter minima are metastable.

The conclusion from this section is that constructing de-Sitter vacua from string theory is possible and one can interpret the small observed dark energy as cosmological constant.

Presently, the standard approach is to start with a class of theory on a particular compact manifold X as internal space and derive a 4-dimensional effective field theory within this class of theories. One obtains an effective potential, which is a function of the various moduli. One has to study for local minima of this potential. The usual problem in this approach is with the potential going to zero at large volume and weak coupling, one has to look for a barrier to this behavior. This approach is explained well for even non-supersymmetric vacua in [63]. One can study effects of warping in de-Sitter vacua or inflationary situations using this effective potential.

1.5 Singularity Theorems

1.5.1 What are Geodesics?

General relativity is a theory of gravity. The main ideas from general relavity can be summerized as 'the spacetime is a manifold (M_d) with Lorentzian metric g_{ab}'. The laws of physics from gravity can be explained with 2 principles, 1) general covariance and 2) The equations of general relativity should reduce to the Special Relativity equations in the limit $g_{ab} \to \eta_{ab}$. The dynamics is governed by Einstein equations. The curvature of metric g_{ab} is related to the matter distribution in the spacetime by

$$R_{ab} - \frac{1}{2} g_{ab} R = 8 T_{ab}.$$

Two concepts : Geodesics and Trapped surfaces from general relativity are very important in order to build non-singular cosmological solutions.
Let's understand the concept of geodesics in this section and trapped surfaces in the next section.

Definition: Let (M, g) be a Riemannian manifold. For a connection ∇, geodesics are defined as curves $\gamma = \gamma(t)$ such that

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0.$$

In usual physics literature, equivalently, geodesic is defined as a curve whose tangent vector V^a is parallel transported along itself.

$$V^a \nabla_a V^b = 0.$$

The important feature of geodesics is the curve of shortest length connecting 2 points on a manifold. The worldlines of particles in a force-free motion satisfy the geodesic equations.

Geodesic Completeness: A geodesic from point $p \in M$ is complete if it can be extended to all values of its affine parameter. Geodesically complete spacetime has all geodesics complete.

With the understanding of geodesic completeness, one can study singularities using the following interpretation: "A spacetime is singular if it is timelike or null geodesically incomplete". Generally the curvature diverges along incomplete geodesics, but geodesic incompleteness can occur with the bounded curvature components or bounded curvature invariants.

1.5.2 Trapped Null Surface

The concept of closed trapped surfaces was first introduced by Penrose in 1965[7]. A trapped surface represents the boundary of a region where any initially expanding null congruence begins to converge. Formation of closed trapped surface leads to singularities.
We discuss the idea of closed trapped surfaces using the procedure developed

by Senovilla[8]. These are closed spacelike co-dimension 2 surfaces, S in the n-dimensional Lorentzian manifold $(\mathcal{M}, g)$. The term 'closed' means the surfaces are compact without boundary.

To understand the properties of such surfaces, assume that there exists a family of (n-2) dimensional spacelike surfaces, Σ_{X^a}, given by

$$\{x^a = X^a\}, \quad a = \{0, 1\} \tag{1.5.1}$$

Here X^a are constants and x^α are local coordinates in $\mathcal{M}$. The metric can be written locally as

$$ds^2 = g_{ab}dx^a dx^b + 2g_{aA}dx^a dx^A + g_{AB}dx^A dx^B \tag{1.5.2}$$

The imbedding $\Phi : \Sigma \to \mathcal{M}$ is given by $\Phi^a(\zeta) = X^a$, $x^A = \Phi^A(\zeta) = \zeta^A$ such that the first fundamental form for each such surface is given by

$$\gamma_{AB} = g_{AB}(X^a, \zeta^C) \tag{1.5.3}$$

while the future null normal one-forms satisfy

$$\begin{aligned} \kappa^{\pm} &= k^{\pm}_b dx^b \\ g^{ab}k^{\pm}_a k^{\pm}_b &= 0 \\ g^{ab}k^{+}_a k^{-}_b &= -1 \end{aligned} \tag{1.5.4}$$

One obtain null expansions by computing

$$\theta^{\pm} = k^{\pm a}\left(\frac{G_{,a}}{G} - \frac{(G\gamma^{AB}g_{aA)})_{,B}}{G}\right) \tag{1.5.5}$$

Here $G = \sqrt{\det g_{AB}}$ and $\mathbf{g}_a = g_{aA}dx^A$. Now we are ready to define mean curvature one-form and the scalar defining the trapping properties,

$$\begin{aligned} H_\mu &= \delta^a_\mu \left((\ln G), a - div(\boldsymbol{g}_a)\right) \\ \kappa &= -g^{bc}H_b H_c \end{aligned} \tag{1.5.6}$$

Σ is trapped (respectively marginally trapped, non-trapped) if κ is positive, (resp. zero, negative) everywhere on Σ.

1.5.3 Hawking-Penrose Singularity

The concept of singularity in General Relativity is very difficult to understand in very concrete framework. One tries to address the singularity in terms of divergent curvature components, but such divergences can be because of a bad choice of coodinates. If one defines singularity using curvature invariants, still there can be singularities. Following the works of Geroch, Hawking and others, singularities can be addressed by curves which cannot be extended in a regular manner and do not take all values of their parameter. Singularity theorems by Hawking and Penrose have played a key role in the development of general relativity since 70s. In this section, we discuss the Hawking-Penrose singularity.

Theorem 1.5.1. *If spacetime (M,g) satisfies following properties:*

- *Energy condition:* $R_{ab}V^aV^b \geq 0$ *for all timelike vectors* V^a*,*
- *M is globally hyperbolic(existence of a Cauchy Surface* $\Sigma \subset M$*),*
- *there is a trapped surface(S) in M,*

then M is geodesically incomplete.

As explained in [19], “timelike and null geodesic completeness are minimum conditions for non-singular space-time”. First let’s justify the assumptions. Using Einstein’s equations, first assumption can be rewritten as $T_{ab}V^aV^b \geq 0$. Strong energy condition is violated in the inflationary universe, but inflationary universe is shown to be geodesically incomplete in past. A Cauchy surface is a spacelike hypersurface of M such that it intersects every smooth, inextendible causal curve exactly once. In previous subsection, we have discussed closed trapped surfaces.

To sketch the proof of this theorem[9], let’s start by assuming (M, g) is null geodesically complete. One can show that $\partial J^+(S)$, the boundary of the causal future of p, is a compact manifold without a boundary. In physical sense, the light rays emitted outwards from points on S should converge since S is a closed trapped surface. In addition, one can show that $\partial J^+(S)$ has a continuous one-to-one mapping to the non-compact hypersurface. This implies either that $\partial J^+(S)$ is non-compact, or that it can have a boundary. Thus we get the contradiction by assuming null geodesic completeness.

1.6 Bouncing Cosmologies

The Big Bang model gives us a predictive description of our universe from nucleosynthesis to present. When one starts understanding Big Bang model back in time, two questions often arise in this cosmological model: 'did our universe have a beginning in the past?' and 'is it possible to make cosmological models with bounces where the scale factor goes through crunch followed by bang?'. These questions are directly connected to the singularity theorems of Penrose and Hawking. According to singularity theorems, a smooth reversal from contraction to expansion is impossible if an energy condition of the form

$$R_{\mu\nu}v^{\mu}v^{\nu} \geq 0 \tag{1.6.1}$$

is imposed. For the null energy condition (NEC), v^{μ} stands for any null future pointing vector. The Null Energy condition is satisfied by all well-known matter and energy sources in the Universe. Let us see how these theorems apply for homogeneous and isotropic Freedmann-Robertson-Walker (FRW) cosmologies

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{1.6.2}$$

The FRW equations are

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{1.6.3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3} \tag{1.6.4}$$

for cosmological constant (Λ).
If NEC is satisfied, then for flat or open ($k = 0$ or $k = -1$) universes, it can be shown that $\dot{H} \leq 0$ and cyclic universes are not possible.

For closed universe ($k = 1$) to prove the singularity theorems, one has to impose SEC. One way to obtain the cyclic universe is to violate SEC but keeping NEC.

Simple Harmonic Universe :
To obtain the Simple Harmonic Universe [11], authors have used the positive curvature ($k = +1$), negative cosmological constant ($\Lambda < 0$) and a matter source such that $P = w\rho$ and $w = -\frac{2}{3}$. The continuity equation gives us $\dot{\rho} + 3H(\rho + P) = 0$. Thus, in this case, we get

$$\rho = \Lambda + \frac{\rho_0}{a} \tag{1.6.5}$$

Using eq.(1.6.4), one obtains $a(t) = a_0 cos(\omega t + \phi) + c$ such that $\omega = \sqrt{\frac{8\pi}{3}G|\Lambda|}$ and $c = \frac{\rho_0}{2|\Lambda|}$. γ is defined as $\gamma = \frac{3|\Lambda|}{2\pi G\rho_0^2}$. The scalar perturbations give instabilities at short distances, if $c_s^2 (= (dP/d\rho)^2)$ is negative [speed of sound wave]. For a perfect fluid ($w < -1/3$), c_s^2 is negative. But it is possible to find matter sources with required equations of state but with c_s^2 positive and $w = -2/3$ and in such a case, scalar perturbations are stable. For $\gamma \ll 1$, some modes of perturbations become unstable. The model with $\gamma \sim 1$ provides an example of an eternal universe without singularities, using positive curvature and violating the SEC, but keeping NEC. This universe is shown classically and quantum mechanically stable at linearized level for small scalar perturbations.

With the BICEP2 results, we think it is important to address the question of inflation in any cosmological setup. With the stable, eternal non-singular cosmologies, we can think of a Universe which begins in such a non-singular phase, lives there for a long period, and then transitions to a realistic Universe with inflation. This idea is addressed in [10].

1.7 Outline

With the basic understanding of String theory compactifications, Kaluza-Klein reduction and some important aspects of general relativity, we are ready to apply these techniques in various cases.

In 1970s, Hawking and Penrose showed that a globally hyperbolic contracting space admitting a closed trapped surface will lead to a singularity, unless an energy condition is violated [19]. The main challenge given by these theorems is finding a non-singular cosmology with homogeneous and isotropic space. In four dimensions, it is hard to obtain such cosmologies without violating Null energy conditions on matter fields. In chapter 2, using the ideas from extra dimensional gravitational theories, we obtain a family of non-singular time-dependent solutions of a six-dimensional gravitational theory that are warped products of a four dimensional bouncing cosmological solution and a two dimensional internal manifold. The warp factor is time-dependent and breaks translation invariance along one of the internal directions. When the warp factor is periodic in time, the non-compact part of the geometry bounces periodically. The six dimensional geometry is supported by matter that does not violate the null energy condition. We show that this 6D geometry does not admit a closed trapped surface and hence the Hawking-Penrose singularity theorems do not apply to these solutions. Some parts of this work was done in collaboration with Dr Koushik Balasubramanian from Yang Institute, Stony Brook[15].

The standard approach in compactifications is to derive an effective action in 4 dimensions to make connections with real world physics. The effective action obtained after performing the compactification is a functional of the

4-dimensional metric and additional fields parametrizing the extra dimensions such as its metric and the other fields of supergravity or superstring theory. Critical points of this effective action correspond to critical points of the original higher-dimensional theory. In chapter 3, We study warped compactifications of string/M theory with the help of effective potentials, continuing previous work initiated by Michael R. Douglas. The dynamics of the conformal factor of the internal metric, which is responsible for instabilities in these constructions, is explored, and such instabilities are investigated in the context of de Sitter vacua. We prove existence results for the equations of motion in the case of a slowly varying warp factor, and the stability of such solutions is also addressed. These solutions are a family of meta-stable de Sitter vacua from type IIB string theory in a general non-supersymmetric setup. Some parts of this work was done in collaboration with Dr Marcelo Disconzi from Vanderbilt University and Dr Vamsi Pingali from Johns Hopkins[16].

In string compactifications, as soon as one turns on background fluxes to obtain supersymmetric vacua, the internal manifold cannot be Calabi-Yau. Fluxes and string sources which allow to evade standard No-Go theorems lead to warping. In chapter 4, we study supersymmetric AdS_4 compactifications using the idea of SU(3) structure manifolds. We study how to use smooth, compact toric varieties for supersymmetric AdS_4 flux compactifications similar to $\mathbb{CP}^3$ solution. The key feature for supersymmetric compactifications is the existance of non-vanishing globally well defined complex 3-form. Necessary topological conditions associated with such form are understood to put constraints on large class of these manifolds for supersymmetric flux compactification. The approach can be extended with mathematical view to understand if nearly Kahler metrics like $\mathbb{CP}^3$ manifold are present for non-homogeneous toric varieties or not. This will need detailed local analysis and ways to patch these local properties with given topological conditions[17].

Chapter 2

Time-dependent Warping and Bouncing Cosmologies

2.1 Background

In this chapter, we are going to discuss the way to evade Singularity theorems using extra dimensions and time-dependent warping. Hawking and Penrose showed that a globally hyperbolic contracting space admitting a closed trapped surface will collapse into a singularity, unless an energy condition is violated [19]. This result imposes severe restrictions on a smooth transition from a contracting phase to an expanding phase.

There are many phenomenological models incorporating a pre-big bang scenario, in which the singularity is avoided by having matter that violates the null energy condition (NEC) [20, 21] or by violating NEC using modified gravity [22]. For closed universes, it is sufficient to relax the strong energy condition (SEC) to avoid the singularity and such an example was constructed in [11].[1]

Though it is not possible to derive the energy conditions from first principles, it is known that most models of classical matter satisfy the NEC. It has also been argued that violation of NEC in certain models are pathological due to superluminal instabilities [23]. However, violation of such energy conditions need not signal a sickness always. The strong energy condition is violated by a positive cosmological constant and also during inflation. Relaxing strong energy condition seems benign. In this regard, it would be interesting to find a microscopic realization of the fluid stress tensor in [11] using classical fields and a cosmological constant. Quantum effects can lead to violation of the

[1] The Hawking-Penrose singularity theorem [19] assumes the strong energy condition to show the existence of singularities in closed universes.

null energy condition, but an averaged null energy condition must be satisfied. Orientifold planes in string theory can also allow for localized violations of the null-energy condition.

Instead of violating null and strong energy conditions, some researchers have sought to understand the singularity outside the realm of classical Einstein gravity. For instance, there have been a large number of proposals in the literature to understand the initial singularity using string dualities [24]-[28]. We will now review some of these proposals briefly.

In [26], the cosmological solution is obtained by connecting two singular solutions at the singularity. A scalar field with a singular profile provides the stress tensor required to source the metric. Even though the infinite past is described by a smooth perturbative vacuum of string theory, the perturbative description breaks down near the bounce singularity and a non-perturbative string description is required to bridge the post big bang universe and the pre-big bang universe.

A geometric picture of certain big bounce singularities in higher dimensions was presented in [28, 29], where the lower dimensional scalar field uplifts to the higher dimensional radion field.[2] The size of the circle shrinks to zero size when the universe passes through the singularity and expands again when the universe bounces from the singularity. They also considered the case where the compact direction is a line interval instead of a circle. In this case, when the universe approaches a big crunch, the branes at the endpoints of the interval collide with each other, and they pass through each other when the universe expands again [28, 29].[3] In [28, 29], the higher dimensional geometry is simply a time-dependent orbifold of flat space-time.

There are many Lorentzian or null orbifold models of bouncing singularities where the geometry is just obtained by taking quotients of flat spacetime by boost or combination of boosts and shifts [34, 35, 36].[4] In the case of singular orbifolds, there is a circle that shrinks to zero size and then expands, leading to a bounce singularity. Such solutions are unstable to introduction of a single particle as the backreaction of the particle and the infinite number of orbifold images produces regions of large curvatures [35, 37]. In [35, 36], examples of non-singular time-dependent orbifolds were presented. In these examples, size of the compact directions remain non-zero at all times but it becomes infinitely large in the infinite past and infinite future. That is, the extra dimensions are initially non-compact and then go through a compactification-decompactification transition. These null-orbifolds are geodesically incomplete

[2] Also see [30, 31, 32] for related work.

[3] This is slightly different from the original ekpyrotic model [33].

[4] Since these geometries are locally flat, they are exact solutions of classical string theory.

unless the anisotropic directions are non-compact.

In this paper, we present a new class of non-singular bouncing cosmological solutions that has the following features:

1. These are classical solutions of Einstein's equations sourced by a stress-energy tensor that satisfies the null energy condition.[5]

2. The stress-energy tensor sourcing the metric can be realized by classical fields.

3. All non-compact spatial directions are homogeneous and isotropic.

4. These solutions can be embedded in string theory.

We show that our bouncing cosmological solutions evade the Hawking-Penrose singularity theorem because they do not admit any closed trapped surface. Demanding homogeneity and isotropy in all spatial directions (including compact directions) rule out the possibility of finding such geometries. In fact, it can be shown that the metric $ds^2 = -dt^2 + a(t)^2 d\boldsymbol{x}^2$ cannot exhibit a bounce (classically) unless the null-energy condition is violated [25, 29]. Hence, it is essential to include anisotropy or inhomogeneity in the compact extra dimensions to find non-singular bouncing cosmologies. We show that a time-dependent warped metric of the following form can exhibit bouncing behavior (non-periodic as well as periodic):

$$ds^2 = \left[\left(-e^{2A(t,\theta_k)}dt^2 + e^{2B(t,\theta_k)}d\boldsymbol{x}^2\right)\right] + e^{2C(t,\theta_k)}g_{ij}d\theta^i d\theta^j + 2\zeta_i(t,\theta_k)dtd\theta^i \tag{2.1.1}$$

More precisely, we find six dimensional solutions of Einstein-Maxwell-scalar theory in which the metric takes the form in (2.1.1). Note that the non-compact directions are homogeneous and isotropic. The compact directions have finite non-vanishing size at all times. Most higher dimensional resolution of singularities that have appeared in literature rely on reducing along a shrinking circle [30, 31, 32]

We show that our solutions are geodesically complete as they do not admit a closed trapped surface. These geometries are homogeneous and isotropic along the non-compact spatial directions $\boldsymbol{x}$. This non-trivial six-dimensional solution can be uplifted to a trivial solution in 7-dimensions using an $O(2,2)$ transformation. This transformation provides a simple method for generating time-dependent warping. We show that the six-dimensional solution does not admit a time-translation symmetry.

[5] Senovilla [38] found non-singular inhomogeneous geometries sourced by a fluid staisfying the NEC. However, a classical field configuration that produces the fluid stress-energy tensor is not known.

In this paper, we also present an example of a class of solutions where the topology of the internal manifold changes dynamically. Note that we need atleast six-dimensions (3+1 non-compact directions and 2 internal directions) to see a topology change in the internal manifold. We work with a class of six-dimensional solutions for convenience, where the topology changes from a genus one surface to genus zero surfcae. Such solutions do not have any simple four-dimensional description as the topology change involves mixing among an arbitrarily large number of Kaluza-Klein modes.

Rest of the paper is organized as follows: In the section §2, we briefly review scale factor duality and $O(d, d)$ transformations. We present examples of some interesting solutions that can be generated from trivial solutions using dimensional reduction and $O(d, d)$ transformations. In section §3, we use $O(d, d)$ transformations to generate six dimensional solutions of the form (2.1.1) and show that these are geodesically complete as they do not admit closed trapped surfaces. In section §4, we conclude with a discussion on the results of this paper. We also present a short discussion on singular solutions with internal manifolds that dynamically change topology.

2.2 Dimensional reduction, scale factor duality and $O(d, d)$ transformations

In this section, we will briefly review some solution generating techniques and also present a brief survey of some interesting solutions (in the literature) that can be obtained using these solution-generating techniques.

2.1 Generating non-trivial solutions from trivial solutions using Kaluza-Klein reduction

We will now present an example which has appeared multiple times in literature (see for instance [29, 28, 39]) to illustrate the utility of Kaluza-Klein reduction as a solution generating technique. We start with a flat metric written as a product of two-dimensional Milne universe and $\mathbb{R}^{d-1}$:

$$ds^2_{\mathcal{M}_2 \times \mathbb{R}^{d-1}} = -dt^2 + t^2 dy^2 + d\boldsymbol{x}^2, \ \varphi = 0, \ H = 0 \tag{2.2.1}$$

This is a trivial saddle point of the following action:

$$S = \int d^{D+1}x \int d^d y \sqrt{g} e^{-2\varphi} \left(R + 4\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right), \tag{2.2.2}$$

We will now show that dimensional reduction along y direction of $\mathcal{M}_2 \times \mathbb{R}^{d-1}$ produces a non-trivial solution of the $d-$dimensional equations of motion. Using the Kaluza-Klein reduction ansatz, we can write the higher dimensional solution as

$$ds^2 = e^{2\alpha\sigma} ds^2_{E,d-1} + e^{2\beta\sigma} dy^2,$$

where $\sigma = \beta^{-1} \log|t|$; $ds^2_{E,d-1}$ is the lower dimensional line element in Einstein frame, and

$$\alpha^2 = \frac{1}{2(d-1)(d-2)}, \beta = -\sqrt{\frac{d-2}{2(d-1)}}$$

The action in (2.2.2) can be consistently truncated to the following Einstein-scalar action in lower dimensions:[6]

$$S_d = \int d^d x \left(R - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi \right)$$

The lower dimensional solution is

$$ds^2_{E,d-1} = t^{2/(d-2)} \left(-dt^2 + d\boldsymbol{x}^2 \right), \ \sigma = -\sqrt{\frac{2(d-1)}{(d-2)}} \log|t| \tag{2.2.3}$$

Recall that the higher dimensional metric is just a special coordinate patch on $d+1$ dimensional Minkowski space-time. However, the lower dimensional solution is non-trivial and does not admit a time-like killing vector. In fact, the lower dimensional geometry has a curvature singularity. Though the curvature invariants of higher dimensional geometry are all finite, the spacetime is geodescially incomplete [30]. The above $d-$dimensional solution and the uplift to $d+1$ dimensional $\mathcal{M}_2 \times \mathbb{R}^{d-1}$ has been discussed in [28, 29, 39] already.

It is also possible to generate solutions with a non-trivial geometry as the starting point instead of flat space-time. For instance, the Hawking-Turok instanton can be obtained by reducing a bubble of nothing in five-dimensions [41]. Using this trick, it is possible to generate magnetic or charged dilatonic solutions (black holes or expanding cosmologies) starting from known uncharged solutions [39, 42, 43, 44, 45].

Now, we will discuss a different uplift of the lower dimensional solution in (2.2.3). The solution in (2.2.3) can also be uplifted to the following solution

[6]The lower dimensional action is a consistent truncation of the higher dimensional action if all solutions of the lower dimensional equations of motion can be uplifted to solutions of higher dimensional action.

of the higher dimensional equations of motion

$$ds_1^2 = -dt^2 + t^{-2}dy^2 + d\boldsymbol{x}^2,\ \varphi = -\log|t|,\ H = 0. \tag{2.2.4}$$

We will now show that the above solution is related to a particular solution of Belinsky-Khalatnikov type [47]. Recall that the action in (2.2.2) is not the Einstein frame action. The saddle point of the Einstein frame action is obtained by a Weyl rescaling of the metric. After shifting to Einstein frame, the solution is given by

$$ds_E^2 = t^{\frac{4}{(d-1)}}\left(-dt^2 + t^{-2}dy^2 + d\boldsymbol{x}^2\right) \tag{2.2.5}$$

After the coordinate redefinition: $t^2 = 2\tau, \boldsymbol{x} = \sqrt{2}\boldsymbol{X}$, the above solution becomes a special case of Belinsky-Khalatnikov solution [47] (with $d = 3$). In the new coordinates the solution takes the following form

$$ds_E^2 = \left(-d\tau^2 + \tau^{2p_1}dX_1^2 + \tau^{2p_2}dX_2^2 + \tau^{2p_3}dy^2\right),\ \varphi = -\frac{q}{\sqrt{2}}\log(2\tau) \tag{2.2.6}$$

where $p_1 = p_2 = 1/2, p_3 = 0, q = 1/\sqrt{2}$. Note that $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^2 = 1 - q^2$. Belinsky and Khalatnikov [47] found more general time-dependent solutions of the above form where p_i and q satisfy the same relation.

The solution in (2.2.4) is related to the trivial solution in (2.2.1) by an $O(d,d)$ duality transformation. When d translationally invariant directions are compactified, the lower dimensional effective action obtained by dimensional reduction enjoys an $O(d,d)$ duality symmetry [46]. These transformations are generalizations of the Buscher transformations [40]. An $O(d,d)$ transformation maps a classical solution of the equations of motion to a different classical solution [26]. This property is helpful in generating new interesting solutions from known solutions (even from trivial solutions). Let us consider the action of an $O(d,d)$ duality transformation on the following solution

$$ds^2 = g_{ab}dx^a dx^b + G_{ij}dy^i dy^j,\ \varphi = \varphi_0$$

where ∂_{y^i} is a Killing vector. The action of a general $O(d,d)$ transformation is given by

$$M = \begin{bmatrix} G^{-1} & G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{bmatrix} \to \Omega^T M \Omega, \tag{2.2.7}$$

where Ω is a $2d \times 2d$ $O(d,d)$ matrix *i.e.,* Ω satisfies the following condition:

$$\Omega^T \begin{bmatrix} 0 & \mathbb{I}_{d\times d} \\ \mathbb{I}_{d\times d} & 0 \end{bmatrix} \Omega = \begin{bmatrix} 0 & \mathbb{I}_{d\times d} \\ \mathbb{I}_{d\times d} & 0 \end{bmatrix} \doteq \eta \tag{2.2.8}$$

The matrix M is a symmetric $O(d,d)$ covariant matrix. It is possible to write the action in a manifestly $O(d,d)$ invariant fashion using the double field theory formalism (see [48]). In the double field theory formalism, $O(d,d)$ transformations can be written as a generalized coordinate transformation of the generalized metric M. Note that when $\Omega = \eta$, $M \to M^{-1}$ which is a generalization of the scale factor inversion.

Scale factor duality (SFD) transformation is a special case of an $O(d,d)$ duality transformation (with $H = dB = 0$). When $dB = 0$, the action of scale factor duality can be written as follows

$$G_{ij} \to \tilde{G}'_{ij} = G^{-1}_{ij}, \quad \varphi \to \varphi' = \varphi_0 - \frac{1}{2}\log\left(\det G\right), \quad H \to H = dB = 0$$

Scale factor duality maps an expanding universe to a contracting universe. This forms the basis for the pre-big bang scenario of [26]. Note that the solution in (2.2.4) is related to the locally flat solution in (2.2.1) through a SFD transformation.

In the next section, we will show that the solution generating techniques discussed in this section can be used to find non-singular bouncing cosmologies that do not admit any closed trapped surface.

2.3 Non-singular Bouncing Cosmological Solutions

3.1. Solution of six dimensional Einstein-Maxwell-Scalar theory

In this section, we will describe a method to obtain six-dimensional non-singular cosmological solutions with time dependent warping. The basic idea is to use a non-trivial parametrization of flat space that would produce non-trivial solutions after dimensional reduction or $O(d,d)$ transformations. We begin by writing down a line element for a flat metric in seven dimensions (with 3 non-compact spatial directions and 3 compact directions):

$$ds_7^2 = -dt^2\left(1 - r'(t)^2\right) + d\boldsymbol{x}^2 + r(t)^2 d\theta^2 + g_{\phi\phi} d\phi^2 + \left(\alpha^2 g_{\phi\phi} + \beta^2\right) dz^2$$

$$+ 2\beta \cos\theta r'(t) dt dz - 2\beta \sin\theta r(t) d\theta dz + 2\alpha g_{\phi\phi} d\phi dz \qquad (2.3.1)$$

where $g_{\phi\phi} = (R + r(t)\sin\theta)^2$; $\boldsymbol{x}$ denotes the 3 non-compact spatial directions, t denotes a timelike coordinate, θ, ϕ and z are the 3 compact directions; α, β and R are non-zero constants. Note that the metric degenerates when $\beta = 0$. To ensure that t is timelike, we choose $r(t)$ such that $-1 < r'(t) < 1$. The above metric can be transformed to the familiar flat space metric: $ds^2 = -dt'^2 + d\boldsymbol{x}'^2 + d\boldsymbol{y}^2$, by using the following change of coordinates

$$\begin{aligned} &\boldsymbol{x}' = \boldsymbol{x},\ t' = t,\ y_1 = \beta z + r(t)\cos\theta, \\ &y_2 = (R + r(t)\sin\theta)\cos\phi,\ y_3 = (R + r(t)\sin\theta)\sin\phi \end{aligned} \qquad (2.3.2)$$

with $-\infty > t > \infty$, $2\pi > \theta \geq 0$ and $2\pi > \phi \geq 0$. The metric in (2.3.1) extremizes the seven dimensional low-energy string effective action in (2.2.2) (with $\varphi = 0$ and $H = 0$). We will now reduce along z direction to obtain a non-trivial solution in six-dimensions. The six dimensional action can be obtained by writing the 7D line element in the Kaluza-Klein reduction ansatz:

$$ds_7^2 = e^{-\sigma/2} d\hat{s}_6^2 + e^{2\sigma}\left(dz + \hat{A}_\mu dx^\mu\right)^2.$$

When φ and H are trivial, the seven dimensional action can be consistently truncated to the following Einstein-Maxwell-scalar action:

$$S_E^{(6)} = \int d^6x \sqrt{\hat{g}} \left(\hat{R} - \frac{5}{4}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{4} e^{\frac{5}{2}\sigma} \hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right), \qquad (2.3.3)$$

where, $\hat{g}$ is the Einstein frame metric, $\hat{F} = d\hat{A}$ is the field strength and σ is the radion field. The six dimensional solution is given by

$$\begin{gathered} e^{2\sigma} = \alpha^2 g_{\phi\phi} + \beta^2, \\ \hat{g}_{tt} = -e^{\frac{\sigma}{2}}\left(1 - r'(t)^2\right) - \beta^2 r'(t)^2 e^{-\frac{3\sigma}{2}}\cos^2\theta,\ \hat{g}_{t\theta} = -e^{5\sigma/2}\hat{A}_t\hat{A}_\theta,\ \hat{g}_{t\phi} = -e^{5\sigma/2}\hat{A}_t\hat{A}_\phi, \\ \hat{g}_{\theta\theta} = r(t)^2 e^{\frac{\sigma}{2}}\left(1 - \beta^2 e^{-2\sigma}\sin^2\theta\right),\ \hat{g}_{\theta\phi} = -e^{5\sigma/2}\hat{A}_\theta\hat{A}_\phi, \qquad (2.3.4) \\ \hat{g}_{\phi\phi} = \beta^2 e^{\frac{-3\sigma}{2}} g_{\phi\phi},\ \hat{g}_{ij} = e^{\frac{\sigma}{2}}\delta_{ij} \\ \hat{A}_t = \beta r'(t)\cos\theta e^{-2\sigma},\ \hat{A}_\theta = -e^{-2\sigma}\beta r(t)\sin\theta,\ \hat{A}_\phi = e^{-2\sigma}\alpha g_{\phi\phi} \end{gathered}$$

Other components of the gauge field and the metric are trivial. This six dimensional solution describes a $\mathbb{T}^2$ fibered over $\mathbb{R}^{3,1}$. Note that the metric on the $\mathbb{T}^2$ is not flat. The above solution can be uplifted to a different classical solution of a 7D theory described by (2.2.2). This non-trivial solution is related to the trivial seven dimensional solution in (2.3.1) by an $O(2,2)$ transformation (Buscher transformations). The details of this solution can be found in appendix A (see A.2.3). Note that the 7D solution is regular if the six-dimensional solution is regular. The six-dimensional solution can be regular only if the size of the compact directions do not shrink to zero size. This is ensured by choosing $r(t)$ such that $R > r(t) > 0$ for all t, and $\beta > 0$. With these conditions, the components of the metric and inverse metric are regular everywhere. All derivatives of the metric are also regular everywhere. All curvature invariants can be built from product of the derivative of metric components and inverse metric. Since the metric, inverse metric and their derivatives are all regular, all curvature invariants are finite. However, finiteness of curvature invariants does not imply the geometry is free of singularities. In order to show the six-dimensional solution in (2.3.4) is non-singular, we have to prove that it is geodesically complete [49]. We will prove this at the end of the next sub-section.

3.2. Absence of time-translation symmetry

In this subsection, we will show that our solution in (2.3.4) does not admit a time-translation symmetry. In the process of showing this, we found a simple trick to prove our solution is geodesically complete. We will present this discussion at the end of this sub-section.

We begin with a discussion on time translation symmetry. ξ is a symmetry generator if the following equations are satisfied

$$\delta_\xi \sigma = \xi^\mu \partial_\mu \sigma = 0, \quad \delta_\xi A_\mu = \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\lambda A_\lambda = \partial_\mu \Lambda, \quad \delta_\xi \hat{g}_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\mu \xi_\nu = 0 \tag{2.3.5}$$

where Λ denotes the gauge shift. We can rewrite the second condition as follows:

$$\xi^\lambda \left(-\partial_\mu A_\lambda + \partial_\lambda A_\mu\right) = \partial_\mu \Lambda - \partial_\mu \left(A_\lambda \xi^\lambda\right)$$
$$\delta_\xi A_\mu = \xi^\nu F_{\nu\mu} = \partial_\mu \tilde{\Lambda} \tag{2.3.6}$$

where $\tilde{\Lambda} = \partial_\mu \Lambda - \partial_\mu \left(A_\lambda \xi^\lambda\right)$ is just a redefinition of the gauge shift.

We will now show that there is no time-like vector satisfying the above conditions. Note that ξ^t must be non-trivial for ξ to be time-like. The first two conditions and the trace of the third condition implies that ξ should take the following form

$$\xi = \frac{A_0}{\sqrt{\hat{g}}} \left(\partial_\theta \sigma \partial_t - \partial_t \sigma \partial_\theta + \frac{F_{t\theta}}{F_{t\phi}} \partial_t \sigma \partial_\phi\right) + \frac{1}{\sqrt{\hat{g}}} \frac{\partial_t \left(\tilde{\Lambda}(\sigma)\right)}{F_{t\phi}} \partial_\phi + B_0^i \partial_i$$

where $\tilde{\Lambda}(\sigma)$ is a function of σ, A_0 and B_0^i are constants. Note that we have used the isotropy and homogeneity of the non-compact spatial directions to write down the above expression. The variation of σ, A_μ and the trace of the Killing equation seems to fix ξ uniquely unto some unknown constants and an unknown function of σ. The only freedom in ξ is in the choice of $\tilde{\Lambda}$. The form of Λ should be fixed by using the other Killing equations. We can verify that there exists *no* $\tilde{\Lambda}(\sigma)$ for which $\delta_\xi \hat{g}_{t\phi}$, $\delta_\xi \hat{g}_{t\theta}$, $\delta_\xi \hat{g}_{\theta\phi}$, $\delta_\xi \hat{g}_{tt}$ and $\delta_\xi \hat{g}_{\theta\theta}$ all vanish when $A_0 \neq 0$. We also know that ξ is not time-like if $A_0 = 0$. This implies that the 6D solution does not admit a time-translation symmetry. Note that when $r(t)$ is periodic, the geometry is invariant under discrete time translation invariance.

We will now show that the 6D geometry is geodesically complete for all choice of $r(t)$ satisfying the conditions: $0 < r(t) < R \ \forall \ t$, and $\beta > 0$. To show this, we will first construct a vector ζ that satisfies $\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu = 0$, but $\delta_\zeta \sigma \neq 0$. Note that such a vector is not a symmetry of the theory. For instance, linear dilaton solutions ten-dimensional supergravity theories admit such a vector [50, 51]. In the linear dilation solutions, translation invariance (along a particular direction) is manifestly broken by the dilaton, while the string frame metric is invariant under spatial translations.[7]

We will now return to our discussion on geodesic completeness. We can verify that the $\zeta_\mu = e^{\sigma/2} \delta_\mu^0$ satisfies $\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu = 0$ but,

$$\delta_\zeta \sigma = \zeta^\mu \partial_\mu \sigma = 2 e^{-\sigma/2} \frac{\alpha^2 r'(t)(\beta^2 + \alpha^2 r(t)^2) \sec\theta (R + r(t)\sin\theta) \tan\theta}{\beta^2 r(t)^2} \neq 0.$$

We would like to emphasize that ζ does not generate time translation symmetry. However, the existence of this vector simplifies the proof of geodesic completeness. Let u^μ denote the tangent vector to a geodesic and λ be an affine parameter. To prove geodesic completeness, we have to show that the affine parameter λ can take all values in $(-\infty, \infty)$. Using the fact $\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu = 0$ and the geodesic equation ($u^\mu \nabla_\mu u^\nu = 0$), we can show that $u^\mu \zeta_\mu$ is a constant. This implies

$$\frac{dt}{d\lambda} = \text{constant} \equiv E \implies \lambda = \frac{t}{E} + \text{constant}$$

This shows that λ can take all values in $(-\infty, \infty)$. To study the derivative of

[7] Also see [52] for an example of a solution of where translation invariance is broken by a complex scalar field, but not by the metric.

θ, we proceed by writing down the geodesic equations:

$$\hat{g}_{tt}\left(\frac{dt}{d\lambda}\right)^2 + \hat{g}_{\theta\theta}\left(\frac{d\theta}{d\lambda}\right)^2 + e^{-\hat{\phi}/2}\boldsymbol{p}^2 + \hat{g}_{\phi\phi}^{-1}L^2 + 2\hat{g}_{t\theta}\left(\frac{dt}{d\lambda}\right)\left(\frac{d\theta}{d\lambda}\right)$$

$$+2\hat{g}_{t\phi}\left(\frac{dt}{d\lambda}\right)\left(\frac{L}{\hat{g}_{\phi\phi}}\right) + 2\hat{g}_{\phi\theta}\left(\frac{L}{\hat{g}_{\phi\phi}}\right)\left(\frac{d\theta}{d\lambda}\right) = k$$

where, $k = 0$ for null geodesics and $k = -1$ for timelike geodesics, $\boldsymbol{p}$ and L are conserved quantities associated with the Killing vectors $\boldsymbol{\partial}$ and ∂_ϕ. We have already showed that $\frac{dt}{d\lambda} = constant$ and all metric components and $e^{\hat{\phi}}$ are finite and bounded, from above equation, it is clear that $\frac{d\theta}{d\lambda}$ is bounded. Hence the six-dimensional geometry is geodesically complete. In the next section, we show that our solution evades the Hawking-Penrose singularity theorem because it does not admit any closed trapped surface.

3.3. Absence of Trapped Surface

The existence of closed trapped surface (CTS) is an essential ingredient in the proof of Hawking-Penrose singularity theorems. A closed trapped surface is a compact codimension-two spacelike surface, where both "ingoing" and "outgoing" null-congruence normal to the surface are converging. In this section, we show that the geometry described by (2.3.4) does not admit such a trapped surface (see Fig. 2.1). To prove the non-existence of CTS, we have to show that the product of the trace of the two null second fundamental forms is not positive.

Before we proceed to the calculations, we will provide a simple argument for the non-existence of closed trapped surfaces in (2.3.4). The six dimensional solution is obtained by reducing (2.3.1) along z direction. The existence of a CTS in six-dimensions would imply the existence of a CTS in seven dimensions because a CTS in 6D ($\mathcal{M}^{6D}_{CTS}$) will simply uplift to a CTS in seven dimensions ($\mathcal{M}^{7D}_{CTS} \equiv$ circle fibered over $\mathcal{M}^{6D}_{CTS}$) . But, the seven dimensional geometry does not admit a CTS since it is just a global coordinate patch covering entire flat space-time (which does not admit a CTS). Hence the six-dimensional solution does not admit a closed trapped surface. This argument relies on the fact that the size of the Kaluza-Klein circle is non-vanishing and finite.

We will now show that the six-dimensional geometry does not admit a CTS by explicitly computing the product of the expansion factors. This also implies the non-existence of a CTS in seven-dimensions. First, we rewrite the

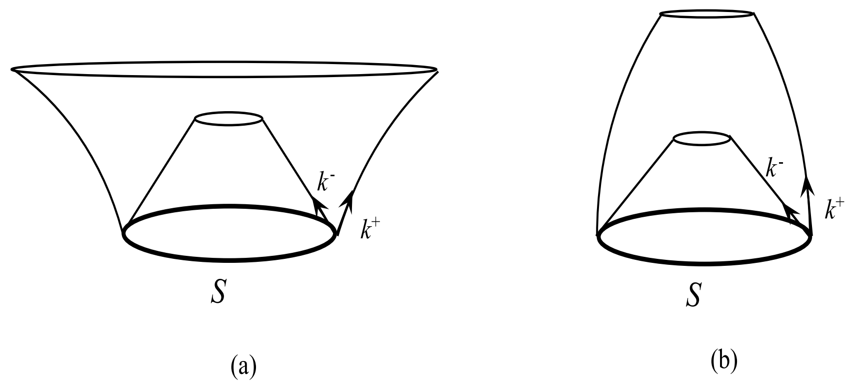

Figure 2.1: Shows (a) an untrapped surface ($\kappa < 0$) and (b) future trapped surface ($\kappa > 0$). k^+ and k^- are the null-vectors associated with the ingoing and outgoing null-congruences normal to the surface of S.

seven-dimensional metric in the following form for convenience.

$$ds^2 = -\hat{g}_{tt}dt^2 + e^{-\sigma/2}\left(d\rho^2 + \rho^2 d\theta_2^2 + \rho^2 \sin\theta_2^2 d\phi_2^2\right) + 2\hat{g}_{t\theta}dtd\theta + 2\hat{g}_{t\phi}dtd\phi$$
$$+2\hat{g}_{\theta\phi}d\theta d\phi + \hat{g}_{\theta\theta}d\theta^2 + \hat{g}_{\phi\phi}d\phi^2 \tag{2.3.8}$$

Since the non-compact spatial directions are homogeneous and isotropic, it is sufficient to show that a surface S, described by $t = t_0,\ \rho = \rho_0$ cannot be trapped, where t_0 and ρ_0 are some constants. The first fundamental form associated with the surface $t = t_0,\ \rho = \rho_0$ is

$$\gamma_{AB}dx^A dx^B = e^{-\sigma(t_0)/2}\left(\rho_0^2 d\theta_2^2 + \rho_0^2 \sin\theta_2^2 d\phi_2^2\right) + 2\hat{g}_{\theta\phi}(t_0)d\theta d\phi + \hat{g}_{\theta\theta}(t_0)d\theta^2 + \hat{g}_{\phi\phi}(t_0)d\phi^2$$

where $A, B \in \{\theta_2, \phi_2, \theta, \phi\}$. Note that this surface is a $\mathbb{T}^2$ fibered over a two-sphere. Now, we can define the future-directed ingoing and outgoing null 1-forms normal to this surface as follows

$$k^{\pm} \doteq e^{\pm 2\nu} e^{\sigma/4}\left(\frac{1}{\sqrt{2}}, 0, 0, 0, \pm\frac{1}{\sqrt{2}}, 0, 0\right) \tag{2.3.9}$$

where ν is an arbitrary function on the surface S. We can now compute the second fundamental form as follows:

$$\chi^{\pm}_{AB} = k^{\pm}_{\mu}\Gamma^{\mu}_{AB}\bigg|_S = \frac{k^{\pm}_{\mu} g^{\mu\rho}}{2}\left(\partial_A g_{\rho B} + \partial_B g_{A\rho} - \partial\rho g_{AB}\right)\bigg|_S \tag{2.3.10}$$

Now, let us define $\kappa = 2\left(\gamma^{AB}\chi^{+}_{AB}\right)\left(\gamma^{CD}\chi^{-}_{CD}\right)$. A simple procedure for computing κ can be found in [53]. The product of the trace of χ^{+}_{AB} and χ^{-}_{AB} is given by

$$\kappa = \Bigg[\frac{r'(t_0)^2 e^{-17\sigma/2}}{\alpha^6\rho_0^4 g_{\phi\phi}{}^2 r(t_0)^2}\Bigg(\alpha^3\rho_0^2\sin^2(\theta)r(t_0)^2\left(2e^{4\sigma}-\alpha\beta e^{2\sigma}\sqrt{g_{\phi\phi}}+2\alpha\beta^3\sqrt{g_{\phi\phi}}\right)+\alpha^3 e^{4\sigma}\rho_0^2 R^2$$

$$+e^{2\sigma}r(t_0)\Bigg(\beta\left(e^{2\sigma}-\beta^2\right)^2\cos(\theta)\cot(\theta_2)+\alpha^2\rho_0^2\sin(\theta)\left(-2\beta^3+2\beta e^{2\sigma}+3\alpha R e^{2\sigma}\right)\Bigg)\Bigg)^2$$

$$-\frac{4}{\rho_0^2 e^{\sigma/2}}\Bigg]_S$$

Note that κ is independent of ν. We will now show that κ cannot be positive everywhere if S is compact (S is compact only if ρ_0 is finite). First, note that when $r'(t_0) = 0$, κ is negative for all values of ρ. Hence, it is sufficient to consider the case where $r'(t_0)$ is non-zero.

Demanding positivity of κ at $\theta = \pi$ we get,

$$\rho_0^2 > \frac{e^{-4\tilde{\sigma}}}{r'(t_0)^2}\Bigg[2e^{3\tilde{\sigma}}r(t_0)^{3/2}\left(e^{2\tilde{\sigma}}r(t_0)+\alpha\beta R^2\cot(\theta_2)r'(t_0)^2\right)^{1/2}+$$

$$2e^{4\tilde{\sigma}}r(t_0)^2+\alpha\beta e^{2\tilde{\sigma}}R^2\cot(\theta_2)r(t_0)r'(t_0)^2\Bigg]$$

where $e^{2\tilde{\sigma}} = \alpha^2R^2+\beta^2$. Note that when $\theta_2 \to 0$, $\rho_0 \to \infty$ (α and β are non-zero). Similarly, ρ_0 diverges when $\theta_2 \to \pi$. Hence, κ cannot be positive when $\theta = \pi$ and $\theta_2 = 0$ or π unless ρ_0 is infinite. This shows that a trapped surface cannot be compact and hence the 6D solution in (2.3.4) does not admit a closed trapped surface.

2.4 Discussion

In this note, we studied a family of six-dimensional (and 7D) nonsingular cosmological solutions that can be obtained from 7D flat spacetime using simple solution generating techniques. We have shown that our solutions are free of closed trapped surfaces and hence they evade the Hawking-Penrose singularity theorems. Since, these solutions can be generated from flat space, it is straightforward to embed these solutions in string theory. In particular, the 7D solutions in appendix A can be obtained from solutions of type II supergravity by reducing along a $\mathbb{T}^3$ (with all RR field strengths set to zero).

In order to understand the physics as seen by a four dimensional observer, it seems essential to study the reduction to four-dimensions. However, it appears that the 6D and 7D solutions discussed in this paper do not have any simple description in four dimensions. When the warp factor is time dependent all Kaluza-Klein modes are excited and it is not clear how the higher Kaluza-Klein modes decouple from the lower dimensional effective action. There has been some work in the literature to understand the quadratic terms appearing in the lower dimensional effective action [55] in a general warped compactification. But at this point it is not clear how one can study the non-linear terms arising from such a reduction. In a general warped compactification, the nonlinear terms lead to mixing between arbitrary number of Kaluza-Klein modes and a procedure for consistently truncating to the lowest Kaluza-Klein modes is not yet known.

In this note, we have only focussed on geometries that are warped products of a $\mathbb{T}^2$ and 3+1 dimensional bouncing cosmology. However, the method used to obtain theses solutions can be used to generate solutions where the topology of the internal manifold is different from $\mathbb{T}^2$. In fact, there are solutions where the topology of the internal manifold changes dynamically. We will provide a simple example of such a solution here. Let us consider the solution in (2.3.4) when $\min(r(t)) < R \leq \max(r(t))$. When $r(t) < R$, the internal manifold is a ring torus and the six-dimensional metric in (2.3.4) describes a $\mathbb{T}^2$ fibered over $\mathbb{R}^{3,1}$, while the internal geometry has topological genus zero when $r(t) \geq R$ (see Fig. 2.2). This topology change can also happen periodically if $r(t)$ is periodic. Such topology changing transitions are singular ($g_{\phi\phi}$ vanishes when $r(t) = R$) even though the scalar field and the gauge field strength do not diverge. The Euler characteristic of the internal manifold is zero even when $r(t) \geq R$ because of the singularities.[8] We would like to point out that the topology changing transitions discussed here are similar to the dynamical topology change discussed in [54]. It would be interesting to study more general topology changes where the internal manifold with topological genus-g changes

[8]The Euler characteristic of a Riemann surface described by an algebraic curve with N_s singular points of multiplicities $m_1, \cdots, m_{N_s}$ and topological genus g is

$$\chi_e = 2 - 2g - \sum_{i=1}^{N_s} m_i(m_i - 1).$$

Note that the topological genus is different from the arithmetic genus for algebraic curves with singularities. The ring torus is topologically equivalent to an elliptic curve with no singularities while the spindle torus (see Fig. 2.2) is equivalent to an elliptic curve with a singularity of multiplicity 2.

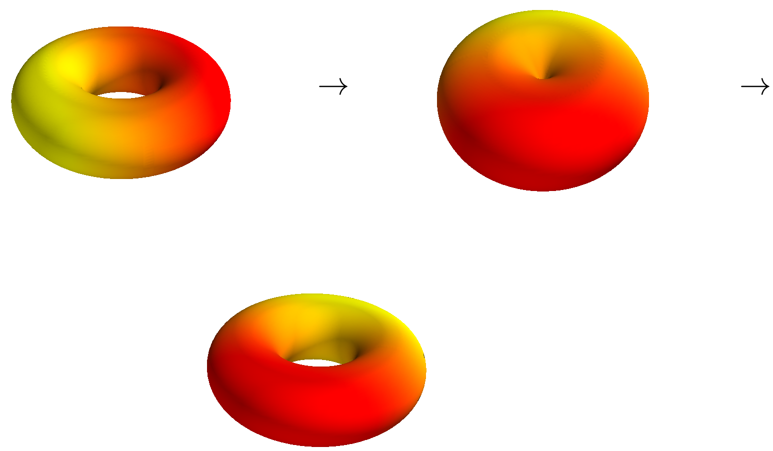

Figure 2.2: Topology change from a surface with topological genus one to a surface with genus (topological) zero.

to a geometry with topological genus-g'. These topology changing transitions suggest the possible existence of tunneling transitions that cannot be described by conventional Coleman-De Luccia instantons [56]. In particular, the lower dimensional effective theory framework used to describe Coleman-De Luccia instantons cannot describe tunneling transitions that involve mixing of an arbitrarily large number of Kaluza-Klein modes.

The family of solutions in (2.3.4) are free of singularities when $\max(r(t)) < R$, but it is not clear if these solutions are all stable. Since these solutions are obtained from flat solutions in higher dimensions, we expect these solutions to be perturbatively stable. It seems worthwhile to analyze the stability of these solutions.

Another concern that needs to be addressed is the following: How can such solutions be consistent with second law of thermodynamics? The gravitational entropy of the universe reaches a minimum when the universe bounces from a contracting phase to an expanding phase. When the geometry does not admit a closed trapped surface the definition of gravitational entropy is not even clear; in particular, it is not possible to define the gravitational entropy as the area of a Killing horizon. It seems that there exists some notion of time's arrow that can be defined using Raychaudhuri equation even when the universe bounces periodically. The arrow of time defined using the Rauchaudhuri equation is related to the the seven-dimensional. However, it is not clear if the thermodynamic arrow of time is actually related to this. It is also not clear, how quantum effects modify the singularity theorems. So a classical bouncing solution that is geodesically complete and stable could be unstable quantum mechanically.

Chapter 3

Positive Energy Vacua and Effective Potentials in String Theory

3.1 Background

It is well known that consistency of String Theory requires a 10-dimensional space-time, while maximal supergravity and its quantum version called “M theory” make sense in 11-dimensional space-time. In both cases, one makes contact with standard 4-dimensional physics by compactifying the extra dimensions to a small $n = 6$ or 7-dimensional compact manifold M — obtaining in this fashion a lower dimensional quantum theory of gravity with matter.

A primary goal of the work on compactifications is to derive an effective action in 4 dimensions — i.e., an action that could reproduce the observed 4-dimensional physics. This effective action is a functional of the 4-dimensional metric and whatever additional data parameterize the extra dimensions — its metric, and the other fields of supergravity or superstring theory — taken as functions on 4-dimensional space-time. Critical points of this effective action, in the usual sense of a variational principle, correspond to critical points of the original higher-dimensional supergravity or superstring action.

Earlier works on compactifications have relied heavily on supersymmetry, and supersymmetric constraints have been used to understand several aspects of string compactifications. The wide physical understanding brought by the study of supersymmetric models notwithstanding, there are at least two good reasons for investigating effective potentials that do not incorporate supersymmetry. The first is that, if it exists, supersymmetry is an exact symmetry of nature only at energy scales far beyond the validity of many of the effective descriptions. The second reason is the strong evidence that the cosmological constant, or vacuum energy, of our universe is positive. In the simplest effective descriptions of string theory, the vacuum energy of 4-dimensional space-time is given by an effective potential V_{eff}. Persistent physical features, like the sign of the cosmological constant, should typically be described by meta-stable local minima of V_{eff}. However, effective potentials with local minima corresponding to positive vacuum energy do not in general allow supersymmetry.

Here we shall be concerned with what can be called cosmological constraints for de-Sitter (dS) vacua. In other words, we consider the case of a maximally symmetric 4-dimensional space-time and seek conditions that guarantee the existence of meta-stable positive local minima of V_{eff}. Our focus will be on compactifications with Dq-branes and/or Oq-planes and Type IIB strings. Therefore, in a nutshell, the main contribution of the present work to the vast literature on effective descriptions of string theory can be summarized as follows: a mathematically rigorous proof of existence of *positive* local minima of V_{eff} under physically reasonable assumptions.

Many authors contributed to our current understanding of effective descriptions in string theory, and a thorough review would be beyond the scope of this manuscript. A detailed and seminal discussion of the matter can be found in [63], with subsequent properties investigated in [61]. The interested reader should also consult [57, 58, 71, 2, 70, 3, 69] and references therein for further details.

3.2 Setting and the Basic Equations

Consider compactification on an $n = D - d$-dimensional compact manifold M to d-dimensional maximally symmetric space-time (Minkowski, AdS, dS). In the D-dimensional space, consider General Relativity coupled to matter, the latter being encoded as usual in a set of field strengths $F^{(p)}$, $p = 1, \ldots, L$ (these are curvature terms, with the standard curvature of the Yang-Mills functional being the canonical example). The full D-dimensional metric is assumed to have the form of a Kaluza-Klein warped metric with a conformal factor,

$$ds^2 = e^{2A(x)}\eta_{\mu\nu}dz^\mu dz^\nu + e^{2B(x)}g_{ij}(x)dx^i dx^j,$$

where $\eta_{\mu\nu}$ is a metric on the 4-dimensional space-time (Minkowski, dS, AdS) with z^μ coordinates on it, g_{ij} is a metric on the internal manifold M, x^i are coordinates on M, and $x \in M$.

We shall adopt the notations and conventions of [61].

assumption: From now on we assume that $d = 4$ and $n = 6$. For simplicity, all quantities involved are assumed to be smooth[1] unless stated differently. Since many of the fields involved are usually distributional quantities, our point of view is that they have been properly smeared by, for example, convoluting against smooth functions.

The above smoothness assumption can certainly be relaxed with no difficulties. In fact, our existence theorems will hold in Sobolev spaces, so it suffices to assume that our fields have only a finite number of derivatives. A possible exception is the "string term" T_{string} (see below). Such a term is, in general, a distribution supported on hyperplanes. Hence, whenever necessary, it will be assumed that fields have been properly smeared or smoothed out, as mentioned above. The smearing of T_{string} notwithstanding, we point out that many of our results will remain valid, if however only suitable "integral bounds" — which allow for distributional coefficients — are imposed on T_{string} similarly to what was done in [61].

Following the construction of V_{eff} as in [63] yields

$$V_{eff} = \frac{1}{2}\int_M \Big(-u^2v^2R - 5\nabla v\cdot\nabla(u^2v) - 3v^2|\nabla u|^2 + \frac{u^2}{2}\sum_{p=0}^{L} v^{(3-p)}|F_p|^2$$

$$-u^2v^{(q-3)/2}T_{string}\Big) + \alpha\Big(\frac{1}{G_N} - \int_M uv^3\Big) + \beta\Big(\mathrm{Vol}(M) - \int_M v^3\Big), \qquad (3.2.1)$$

where $u = e^{2A}$ is called the warp factor, $v = e^{2B}$ is called the conformal factor, R is the scalar curvature of g, α and β are constants[2], G_N is Newton's constant, p, q and L are integers that depend on the particular model under consideration, and integrals are with respect to the natural volume element given by g. The dot "$\cdot$" is the inner product on the metric g, but we shall omit it when no confusion can arise and write simply $\nabla u\nabla v$ etc. The term T_{string} is added ad hoc to incorporate the non-classical contributions to the effective potential coming from string/M-theory.

It is shown in [63] that once the warped constraint is imposed (see below), the Lagrange multiplier α becomes the 4-d space-time scalar curvature. This provides a setup for justifying many effective potentials that have been studied in the context of string/M theory compactifications, especially regarding dS solutions. In particular, a given vacuum corresponds to a dS solution, and thus has a positive vacuum energy, if and only if $\alpha > 0$. The other Lagrange

[1]Orbifolds could also be included, in which case quantities should be smooth away from singularities. We have not treated orbifolds here to avoid technicalities; they will be the focus of a future work [59].

[2]These are in fact Lagrange multipliers in the sense that their variational equations enforce constraints.

multiplier, β, is used to obtain the minimum for the volume modulus related to the conformal factor. Also, it is important to notice that arguments related to Chern-Simons terms and warping done in [63] will also hold here, as those terms do not depend on the conformal factor.

We consider compactifications with Dq-branes or/and Oq-planes, and we want to study critical points of V_{eff}. The first variation of V_{eff} with respect to u and v in the direction of ψ and φ are, respectively,

$$\frac{\delta V_{eff}}{\delta u}(\psi) = \frac{1}{2}\int_M \Big(-2uv^2 R + 10uv\Delta v + 6\nabla(v^2\nabla u) + u\sum_{p=0}^{L} v^{(3-p)}|F_p|^2 \\ -2uv^{(q-3)/2}T_{string}\Big)\psi - \alpha\int_M v^3\psi, \tag{3.2.2}$$

and

$$\frac{\delta V_{eff}}{\delta v}(\varphi) = \frac{1}{2}\int_M \Big(-2u^2 vR + 5\Delta(u^2 v) + 5u^2\Delta v - 6v|\nabla u|^2 \\ +\frac{u^2}{2}\sum_{p=0}^{L}(3-p)v^{(2-p)}|F_p|^2 - \frac{(q-3)}{2}u^2 v^{(q-5)/2}T_{string}\Big)\varphi \\ -3\alpha\int_M uv^2\varphi - 3\beta\int_M v^2\varphi. \tag{3.2.3}$$

We must satisfy $\frac{\delta V_{eff}}{\delta u} = \frac{\delta V_{eff}}{\delta v} = 0$ at each critical point. From the above we obtain the following equations of motion:

$$\begin{cases} 10uv\Delta v + 6\nabla(v^2\nabla u) - 2uv^2 R + u\sum_{p=0}^{L} v^{(3-p)}|F_p|^2 - 2uv^{(q-3)/2}T_{string} = 2\alpha v^3, & (3.2.4a) \\ 5\Delta(u^2 v) + 5u^2\Delta v - 2u^2 vR - 6v|\nabla u|^2 + \frac{u^2}{2}\sum_{p=0}^{L}(3-p)v^{(2-p)}|F_p|^2 \\ \quad -\frac{(q-3)}{2}u^2 v^{(q-5)/2}T_{string} = 6[\alpha uv^2 + \beta v^2]. & (3.2.4b) \end{cases}$$

These are subject to the constraints

$$\begin{cases} \int_M uv^3 = \frac{1}{G_N}, & (3.2.5a) \\ \int_M v^3 = \mathrm{Vol}(M). & (3.2.5b) \end{cases}$$

Equation (3.2.5a) is sometimes referred to as the warped constraint.

The second variation of V_{eff} with respect to v and in the direction φ is

$$\frac{\delta^2 V_{eff}}{\delta v^2}(\varphi) = \frac{1}{2}\int_M \Big(-2u^2 R - 6|\nabla u|^2 + \frac{u^2}{2}\sum_{p=0}^{L}(3-p)(2-p)v^{(1-p)}|F_p|^2$$

$$-\frac{(q-3)(q-5)}{4}u^2 v^{(q-7)/2}T_{string}\Big)\varphi^2 - 5\int_M \nabla\varphi\nabla(u^2\varphi)$$

$$-6\alpha\int_M uv\varphi^2 - 6\beta\int_M v\varphi^2. \tag{3.2.6}$$

definition: The *mass squared of the conformal factor* or *volume modulus*, denoted $\frac{\partial^2 V_{eff}}{\partial v^2}$, is defined by

$$\frac{\partial^2 V_{eff}}{\partial v^2} := \frac{\delta^2 V_{eff}}{\delta v^2}(\varphi)\Big|_{\varphi=1}.$$

Solutions (v, u) of (3.2.4) such that

$$\frac{\partial^2 V_{eff}}{\partial v^2}\Big|_{(v,u)} > 0$$

are called stable[3] and unstable otherwise.

In our case,

$$\frac{\partial^2 V_{eff}}{\partial v^2} = \frac{1}{2}\int_M \Big(-2u^2 R - 6|\nabla u|^2 + \frac{u^2}{2}\sum_{p=0}^{L}(3-p)(2-p)v^{(1-p)}|F_p|^2$$

$$-\frac{(q-3)(q-5)}{4}u^2 v^{(q-7)/2}T_{string}\Big) - 6\alpha\int_M uv - 6\beta\int_M v. \tag{3.2.7}$$

We are interested in understanding the stability associated with the conformal factor v. Mostly in supersymmetric solutions, the warp factor u is related to v, and such solutions are stable. It is known, however, that one has to deal with instabilities in de Sitter vacua obtained from string compactifications. The KK mode mostly responsible for such instabilities is the conformal factor of the metric. Thus, a reasonable strategy is to hold the other fields coming from compactifications fixed and study the dynamics of the conformal factor v

[3]Here we use a slight abuse of language, as such a condition would be better called meta-stable, since tunneling to other vacua can occur. We shall, however, use the terms stable and meta-stable interchangeably.

along with the warp factor u. This can be done for general string/supergravity compactifications, but we mainly study theories related to Type IIB strings in this paper. In [3], it was found that in Type IIB string theory, one can achieve stability by fixing all massless fields to AdS vacuum with the help of non-perturbative effects. The authors add to the potential a term like an Anti-D3 brane. For suitable choices of the added potential term, the AdS minimum becomes a dS minimum, but the rest of the potential does not change significantly (this is the main idea behind the uplifting construction). This minimum is meta-stable. It is unstable to either quantum tunneling or thermal excitations over a barrier, in which case the Universe goes to infinity in moduli space after some time. Since we are not dealing with supersymmetric setups, we would like to stabilize v by finding conditions that imply $\frac{\partial^2 V_{eff}}{\partial v^2} > 0$ in general, and we suggest that non-perturbative effects are very small to disturb the minimum. Notice that as we are primarily interested in dS space, it will be natural to consider $\alpha > 0$ in much of what follows below.

3.3 Slowly varying Warp Factor and (in)stability analysis

One commonly investigated case is that of a slowly varying warp factor, i.e., $\nabla u \approx 0$ (see e.g. [63] and references therein). Here we consider two situations where it is shown that the system (3.2.4) can be solved for u sufficiently close to a constant. In one case, we shall obtain instability of the volume modulus, whereas in the second case, stability will be demonstrated. Our methods are based on the implicit function theorem and they also involve a perturbation of the coefficients of the equations. We comment on the legitimacy of this perturbation at the end. We do not necessarily impose (3.2.5) at this point, and a more thorough investigation of the existence of solutions to (3.2.4) will be carried out in a future work [59].

3.3.1 Unstable solutions

Assume $\alpha > 0$, and consider first the case of a constant warp factor. Plugging $u = \textit{constant}$ in (3.2.4), we see that upon the redefinition $\alpha \mapsto \alpha/u$ and $\beta \mapsto \beta/u^2$ we can assume $u \equiv 1$. Setting $u = 1$ implies that both (3.2.4a) and (3.2.4b) hold if

$$F_p = 0 \text{ except for } p = 1,\ q = 7, \text{ and } \beta = -\frac{2\alpha}{3},$$

and we henceforth suppose so, in which case both equations reduce to

$$10\Delta v - 2Rv + |F_1|^2 v - 2T_{string} v - 2\alpha v^2 = 0, \tag{3.3.1}$$

provided that $v > 0$. Equation (3.3.1) can be solved by the method of sub- and super-solutions (see e.g. [68] or [62] for the case with boundary). Write the equation as

$$\Delta v + f(v) = 0.$$

We seek functions v_- and v_+ such that $v_- \leq v_+$,

$$\Delta v_- + f(v_-) \geq 0,$$

and

$$\Delta v_+ + f(v_+) \leq 0.$$

Let $v_- \equiv constant > 0$. The differential inequality for v_- then becomes

$$v_- \leq \frac{\frac{1}{2}|F_1|^2 - T_{string} - R}{\alpha}.$$

Hence, if

$$\frac{1}{2}|F_1|^2 - T_{string} - R > 0,$$

we can choose v_- so small that the above inequality is satisfied. Similarly, the differential inequality for $v_+ \equiv constant$ becomes

$$v_+ \geq \frac{\frac{1}{2}|F_1|^2 - T_{string} - R}{\alpha},$$

which will be satisfied by choosing v_+ sufficiently large. The method of sub- and super-solutions now implies the existence of a smooth solution v_* to (3.3.1). This solution is positive because it satisfies $v_- \leq v_* \leq v_+$.

Solutions in the neighborhood of $v = v_*$, $u = 1$ can now be obtained with the help of implicit function-type theorems. Consider

$$M(v,u) = 10uv\Delta v + 6v^2\Delta u + 12v\nabla v\cdot\nabla u - 2uv^2 R + u\sum_{p=0}^{L} v^{3-p}|F_p|^2 - 2uv^2 T_{string} - 2\alpha v^3,$$

and

$$N(v,u) = 10u^2\Delta v + 10uv\Delta u + 20u\nabla u\cdot\nabla v + 10uv|\nabla u|^2 - 2u^2 vR + 4v|\nabla u|^2$$

$$+\frac{u^2}{2}\sum_{p=0}^{L}(3-p)v^{2-p}|F_p|^2 - 2u^2 vT_{string} - 6\alpha uv^2 - 6\beta v^2.$$

Let

$$h = -2R + |F_1|^2 - 2T_{string}.$$

A solution to (3.2.4) with $q = 7$ is then given by $M(v,u) = 0 = N(v,u)$. As we are interested in positive solutions, we can factor v from $M(v,u)$ and look equivalently for solutions of $\widetilde{M}(v,u) = 0 = N(v,u)$, where

$$\widetilde{M}(v,u) = 10u\Delta v + 6v\Delta u + 12\nabla v\cdot\nabla u - 2uvR + u\sum_{p=0}^{L} v^{2-p}|F_p|^2 - 2uvT_{string} - 2\alpha v^2.$$

Let $H^s = H^s(M)$ be the standard Sobolev spaces, where s is large. Define a map

$$G : \mathbb{R}\times\mathbb{R}\times H^s\times\ldots\times H^s\times H^s \to H^{s-2}\times H^{s-2},$$
$$(\alpha,\beta,F_0,F_1,\ldots,R,T_{string},v,u)\mapsto(\widetilde{M}(v,u),N(v,u)). \tag{3.3.2}$$

We claim that $D_{v,u}G(0,-\frac{2\alpha}{3},0,\ldots,0,v_*,1)(\chi_1,\chi_2)$ is an isomorphism, where $D_{v,u}$ is the derivative with respect to the last two components. Computing,

$$D_{v,u}G(0,-\frac{2\alpha}{3},0,\ldots,0,v_*,1)(\chi_1,\chi_2) = (P,Q),$$

where

$$P = 10\Delta\chi_1 + 6v_*\Delta\chi_2 + (h-4\alpha v_*)\chi_1 + 12\nabla v_*\cdot\nabla\chi_2 + 2\alpha v_*^2\chi_2, \tag{3.3.3}$$

and

$$Q = 10\Delta\chi_1 + 10v_*\Delta\chi_2 + (h-4\alpha v_*)\chi_1 + 20\nabla v_*\cdot\nabla\chi_2 - 2\alpha v_*^2\chi_2.$$

Given $\psi_1,\psi_2\in H^{s-2}$, we wish to solve

$$\begin{cases} 10\Delta\chi_1 + 6v_*\Delta\chi_2 + (h-4\alpha v_*)\chi_1 + 12\nabla v_*\cdot\nabla\chi_2 + 2\alpha v_*^2\chi_2 = \psi_1, & \text{(3.3.4a)}\\ 10\Delta\chi_1 + 10v_*\Delta\chi_2 + (h-4\alpha v_*)\chi_1 + 20\nabla v_*\cdot\nabla\chi_2 - 2\alpha v_*^2\chi_2 = \psi_2. & \text{(3.3.4b)}\end{cases}$$

Subtracting (3.3.4a) from (3.3.4b) produces

$$4v_*\Delta\chi_2 + 8\nabla v_*\cdot\nabla\chi_2 - 4\alpha v_*^2\chi_2 = \psi_2 - \psi_1. \tag{3.3.5}$$

Since $\alpha > 0$ and $v_* > 0$, a standard argument with the maximum principle and the Fredholm alternative shows that Eqn. (3.3.5) has a unique H^s solution.

Plugging it back into (3.3.4a) gives a scalar equation for χ_1 only, which will, again by the maximum principle and the Fredholm alternative, have a unique solution provided that

$$h - 4\alpha v_* < 0.$$

We conclude that if the above is satisfied, then $D_{v,u}G(0, -\frac{2\alpha}{3}, 0, \dots, 0, v_*, 1)$ is an isomorphism. Invoking now the implicit function theorem and recalling that $v_- \leq v_* \leq v_+$, we conclude the following.

Theorem 3.3.1. *Let $\alpha > 0$, $q = 7$, and fix a sufficiently large real number s. Denote by v_- and v_+, respectively, the minimum and maximum of $\frac{1}{\alpha}(\frac{|F_1|^2}{2} - R - T_{string})$. If β is sufficiently close to $\frac{-2\alpha}{3}$ and F_p is close enough to 0 in the H^s-topology, except possibly for $p = 1$, and if $\frac{v_+}{v_-} < 2$, then there exists a unique H^s solution (v, u) to the system (3.2.4) in a small H^s-neighbourhood of $(v_*, 1)$, where v_* is a solution of (3.3.1) satisfying $v_- \leq v_* \leq v_+$. The solution satisfies $v > 0$, $u > 0$, and depends continuously on α, β, F_p, R, and T_{string}.*

Remark 3.3.2. By taking s sufficiently large and applying the Sobolev embedding theorem, we can replace the H^s-neighborhood with a C^k-neighborhood in the previous statement. A similar statement holds for the other theorems presented below.

We notice that $v > 0$ and $u > 0$ follow by making the H^s-neighborhood of the theorem very small and using $v_* > 0$; since s is assumed to be large, these solutions are in fact continuous and the pointwise inequalities $v > 0$ and $u > 0$ then hold. We also remark that starting with smooth F_1, R, and T_{string} does not necessarily yield smooth solutions. This is because the perturbed F_p produced by the implicit function theorem will, in general, be only in H^s. If they happen to be smooth, however, then v and u are smooth due to elliptic regularity. Indeed, if $M(v, u) = 0 = N(v, u)$, then $vN(v, u) - uM(v, u) = 0$, which takes the form

$$4uv^2 \Delta u = f(u, v, \nabla u, \nabla v),$$

where f is a smooth function of its arguments. Since $v, u \in H^s$, $f(u, v, \nabla u, \nabla v) \in H^{s-1}$ and so $u \in H^{s+1}$ by elliptic regularity. $M(v, u) = 0$ and elliptic regularity then give $v \in H^{s+1}$, and bootstraping this argument, we conclude that v, u are smooth.

We now show that solutions given by Theorem 3.3.1 are in general unstable. Evaluating (**??**) at $(v_*, 1)$ yields

$$\left. \frac{\partial^2 V_{eff}}{\partial v^2} \right|_{(v_*,1)} = \frac{1}{2} \int_M (h - 2\alpha v_*),$$

where h is as above. Multiplying (3.3.1) by v_* and integrating by parts,

$$\int_M (h - 2\alpha v_*) = -\int_M \frac{1}{v_*^2} |\nabla v_*|^2 \leq 0,$$

so that $\left.\frac{\partial^2 V_{eff}}{\partial v^2}\right|_{(v_*,1)} \leq 0$, and the inequality is in fact strict if v_* is not constant. Notice that $v_* = constant$ will not be a a solution of (3.3.1) unless further relations among the scalar curvature, the gauge fields, and the string term hold. It follows that the strict inequality will be preserved for solutions very close to $(v_*, 1)$, and we conclude:

Corollary 3.3.3. *If v_* is not constant, then the solutions (v, u) given by theorem 3.3.1 are unstable in the sense of definition 3.2, provided that (v, u) is sufficiently close to $(v_*, 1)$.*

3.3.2 Stable solutions and applications to Type IIB strings

Using different hypotheses than those of the previous section, here we prove existence of stable solutions to (3.2.4). The key ingredient is to balance the contribution of the gauge fields with that of the source T_{string}. This is, in fact, an idea that goes back to [2, 3] and has been extensively used in moduli stabilization [71, 70]. We shall impose conditions that lead to a direct application to dS-vacuum in Type IIB strings. In fact, we shall provide a set of slightly different theorems applicable to different Type IIB scenarios[4]. For the rest of this section we suppose the following.

Assumption 3.3.4. Let $q = 3$, and we suppose from now on that $F_0 = F_2 = F_4 = F_6 = 0$, as these fluxes are not present in Type IIB compactifications.

The arguments that follow will be similar to the proof of theorem 3.3.1, and they are all more or less analogous to each other. Hence, in order to avoid being repetitive, we shall go through them rather quickly. Set $q = 3$, $\alpha = \beta = R = 0$ and $F_p = 0$ for $p \neq 3$ in (3.2.4), and plug in $v = u = 1$. Then (3.2.4b) holds identically and (3.2.4a) is satisfied provided that

$$-2T_{string} + |F_3|^2 = 0, \tag{3.3.6}$$

so we hereafter assume this condition. We point out that, other than being similar to previous balance conditions used in uplifting [70, 2, 71, 3], (3.3.6) also resembles a local version of the tadpole cancellation appearing in [2, 67]. In

[4] It will be clear from what follows that similar arguments can be constructed in other settings.

fact, in supersymmetric solutions, like in [67], one can see that the contribution of the 3-flux to the overall potential gets a cancellation from localized sources. This follows as an application of the integrated Bianchi identity. In our case, the assumption shows the cancellation locally. Thus, it would be interesting to explore the relationship between this assumption and the Bianchi identity.

Theorem 3.3.5. *Let $q = 3$, assume that $|F_3|^2 - 2T_{string} = 0$, and fix a sufficiently large real number s. If α, β are close enough to zero, and R, F_p are sufficiently close to zero in the H^s topology, except possibly for $p = 3$, then the system (3.2.4) has a H^s solution (v, u) in a small H^s-neighborhood of $(1, 1)$. Such a solution satisfies $v > 0$, $u > 0$, and it depends continuously on α, β, R, F_p, T_{string}, and on two real parameters, l_1, l_2 (these parameterize the kernel of (3.3.7) below).*

Proof. Let

$$W = 10uv\Delta v + 6v^2\Delta u + 12v\nabla v\cdot\nabla u - 2uv^2R + u\sum_{p=1,3,5} v^{(3-p)}|F_p|^2 - 2uT_{string} - 2\alpha v^3,$$

and

$$Z = 10u^2\Delta v + 10uv\Delta u + 20u\nabla u\cdot\nabla v + 10uv|\nabla u|^2 + 4v|\nabla u|^2 - 2u^2vR$$
$$+\frac{u^2}{2}\sum_{p=1,3,5}(3-p)v^{2-p}|F_p|^2 - 6\alpha uv^2 - 6\beta v^2.$$

Consider the map

$$J : \mathbb{R}\times\mathbb{R}\times H^s\times\cdots\times H^s\times H^s \mapsto H^{s-2}\times H^{s-2}, (\alpha, \beta, |F_1|^2, |F_3|^2, |F_5|^2, R, T_{string}, v, u)$$
$$\mapsto (W, Z).$$

Solutions are given by $W = 0 = Z$, and $u = v = 1$ is a solution when $\alpha = \beta = F_1 = F_5 = R = 0$ and (3.3.6) holds. Linearizing at $(1, 1)$ and in the direction of (χ_1, χ_2), and setting it equal to (ψ_1, ψ_2) gives

$$\begin{cases} \Delta\chi_1 = \psi_1, \\ \Delta\chi_2 = \psi_2. \end{cases} \tag{3.3.7}$$

As M is compact without boundary, its harmonic functions are constant and (3.3.7) has a unique solution modulo additive constants. The implicit function theorem is not, therefore, directly applicable, but we can still rely on it after restricting the linearization to the L^2-orthogonal to its kernel, what produces solutions near $(1, 1)$. A parameterization of the kernel of (3.3.7) yields the parameters (l_1, l_2) of the theorem. As in the previous section, these solutions are positive if the H^s-neighborhood is sufficiently small. □

Next, we investigate stability. With $q = 3$, $\alpha = \beta = R = 0$, and $F_p = 0$ for $p \neq 3$, evaluating (**??**) at $(1, 1)$ yields

$$\left.\frac{\partial^2 V_{eff}}{\partial v^2}\right|_{(1,1)} = 0. \tag{3.3.8}$$

The "uplifting" approach [2, 3] to constructing positive energy vacua consists, in a nutshell, of starting with an AdS supersymmetric vacuum ($\alpha < 0$), where stability can be achieved, and deforming the data to a dS ($\alpha > 0$) vacuum. With proper control of this deformation, the new vacuum can be shown to be stable. Furthermore, experience shows that $\alpha < 0$ generally favors stability [61, 71]. Therefore, on physical grounds, we expect the continuous dependence on the data guaranteed by theorem 3.3.5 to allow us to continue solutions from $\alpha = 0$ to $\alpha > 0$, while changing the equality on (3.3.8) to a strict "greater than" inequality. The favorable physical arguments and apparent absence of a preventative mechanism notwithstanding, in order to prove stability, further hypotheses are needed. Interestingly enough, such hypotheses concern the value of some natural constants that arise in elliptic theory and which are ultimately tied to the topology and geometry of M. This is consistent with experience in compactifications, where global properties of the compact manifold play an important role in moduli stabilization.

Theorem 3.3.6. *(dS stability in Type IIB with slowly varying warp factor) Assume the same hypotheses of theorem 3.3.5. Let K_1 be the norm of the map $\Delta : H^s_0 \to H^{s-2}$, where $H^s_0 := H^s / \ker \Delta$, and let K_2 be the best Sobolev constant of the embedding $H^s \hookrightarrow C^1$. If $\frac{K_2}{K_1} < 26$ holds[5], then it is possible to choose $\alpha > 0$, $\beta < 0$, F_1, F_5, and R all sufficiently small, such that the corresponding solutions (v, u) with $l_1 = l_2 = 0$ given by theorem 3.3.5 are stable in the sense of definition 3.2.*

Proof. Compute

$$D_x J(x_0, 1, 1)\chi = \begin{pmatrix} -2 & 0 & 1 & 1 & 1 & -2 & -2 \\ -6 & -6 & 1 & 0 & -1 & -2 & 0 \end{pmatrix} \cdot \chi,$$

where x is shorthand for $(\alpha, \beta, |F_1|^2, |F_3|^2, |F_5|^5, R, T_{string})$, x_0 stands for $(0, 0, 0, 2T_{string}, 0, 0, T_{string})$, and $\chi = (\chi_1, \ldots, \chi_7) \in \mathbb{R} \times \mathbb{R} \times H^s \times \cdots \times H^s$.

[5] This will be the case, for example, if g_{ij} is sufficiently close to the Euclidean metric.

From this expression it follows that

$$\partial D_x J \partial < 26,$$

if (x, v, u) is sufficiently close to $(x_0, 1, 1)$. By theorem 3.3.5, we have solutions $J(x, v(x), u(x)) = 0$, and the implicit function theorem guarantees that the map $x \mapsto (v(x), u(x))$ is differentiable. The map $D_{v,u}J$, being an isomorphism at $(x_0, 1, 1)$, will be invertible nearby, thus

$$D_x \begin{pmatrix} v \\ u \end{pmatrix} = -\left(D_{v,u}J\right)^{-1} D_x J.$$

Keeping $\alpha = F_1 = F_5 = R = 0$ and $|F_3|^3 = 2T_{string}$, we find, under the assumptions of the theorem and invoking the Sobolev embedding theorem, that

$$\partial u - 1 \partial_{C^1} \leq |\beta|,$$

where $\partial \cdot \partial_{C^1}$ is the standard C^1 norm. Furthermore, since $u = v \equiv 1$ when $\beta = 0$, we can choose $\beta < 0$ so small that

$$\frac{1}{2} \int_M -6|\nabla u|^2 - 6\beta \int_M v > 0.$$

From (**??**), it now follows that we can pick $\alpha > 0$ and the remaining fields so small that

$$\frac{\partial^2 V_{eff}}{\partial v^2} > 0,$$

finishing the proof. □

From (**??**), we see that $\beta < 0$ favors stability, hence it would be natural to expect that $\frac{\partial^2 V_{eff}}{\partial v^2} > 0$ if $\beta \ll -1$. Theorem 3.3.5, however, only guarantees the existence of solutions for β close to zero. We therefore consider another condition which allows β to be considerably negative, namely,

$$|F_3|^2 - 2T_{string} = 2R - |F_1|^2 = -6\beta. \tag{3.3.9}$$

We readily check that if (3.3.9) holds and $\alpha = F_5 = 0$, then $v = u = 1$ is a solution of (3.2.4). Arguing similarly to the proof of theorem 3.3.5 and recalling (**??**) leads to:

Theorem 3.3.7. *(dS stability in Type IIB with slowly varying warp factor and $\beta \ll -1$) Let $q = 3$, assume (3.3.9), and fix a sufficiently large real number s. If α is close enough to zero, F_5 is sufficiently close to zero in the H^s topology,* *and β is sufficiently negative, then the system (3.2.4) has a H^s solution (v, u) in a small H^s-neighborhood of $(1, 1)$. Such a solution satisfies $v > 0$, $u > 0$, depends continuously on α, β, R, F_p, T_{string}, and on two real numbers l_1, l_2 parameterizing the kernel of (3.3.7). Furthermore, this solution is stable in the sense of definition 3.2, provided that we choose $l_1 = l_2 = 0$.*

Remark 3.3.8. Although upon setting $\beta = R = F_1 = 0$, (3.3.6) can be obtained from (3.3.9), the interest in the latter is, of course, when β is large negative, as in theorem 3.3.7, in which case, theorem 3.3.6 does not apply.

A clear limitation of theorem 3.3.7 is the lack of a precise bound on how negative β ought to be. This is important because, since β is related to $\mathrm{Vol}(M)$, a large $|\beta|$ might be out of the supergravity limit in parameter space. We still find it interesting to state theorem 3.3.7, however, because a closer inspection suggests that a moderately negative β should suffice, as long as something like (3.3.9) holds. Confirming this requires a sharper understanding of the solutions to (3.2.4), which will be carried out elsewhere [59]. Furthermore, we can still study compactifications in the low energy limit without imposing supersymmetry, and hence considering supergravity, provided that we solve the full higher dimensional Einstein's equations coupled to matter. We could then start with solutions with $\beta \ll -1$, and by varying β towards zero, investigate how far in parameter space the low energy description remains valid.

Together, theorems 3.3.5, 3.3.6, and 3.3.7 establish satisfactory properties of the effective description with a slowly varying warp factor; they give conditions for existence of solutions to the equations of motion and their stability. Many times, however, it is important to have a more explicit account of the functions u and v. In particular, one is interested in expanding $u = 1+\varepsilon$, where ε is a fairly tractable (perhaps explicit) small function. To accommodate this situation, we turn our attention to another existence result.

Set $\mathcal{F}_p = |F_p|^2$, $a = 2R - \mathcal{F}_1$, and $b = \mathcal{F}_3 - 2T_{string}$. Let

$$X = 10uv\Delta v + 6v^2\Delta u + 12v\nabla v \cdot \nabla u - auv^2 + uv^{-2}\mathcal{F}_5 + bu - 2\alpha v^3,$$

and

$$Y = 10u^2\Delta v + 10uv\Delta u + 20u\nabla u \cdot \nabla v + 10uv|\nabla u|^2 - au^2v + 4v|\nabla u|^2$$
$$-u^2v^{-3}\mathcal{F}_5 - 6\alpha uv^2 - 6\beta v^2,$$

so that solutions to (3.2.4) are given by $X = 0 = Y$. Notice that $u = v = 1$ is a solution to $X = Y = 0$ if $\mathcal{F}_5 = 0$, $\alpha = 0$, and $a = b = -6\beta > 0$.

It is easily seen that if $X = Y = 0$, then by inverting a matrix one can solve for a and b in terms of $(\alpha, \beta, \mathcal{F}_5, u, v, \nabla u, \nabla v, \Delta u, \Delta v)$. In other words,

Lemma 3.3.9. *Let $q = 3$, assume that $F_0 = F_2 = F_4 = F_6 \equiv 0$, and fix a sufficiently large real number s. If (v, u) is sufficiently close to $(1, 1)$ in the H^s topology, then the following holds. There exist H^{s-2} functions a and b, depending continuously on $\mathcal{S} := (\alpha,\, \beta,\, F_5,\, v,\, u)$ such that the system (3.2.4) is satisfied upon replacing $2R - |F_1|^2$ and $|F_3|^2 - 2T_{string}$ with a and b, respectively, and the remaining data taking the values given by $\mathcal{S}$.*

We comment on the difference between theorem 3.3.5 and lemma 3.3.9, which consists basically in what is treated as independent or dependent data. Theorem 3.3.5 states that if we start with the solution $(1, 1)$ and slightly perturb the other fields, then it is possible to find functions near $(1, 1)$ that satisfy the perturbed equation. Lemma 3.3.9, on the other hand, says that if we choose any two functions u and v close to one, then we can find functions fitting the remaining data of the equation (namely, a and b) in order to force v and u to be solutions. Although theorem 3.3.5 is a more standard existence theorem, lemma 3.3.9 has the advantage of allowing one to explicitly construct solutions of the form $u = 1 + \varepsilon_1$ and $v = 1 + \varepsilon_2$, which are useful in asymptotic analysis of the problem.

Corollary 3.3.10. *It is possible to choose $\alpha > 0$, $\beta < 0$, F_5, all sufficiently small, and (v, u) sufficiently close to $(1, 1)$, such that the corresponding solutions given by lemma 3.3.9 are stable in the sense of definition 3.2.*

Proof. This corollary is a direct consequence of the following identity, which is, in turn, proven by dividing equation (3.2.4b) by v and integrating,

$$\frac{\partial^2 V_{eff}}{\partial v^2} = -10 \int_M \frac{u^2 |\nabla v|^2}{v^2} + \int_M u^2 R + 3 \int_M |\nabla u|^2 - \frac{1}{2} \int_M u^2 \mathcal{F}_1 + \frac{5}{2} \int_M u^2 v^{-4} \mathcal{F}_5.$$

□

We finally comment on the fact that, in order to construct solutions, we allowed the several fields in (3.2.4) to vary. In other words, we are not solving for v and u given fixed data α, β, R, F_p and T_{string}, but these data themselves are allowed to change slightly so that solutions are found. Had we been investigating a set of equations taken as the fundamental equations of a theory, this approach would certainly be problematic. The warped-conformal factor system (3.2.4), however, is only an approximation to the fundamental set of equations of string/M-theory, and as such, they can be slightly adjusted as long as their relevant physical content is kept unchanged. This possibility of tweaking the several fields involved is, in fact, what has allowed physicists to pursue programs like moduli stabilization and the construction of meta-stable positive energy vacua.

3.4 Volume estimates and non-perturbative effects

In this section we derive some basic identities and inequalities that relate Vol(M) with the other quantities of the problem. We assume throughout this sections that we are given positive solutions u and v of (3.2.4). Because $n = 6$, we must have $L \leq 6$. Notice also that many of the bounds below involve integral quantities, and, therefore, the smoothness of Assumption 3.2 can be relaxed, as mentioned in Remark 3.2.

Lemma 3.4.1. *The following identity holds:*

$$\int_M \left(\frac{u^2}{2}\sum_{p=0}^{L}(1-p)v^{3-p}|F_p|^2 + \frac{7-q}{2}u^2 v^{(q-3)/2}T_{string}\right) = \frac{4\alpha}{G_N} + 6\beta\, Vol(M). \tag{3.4.1}$$

Proof. Equations (3.2.4) can be written as

$$12u\nabla v\cdot\nabla u + 6uv\Delta u + 10u^2\Delta v - 2u^2 vR + u^2\sum_{p=0}^{L} v^{2-p}|F_p|^2 - 2u^2 v^{(q-5)/2}T_{string} = 2\alpha v^2 u,$$

and

$$\begin{aligned} &20u\nabla v\cdot\nabla u + 10uv\Delta u + 10u^2\Delta v - 2u^2 vR + 4|\nabla u|^2 v \\ &+\frac{u^2}{2}\sum_{p=0}^{L}(3-p)v^{2-p}|F_p|^2 - \frac{q-3}{2}u^2 v^{(q-5)/2}T_{string} = 6(\alpha v^2 u + \beta v^2). \end{aligned}$$

Subtracting the first from the second,

$$\begin{aligned} &8u\nabla v\cdot\nabla u + 4uv\Delta u + 4|\nabla u|^2 v + \frac{u^2}{2}\sum_{p=0}^{L}(1-p)v^{2-p}|F_p|^2 \\ &+\frac{7-q}{2}u^2 v^{(q-5)/2}T_{string} = 4\alpha v^2 u + 6\beta v^2. \end{aligned} \tag{3.4.2}$$

Multiplying equation (**??**) by v, integrating and using (3.2.5) gives (3.4.1). □

Lemma 3.4.1 can be used to derive useful relations among α, β, and the volume of compact dimensions. The only positive contribution from the gauge fields to the right hand side of (3.4.1) comes from F_0, hence one expects that it is possible to tune the gauge fields' contribution in order to balance that of the string term so that

$$\frac{4\alpha}{G_N} + 6\beta \mathrm{Vol}(M) \leq 0. \tag{3.4.3}$$

It follows that if $\alpha > 0$ (which is the main case of interest, as previously pointed out), then necessarily $\beta < 0$. This is important because, in general, no simple physical condition fixes β, and in this case we also have the bound

$$\mathrm{Vol}(M) \geq \frac{2\alpha}{3G_N|\beta|}.$$

The standard strategy in string compactification is to perform the analysis in the supergravity limit and consider string theory as the UV cutoff for the effective field theory. This imposes the radius R_c, and hence the volume of M, of the compactified six dimensions to be much larger than the string length, $\ell_{string} = \sqrt{\alpha'}$, and moderately weak coupling, $g_s \to 0$. The effective 4-dimensional Planck scale, $M^2_{Pl,4} = \frac{1}{G_N}$, is determined by the fundamental 10-dimensional Planck scale, $M_{Pl,10}$ (set to be equal to one in [63]), and the geometry of the extra dimensions with warping. Thus, $\frac{1}{G_N} = \mathrm{Vol}(M)_{warped}$. We have no experimental signs of the extra dimensions because the compactification scale, $M_c \sim 1/(\mathrm{Vol}(M)_{warped})^{1/n}$ would have to be smaller than the observable particle physics scale. Thus, in general, we would like to set β such that the supergravity limit is valid (i.e. $\mathrm{Vol}(M) >> \ell^6_{string}$), and the compactification scale is beyond the current observable scale (which is TeV scale in standard units). A detailed physics discussion can be found in [71].

As stated, Lemma 3.4.1 is quite general, and hence inequality (3.4.3) should be valid under a wide range of scenarios. Given a particular model, however, it may be difficult to verify that the integral on the left hand side of (3.4.1) is negative. We therefore point out two further situations involving pointwise rather than integral conditions, where (3.4.3) holds, and in which the verification of the hypotheses is more direct. One is when $q = 3$, $F_0 = 0$, and $2T_{string} - |F_3|^2 \leq 0$. The other is when $q = 7$ and $F_0 = 0$. In both cases, it follows at once that the left hand side of (3.4.1) is non-positive.

Any geometric compactification of string/M-theory has a large volume limit which approaches ten or eleven dimensional Minkowski space. In this limit, the four dimensional effective potential vanishes. De Sitter models from string compactifications are difficult to construct because, as non-supersymmetric vacua, they are isolated points in the moduli space with all moduli stabilized. To understand de Sitter solutions, one must have sufficient understanding of non-perturbative effects to show that such potentials could come from string theory, and, moreover, could be computed in some examples in order to make real predictions. Such effects usually give AdS vacua with all moduli stabilized, and then uplifting is achieved with a proper classical contribution. This has been an active research topic within string compactifications.

Since we are looking at Type IIB theory mainly, non-perturbative effects are added to stabilize the Kähler moduli. In our approach, we have not included such effects yet. Non-perturbative effects give a contribution to the overall potential in the following way [65]:

$$V_{non-pert} = Be^{2av^4}/v^s,$$

such that the constants a and s arise from Euclidean D3 brane instantons or gluino condensation, as explained in [3]. We want to study the conformal factor classically, and we claim that non-pertubative effects are added such that they do not affect the critical point and its mass noticeably. This can be written qualitatively as

$$\frac{\partial^2 V_{eff}}{\partial v^2} \gg \frac{\partial^2 V_{non-pert}}{\partial v^2}$$

at critical points. Thus, we should expect to find locally stable minima with positive cosmological constant.

Chapter 4

Supersymmetric Compactifications with Toric Manifolds

4.1 Background

The widely studied supersymmetric compactifications use Calabi-Yau manifolds as the internal space. As soon as one turns on background fluxes to obtain supersymmetric vacua, the internal manifold cannot be Calabi-Yau. The main ideas of string compactification with fluxes are discussed in [70, 71, 72], to name few. By now several supersymmetric AdS_4 flux vacua are known. Constructing a string vacuum with a positive cosmological constant poses lots of difficulties, such as use of orientifolds to evade No-go theorems. For Type IIB solutions, one needs to play with non-perturbative effects for Kahler moduli potential. In case of type IIA compactifications, one can turn on fluxes and all geometric moduli fields can be stabilized classically with O6-planes where the supergravity description is valid[73]. Recently, some issues with such moduli stabilization with O6-planes are discussed in [74]. A lot of progress is happening in Type II/Heterotic flux compactification and uplifting to dS vacua, but here we will focus on string vacua with negative cosmological constant with some supersymmetry.

Toric varieties have played an important role in string compactifications and mirror symmetry: as Calabi-Yau manifolds are embedded in them as hypersurfaces. Once fluxes are turned on, the three-dimensional smooth, compact toric varieties can be used for compactifications with the help of SU(3)-structures, instead of considering them as embedding spaces. In string theory/M-theory compactifications, $\mathbb{CP}^3$ has played a great role in constructing explicit examples. We will study how to use more general smooth toric manifolds for flux compactification following the procedure given in [78, 79].

We will mainly deal with the symplectic quotient description of smooth toric variety and the construction of SU(3)-stucture on it. Massive type IIA vacua with $\mathbb{CP}^3$ was obtained by considering $\mathbb{CP}^3$ as a twistor fibration of S^2 on S^4 with unusual almost complex structure which is not integrable. In section IV, we will discuss topological restrictions for carrying out the procedure of changing almost complex structure similar to $\mathbb{CP}^3$ on smooth toric manifolds. This puts the constraint on large number of toric manifolds in order to use them for flux compactifications. The first Chern class is commonly used to study string compactification, in section IV.C, we study Top Chern class and propose its use in the compactifications with SU(3)x SU(3) or strict SU(3) structure manifolds.

In section V, we will carry out the local analysis of SU(3) structure conditions and will show that we have many parameters to change the torsion classes associated with SU(3)-structure, this leads to the possibilty of obtaining more Type IIA flux vacua using smooth, compact toric manifolds. Also, one can use the procedure for compactification of Heterotic theories [91]. The similar idea of adjusting torsion classes for Heterotic solutions is explored in [90].

4.2 Basics of G-structures

Supersymmetry requires the existence of a nonvanishing, globally well defined spinor on the internal manifold. This condition puts some topological restrictions on the internal manifold. This is very well understood and various cases are known for supersymmetric vacua [75]. Numerous cases for Type II supergravity with such restrictions are known by now. For our purpose some useful cases are mentioned in [80, 82, 83, 84, 85]. We mainly focus on strict SU(3) and dynamic SU(3)×SU(3) structures. SU(3) structure manifolds play key roles in $N = 1$ compactification of Heterotic strings and Type IIA compactifications.

4.2.1 Mathematical Terminology

Let's first understand mathematics associated with G-structures. In this section, we use the conventions of [76, 77].

The tangent frame bundle FM, associated to the tangent bundle TM is the bundle over the manifold M with fiber in each point $p \in M$ the set of ordered bases of the tangent space T_pM. M has a G-structure with $G \subset GL(d, \mathbb{R})$ if by an appropriate choice of local frame in the different patches one can obtain reduced tangent frame bundle which has structure group G.

For this presentation, we restrict to SU(3) structure on 6-manifolds.

4.2.2 Strict SU(3)-structure

Here we discuss the idea of SU(3) structure in more details.Manifolds with SU(3) structures admit one globally defined, nonvanishing spinor η. This structure can be understood through (J,Ω) forms. J is a real (1,1) form and Ω is a complex (3,0) form such that $J \wedge \Omega = 0$ and $i\Omega \wedge \bar{\Omega} = \frac{4}{3}J^3 \neq 0$. In terms of spinors, one can write $J_{ab} = i\eta_-^\dagger \gamma_{ab}\eta_-$ and $\Omega_{abc} = \eta_-^\dagger \gamma_{abc}\eta_+$.

1. $dJ = \frac{3}{2}Im(\bar{W}_1\Omega) + W_4 \wedge J + W_3$,

2. $d\Omega = W_1 J^2 + W_2 \wedge J + \bar{W}_5 \wedge \Omega$.

Here W_1 is a complex scalar, W_2 is a complex primitive (1,1) form, W_3 is a real primitive (1,2)+(2,1) form, W_4 is a real one form and W_5 is a complex (1,0) form.

If $W_1 = W_2 = 0$, then the manifold is complex and $W_1 = W_3 = W_4 = 0$ will be a symplectic manifold. All $W_i = 0$ lead to Calabi-Yau.

4.3 G-structures on smooth, compact Toric manifolds

The toric manifold M_6 is Kähler and admits a global U(3) structure naturally.

4.3.1 SU(3)-structure

In this section, we discuss a general procedure for constructing string compactifications on smooth toric varieties via a method for producing SU(3)-structures[78],[79].

Consider the quotient description of the toric variety(Real dimension $2d$)[81]. If $\{z_i, i = 1,..,n\}$ are the holomorphic coordinates of ambient space $\mathbb{C}^n$ such that toric action is $\{z_i \to e^{iQ_i^a \alpha_a z^i}\}$, then the toric variety is described as

$$\mathcal{M}_{2d} = \{z^i \in \mathbb{C}^n | \sum_i Q_i^a |z^i|^2 = \xi^a\}/U(1)^s. \tag{4.3.1}$$

The toric variety has the induced real form $\tilde{J}_{FS}$ and a complex d-form $\tilde{\Omega}_{FS}$.

$$\tilde{\Omega}_{FS} = (det(g_{ab}))^{-1/2} \Pi_a \imath_{V^a} \Omega_{\mathbb{C}}. \tag{4.3.2}$$

Here $\Omega_{\mathbb{C}}$ is a holomorphic top form on the ambient space. It is easy to see that $\tilde{\Omega}_{FS}$ is vertical and regular with no poles. Consider a (1,0) form K with respect to complex structure on the ambient space $\mathbb{C}^n$ and the holomorphic vector fields generating the $U(1)^s$ action, $V^a = \sum_i Q_i^a z^i \partial_{z_i}$ such that K satisfies following conditions:

(a) K is vertical. $\imath_{V^a} K = 0$.

(b) It has a definite Q^a-charge. $\mathcal{L}_{ImV^a} K = q^a K$ where $q^a = \frac{1}{4}\sum_i Q_i^a$. This condition is required to have well-defined 3-form on $\mathcal{M}_6$.

(c)*. K is nowhere-vanishing. This condition needs some special attention.

Conditions (a) and (b) tell us that K is not well-defined on $\mathcal{M}_{2d}$. But the local SU(2)-structure comprising of a real two-form J and a complex two-form ω can be obtained using K.

$$\omega \quad = -\tfrac{i}{2} K^* \cdot \tilde{\Omega}_{FS}, \tag{4.3.3}$$

$$j \quad = \tilde{J}_{FS} - \tfrac{i}{2} K \wedge K^*. \tag{4.3.4}$$

We note that the construction for SU(3) structure suggested in [79] using local SU(2) structure is obtained using j, ω and K. We argue that condition 3: K is nonvanishing everywhere should be understood carefully by studying the topological condition $c_1(\mathcal{M}) = 0$.

$$J \quad = \alpha j - \tfrac{i\beta^2}{2} K \wedge K^*, \tag{4.3.5}$$

$$\Omega \quad = e^{i\gamma} \alpha\beta K^* \wedge \omega. \tag{4.3.6}$$

Here α,β and γ are real, gauge invariant, nowhere-vanishing functions and $\tilde{\Omega}_{FS}$ is a complex d-form on $\mathcal{M}_6$. Ω obtained with such K is well-defined on 6-manifold.

4.3.2 Comment on static SU(2)-structure

Let's explore at what happens in SU(2)-structures. The manifold with static SU(2)-structure admits two nonvanishing globally-defined spinors $\eta_i, i = 1, 2$, that are linearly independent and orthogonal at each point. From the supergravity point of view, such manifolds in general lead to $N = 4$ SUGRA in 4 dimensions [82][89]. For Type IIA point of view, there is no solution on manifolds with static SU(2)-structure.

The SU(2)-structure on 6 manifolds is characterized by a non-vanishing complex one-form K, a real two-form J and a complex two-form Ω. For our purpose, consider the one-form K, it follows

$$K \cdot K = 0,$$
$$K^* \cdot K = 2,$$
$$K_j = \eta_2^c \gamma_j \eta_1. \tag{4.3.7}$$

To study SU(2)-structures on smooth, compact toric manifolds, we need to have a nonvanishing section of cotangent bundle. It is known in mathematics literature that if section of a tangent bundle (E) is nonvanishing, the Euler class, $e(E) = 0$[87]. The top Chern class of a complex vector bundle is the Euler class [81]. We know that top Chern class of a smooth, complete toric variety is $c_n = |\Sigma(n)|[pt]$, where $[pt] \in H^{2n}(M, \mathbb{Z})$[88]. See Appendix for more details using the cones associated with Toric varieties. $|\Sigma(n)|$ is non-zero for smooth toric manifolds. Hence, we cannot have static SU(2)-structure on smooth, compact toric varieties. Thus we cannot obtain $N = 4$ supergravity compactification on such manifolds.

4.4 Topological conditions for Toric compactifications

In this section, we study topological ideas involved in Toric manifolds and SU(3) structures. First we look at $\mathbb{CP}^3$ carefully. The computations use mathematical formulas explained in Appendices A and B, which are well explored in mathematical context [87, 88, 93].

4.4.1 $\mathbb{CP}^3$ case

Consider $\mathbb{CP}^3$ as a twistor fibration on S^4 with S^2 as a fiber.

$$S^2 \hookrightarrow CP^3 \to S^4$$

. We know S^2 is diffeomorphic to $\mathbb{CP}^1$. Naturally one can consider almost complex structure on $\mathbb{CP}^3$, given by This almost complex structure is integrable. Locally $TM_{\mathbb{C}} = TM_{\mathbb{R}} \otimes \mathbb{C} = T^{(1,0)}M \otimes T^{(0,1)}M$. We know that locally $T^{(1,0)}\mathbb{CP}^3 = T^{(1,0)}\mathbb{CP}^1 \oplus \xi^{(1,0)}$, where ξ is a four dimensional normal bundle of $\mathbb{CP}^1$. Thus, for integrable almost complex structure, using $c(T^{(1,0)}\mathbb{CP}^3) = c(T^{(1,0)}\mathbb{CP}^1)c(\xi)$, if g is the element in $H^2(\mathbb{CP}^1)$, we have

$$(1 + 4g + 6g^2 + 4g^3) = (1 + 2g)(1 + c_1(\xi) + c_2(\xi)). \tag{4.4.1}$$

We get $c_1(\xi) = 2g$ and $c_2(\xi) = 2g^2$.

To obtain non-integrable almost complex structure, we will consider Let's study what happens when we make this change in I_2. We have

$$(1 - 2g)(1 + 2g + 2g^2) = (1 + c_1^{new} + c_2^{new} + c_3^{new}), \qquad (4.4.2)$$

and we get $c_1^{new} = 0$ and $c_2^{new} = -2g^2$ and $c_3^{new} = -4g^3$.

Even though we have the same real tangent bundle, due to the choice of our almost complex structure, we have modified the complex tangent bundle. It was known to mathematicians that this change leads to vanishing 1st Chern class, but it is important to notice that c_{top} does not vanish, which is expected as it is equal to the Euler class of M which doesn't depend on the choice of almost complex structure. It just picks up a sign based on orientation. This new almost complex structure leads to $c_1 = 0$ and there is a globally defined 3-form.

4.4.2 Smooth, Compact Toric varieties

In this section, we see how constrained such change in almost complex structure is for smooth, compact Toric varieties. Consider a smooth, compact Toric variety $\mathcal{M}_6$, with a four-two split of tangent bundle, not necessarily restricted to twistor space or product manifold. This is obtained using the almost product structures[75]. The tangent bundle at a point can be split into two parts. Following previous section, $T^{(1,0)}\mathcal{M}_6 = T^{(1,0)}\mathcal{M}_2 \oplus \xi$.
Thus, for such almost complex structure, using $c(T^{(1,0)}\mathcal{M}_6) = c(T^{(1,0)}\mathcal{M}_2)c(\xi)$, we have

$$c_1(T^{(1,0)}\mathcal{M}_6) = c_1(T^{(1,0)}\mathcal{M}_2) + c_1(\xi^{(1,0)}). \qquad (4.4.3)$$

Now let's understand the flip (the sign change) in the almost complex structure which leads to $c_1^{new}(T^{(1,0)}\mathcal{M}_6) = 0$.

$$0 = -c_1(T^{(1,0)}\mathcal{M}_2) + c_1(\xi^{(1,0)}). \qquad (4.4.4)$$

From Equations (4.4.3) and (4.4.4), we get $c_1(T^{(1,0)}\mathcal{M}_6) = 2 \times c_1(T^{(1,0)}\mathcal{M}_2)$. In terms of divisors, $c_1(T^{(1,0)}\mathcal{M}_6) = \sum_i D_i$.

Thus, in order to get vanishing 1st chern class, it is important to notice that the four-two split satisfies above condition. Then one can do the compactification of String theory on smooth, compact Toric variety.

We should see this condition with an example: $\mathbb{CP}^1$ bundle over $\mathbb{CP}^2$, discussed in Denef's review[81]. It can be described as

In this case, n accounts for the "twisting" and action of $U(1)^2$ is given by

$$(z_1, z_2, z_3, z_4, z_5) \to (e^{i\phi_1} z_1, e^{i\phi_1} z_2, e^{i\phi_1} z_3, e^{i(\phi_2 - n\phi_1)} z_4, e^{i\phi_2} z_5).$$

The divisors for this smooth toric manifold are $D_1 = D_2 = D_3$ and $D_4 = D_5 - nD_1$. The first Chern class is given by $c_1 = (3-n)D_1 + 2D_5$. For $n = 3$, $c_1(M) = 2D_5$, thus in this case, we can obtain the vanishing first cherm class, by choosing proper divisor and corresponding sign flip. For $n = 2$, $c_1(M) = D_1 + 2D_5$, in this case, we cannot obtain the vanishing first Chern class. In general, all odd twistings are allowed. For a complicated case, the first Chern class is given by $c_1(M) = \sum_i D_i$. Note that in this discussion, D_i represents Poincarè dual associated with the divisor $D_i : z_i = 0$.

The relation obtained between first Chern classes should hold for any twistor space considered for the compactification where we intend to use two-four split. Thus, in this subsection, even though the 2-4 split of tangent space followed by change in almost complex structure gives a vanishing first Chern class for $\mathbb{CP}^3$ case, one cannot perform similar modifications on any general smooth toric manifold.

4.4.3 More about 1-form and Holomorphic 3-form

In this section, we study 1-form K which plays a central role in SU(3)-structure we are considering from the 6-manifold perspective[75, 84]. Using the Chern classes for modified almost complex structure from previous subsection 4, We know $c_1^{new} = 0$.
Let's understand more about top Chern class.

$$c(T^{(1,0)}M) = (1 - c_1)(1 + c_1 + c_2) = 1 + c_1^{new} + c_2^{new} + c_3^{new}. \tag{4.4.5}$$

This gives $c_3^{new} = -c_1 \times c_2$ and $c_2^{new} = -c_1^2 + c_2$. Let's compute the top Chern class of holomorphic cotangent bundle twisted with a line bundle using equations from Appendix A.4:

$$\begin{aligned} c_3(T^{*(1,0)} \otimes L) &= y^3 + c_2^{new}(T^{*(1,0)})y + c_3^{new}(T^{*(1,0)}) \\ &= y^3 + c_2^{new}(T^{(1,0)})y - c_3^{new}(T^{(1,0)}) \end{aligned} \tag{4.4.6}$$

$$c_3(T^{*(1,0)} \otimes L) = y^3 - (c_1^2 - c_2)y + c_1 \times c_2. \tag{4.4.7}$$

Now we should ask whether it is possible to have a non-vanishing holomorphic section of this twisted bundle. Firstly, we should see what happens in $\mathbb{CP}^3$

case. It has $c_1(T^{(1,0)}) = 2g, c_2(T^{(1,0)}) = 2g^2$ for tangent bundle, thus

$$c_3(T^{*(1,0)} \otimes L) = y^3 - (4g^2 - 2g^2)y + 4g^3.$$

In order to have $c_3(T^{*(1,0)} \otimes L) = 0$ in $\mathbb{CP}^3$, $y = -2g$, there exists a solution for y. In general, it is important to observe that $y = -c_1$ is a solution and such a line bundle is easy to find out for various smooth toric manifolds. We observe that the 1-form needed in the construction explained in section 4.3.1 should be obtained as mentioned above with a proper choice of line bundle. Furthermore, it is interesting to observe that

$$\begin{aligned} c_1(T^{*(1,0)}M_2 \otimes L) &= c_1(T^{*(1,0)}M_2) + c_1(L) \\ &= -c_1(T^{(1,0)}M_2) + y \\ &= c_1 - c_1 = 0. \end{aligned}$$

This is similar to what happens in SU(2) structure where 1-form lives in 2-dimensional part of cotangent bundle, here it is associated with twisted cotangent bundle of M_2. Thus, the non-vanishing holomorphic section of such a twisted bundle is associated with the 1-form, one uses for wedging it non-vanishing 2-form on dual of ξ. This fact is always known, but here we say how to obtain such form with T_2^*.

In this section, we showed that with the 2-4 split of tangent space and flipped sign of almost complex structure in 2 of those directions leads to the fact that one can twist the cotangent bundle with appropriate line bundle and one obtains non-vanishing holomorphic 1-form which one can use later for getting nowhere vanishing holomorphic 3-form. This is always understood through supersymmetry conditions, but here we obtain the proper understanding using Chern classes. We have obtained a better procedure to obtain an 1-form which can keep vanishing 1st Chern class following section 4.3.1.

4.5 Local Analysis for SU(3)-structure

In this section, we study the conditions to construct SU(3) structure on Toric manifolds explained in Section 4.3 in order to obtain massive Type IIA flux vacua.

4.5.1 AdS_4 flux vacua in Type IIA theories

In this section, we will study the 4d flux compactification of (massive) Type IIA supergravity on SU(3) structure manifolds. Bosonic fields of massive IIA theory are a metric $g_{\mu\nu}$, an RR 1-form potential A and 3-form potential C, a NSNS 2-form potential B and a dilaton ϕ. In this note, we are interested in the supersymmetric vacua from 10d point of view. We will consider a 10 dimensional background, a warped product of four dimensional space and an internal six dimensional manifold.

$$ds^2_{10d} = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + ds^2_6. \tag{4.5.1}$$

In order to preserve the symmetry of 4d space-time, fluxes needs to be chosen appropriately.

In massive Type IIA, AdS_4 vacuum can be obtained using following choice of internal fluxes, (We will follow conventions of [80] for our discussion):

$$\begin{aligned} H &= 2mRe\Omega \\ g_s F_6 &= -\tfrac{1}{2}\tilde{m}J^3, \\ g_s F_4 &= \tfrac{3}{2}mJ^2, \\ g_s F_2 &= -W_2^- + \tfrac{1}{3}\tilde{m}J, \\ g_s F_0 &= 5m. \end{aligned} \tag{4.5.2}$$

and

$$\begin{aligned} dJ &= 2\tilde{m}Re\Omega, \\ d\Omega &= i(-\tfrac{4}{3}\tilde{m}J^2 + W_2^- \wedge J). \end{aligned} \tag{4.5.3}$$

Also, $3A = \phi =$ constant. The cosmological constant is given by $\Lambda = -3(m^2 + \tilde{m}^2)$. To obtain all equations of motion, supersymmetric equations have to be complemented with Bianchi identities for fluxes. For F_n, bianchi identity is $dF_n = H \wedge F_{n-2} + Q\delta(sources)$ where the source contribution comes from D-branes or O-planes.

Let's restrict our discussion to the sourceless case, $dF_n = H \wedge F_{n-2}$. The only complication occurs when $n = 2$, which places a restriction on W_2^- such that

$$dW_2^- = (\frac{1}{3}\tilde{m}^2 - 5m^2)2Re\Omega. \tag{4.5.4}$$

To obtain AdS_4 flux vacuum without localised sources, it is important to satify all equations (4.5.2),(4.5.3) and (4.5.4). One should notice that massive Type IIA solutions have $W_3 = W_4 = W_5 = 0$.

Since this setup and corresponding solution of massive Type IIA were achieved for $\mathbb{CP}^3$ with the help of almost complex structure explained in Section IV A, we should try to see whether we can obtain this setup for other possible smooth Toric manifolds.

4.5.2 Analysis

Let's perform the local analysis of differential system which K satisfies. We assume that smooth toric variety is chosen such that it satisfies the condition from section 4.4.2. We choose coordinates such that $z^i = e^{t^i}$. In new coordinates, the vector fields look like $V^a = \sum_i Q_i^a \partial_{t_i}$ and $K = K_i dt^i$. From condition (a),

$$\sum_i Q_i^a K_i = 0. \tag{4.5.5}$$

Using Cartan's magic formula, condition (b) can be simplified further.

$$\begin{aligned}\mathcal{L}_{ImV^a}K &= (d \circ \imath_{ImV^a} + \imath_{ImV^a} \circ d)K \\ &= \imath_{ImV^a}(dK) \\ &= \frac{1}{2}\left[\sum_{j,j\neq i} Q^{aj}\partial_j K_i dt^i - \sum_{i,j\neq i} Q^{ai}\partial_j K_i dt^j - \sum_j Q^{aj}\partial_{\bar{j}} K_i dt^i\right].\end{aligned} \tag{4.5.6}$$

One can use condition(a) and it gives locally $\sum_i Q^{ai}\partial_j K_i = 0$. Thus we get

$$\mathcal{L}_{ImV^a}K = \frac{1}{2}\left[\sum_j Q^{aj}(\partial_j K_i - \partial_{\bar{j}} K_i)dt^i\right]. \tag{4.5.7}$$

Let's understand the eigenvalue relation of condition (b) component-wise in these coordinates.

$$\frac{1}{2}\left[\sum_j Q^{aj}(\partial_j K_i - \partial_{\bar{j}} K_i)\right] = \frac{1}{4}\left(\sum_k Q_k^a\right) K_i \tag{4.5.8}$$

Suppose $K_i = f \cdot G_i$ such that f is given by $\sum_j Q^{aj}(\partial_j - \partial_{\bar{j}})f = (\frac{1}{2}\sum_k Q_k^a)f$. Thus, f will be of form $e^{\lambda\cdot\bar{t}}$ such that $\sum_j Q^{aj}\lambda_j = \sum_k Q_k^a$. This allows λ to take the following form,

$$\lambda_j = -\frac{1}{2} + \frac{1}{2}p(t^i)\sigma_j, \tag{4.5.9}$$

such that $\sigma \in Kernel(Q)$ and $p(t^i)$ is a complex-valued scalar function.

To have $K_i = f \cdot G_i$ as the local description, the restriction on G_i is following:

$$\sum_j Q^{aj}(\partial_j - \partial_{\bar{j}})G_i = 0. \tag{4.5.10}$$

The simplest solution for K_i can have is $fG_i(Re(t^j))$.

4.5.3 Changing the Torsion classes

In this section, we try to modify SU(3) structures by using K. We would like to see locally if we can find torsion classes for Type IIA flux vacua.

The natural question to ask is whether we can change the torsion classes for the general smooth, compact toric manifold or not. Suppose we have K with a $p = 0$ in eq. (4.5.9) and corresponding real 2-form J and three form Ω can be computed using Equations (4.3.5-4.3.6) with $\alpha = \beta = 1$ and $\gamma = \pi/2$. Let's assume this situation leads to

$$dJ^{old} = \frac{3}{2}Im(\bar{W}_1^{old}\Omega^{old}) + W_4^{old} \wedge J^{old} + W_3^{old}, \tag{4.5.11}$$

$$d\Omega^{old} = W_1^{old}J^{old} \wedge J^{old} + W_2^{old} \wedge J^{old} + \bar{W}_5^{old} \wedge \Omega^{old}. \tag{4.5.12}$$

Now, the goal is to modify K with the help of Eq. (4.5.9). $K^{new} = e^{p\sigma_i Imt^i}K^{old} = e^{p\Sigma}K^{old}$. In this section, we keep α, β^2 and γ as real, gauge invariant functions on the toric variety, but $p(t^i)$ is purely imaginary function, this choice is made just to have compatible J and ω as explained in section II.A .

$$J^{new} = \alpha(J_{FS} - \tfrac{i}{2}K^{old} \wedge K^{*old}) - \tfrac{i\beta^2}{2}(K^{old} \wedge K^{*old}), \tag{4.5.13}$$

$$\Omega^{new} = \alpha\beta e^{i\gamma + p^*\Sigma}\Omega^{old}. \tag{4.5.14}$$

Let's compute the torsion classes for this case.

$$dJ^{new} = \frac{1}{2}[d(\alpha - \beta^2) \wedge J_{FS} + d(\alpha + \beta^2) \wedge J^{old} + (\alpha + \beta^2)dJ^{old}], \tag{4.5.15}$$

$$d\Omega^{new} = d\ln(\alpha\beta e^{i\gamma + p^*\Sigma}) \wedge \Omega^{new} + \alpha\beta e^{i\gamma + p^*\Sigma}d\Omega^{old}. \tag{4.5.16}$$

Using eq. (4.5.11) and (4.5.12), we get

$$\begin{aligned} dJ^{new} &= \frac{(\alpha+\beta^2)}{2}\{(\frac{3}{2}Im(\bar{W}_1^{old}\Omega^{old}) + W_4^{old}\wedge J^{old} + W_3^{old})\} + \frac{1}{2}d(\alpha-\beta^2)\wedge J_{FS} \\ &+\frac{1}{2}d(\alpha+\beta^2)\wedge J_{old} \qquad (4.5.17)\\ &= \frac{(\alpha+\beta^2)}{2}(\frac{3}{2}Im(\bar{W}_1^{old}\Omega^{old}) + (W_4^{old}\wedge J^{new} + \frac{d(\alpha+\beta^2)}{\alpha+\beta^2}) + \frac{\alpha+\beta^2}{2}W_3^{old} \\ &- \frac{(\alpha-\beta^2)}{2}d\ln(\alpha+\beta^2)\wedge J_{FS} + \frac{1}{2}d(\alpha-\beta^2)\wedge J_{FS} \\ &-\frac{1}{2}(\alpha-\beta^2)W_4^{old}\wedge J_{FS}. \qquad (4.5.18)\end{aligned}$$

$$\begin{aligned} d\Omega^{new} &= d\ln(\alpha\beta e^{i\gamma+p^*\Sigma})\wedge\Omega^{new} + \alpha\beta e^{i\gamma+p^*\Sigma}d\Omega^{old} \qquad (4.5.19)\\ &= [d\ln(\alpha\beta e^{i\gamma+p^*\Sigma}) + \bar{W}_5^{old}]\wedge\Omega^{new} + \frac{\alpha\beta e^{i\gamma+p^*\Sigma}}{(\alpha+\beta^2)^2}W_1^{old}[J_{new}\wedge J_{new} \\ &- 2\beta^2 J_{FS}\wedge J_{new} + \beta^4 J_{FS}\wedge J_{FS}] + \frac{\alpha\beta e^{i\gamma+p^*\Sigma}}{(\alpha+\beta^2)}[W_2^{old}\wedge J_{new} \\ &- W_2^{old}\wedge J_{FS}]. \qquad (4.5.20)\end{aligned}$$

Now, we will see how to change the torsion classes and restrictions associated with the change.

$$W_5^{new} = W_5^{old} + d\ln(\alpha\beta e^{-i\gamma+p\Sigma}), \qquad (4.5.21)$$

$$W_4^{new} = W_4^{old} + d\ln(\alpha+\beta^2), \qquad (4.5.22)$$

$$\begin{aligned} W_3^{new} &= \frac{1}{2}(\alpha+\beta^2)W_3^{old} + \frac{d(\alpha-\beta^2)}{2}\wedge J_{FS} + \frac{(\alpha-\beta^2)}{2}W_4^{old}\wedge J_{FS} \\ &- \frac{\alpha-\beta^2}{2}d\ln(\alpha+\beta^2)\wedge J_{FS}. \qquad (4.5.23)\end{aligned}$$

Since we are working with toric manifolds, we know that $H_1(M)=0$, thus we know that if one form is closed, then it is exact. If W_4 is exact, we can choose function $(\alpha+\beta^2)$ such that eq.(4.5.22) gives $W_4^{new}=0$. With this condition, eq.(4.5.23) gives

$$W_3^{new} = \frac{1}{2}(\alpha+\beta^2)W_3^{old} + \frac{d(\alpha-\beta^2)}{2}\wedge J_{FS}$$

W_3 is a primitive (1,2)+(2,1) form and in order to enforce primitivity, one option is to impose $\alpha=\beta^2$. We have a function α to adjust W_4 to zero locally,

one does not have enough functions with the chosen ansatz to tune W_3 to zero. This ansatz might be more useful in finding classical dS solutions mentioned in [92].
The general idea on the lines of Calabi-Yau compactifications (Calabi conjecture) is to understand global properties with topological conditions and find a solution locally. Here we see that for an arbitrary toric case, we cannot find massive Type IIA solution locally. Similarly, we can change the W_5 by adding an exact form with the help of σ of eq.(23). This process does change W_1 and W_2 beyond multiplying by functions, but one has to fix coefficients appropriately. In general, we have $\alpha, \gamma, \sigma, p$ and freedom in J_{FS} to adjust W_i. We have shown that one can tune torsion classes on a case-by-case basis and in general when W_4 is closed. At this stage, we can hope to find more solutions of massive Type IIA by adjusting W_i suitably for smooth toric manifolds when conditions above are matched.

4.6 Discussions

We have showed that first Chern class computations for new almost complex structure can vanish on smooth toric manifolds. We also had to study top Chern class properties for using this construction which played an important role in getting nowhere vanishing holomorphic 3-form. After understanding global topological conditions, we carried out local analysis of differential system and showed that it is possible to change torsion classes associated with the SU(3) structures. Here we are trying to find more Type IIA flux vacua. We would like to conclude that in certain cases, torsion classes can be changed appropriately, but there is no explicit argument for the class of toric manifolds in general. Type IIA vacua obtained in such cases would be AdS_4. One might find this technique useful to explore classical dS solutions[92] with smooth, compact toric manifolds.

Chapter 5

Conclusion

The warped flux compactification is playing a central role in the phenomenology and cosmology coming from string theory. The systematic treatment of such compactifications is very important. With data from Planck-BICEP2 and such missions, we know that Inflation scale is much higher than those of LHC phenomenology, hence cosmology data can give us concrete constraints on string compactifications. We mainly understand three different aspects arising from warping: de Sitter vacua in string theory, non-singular cosmologies and moduli stabilization.

In this thesis, we presented a family of six-dimensional nonsingular cosmological solutions that can be embedded in seven-dimensional spacetime. The standard singularity theorems are evaded by absence of closed trapped surface, without violating null energy condition. One can understand these solutions through string theory. The important question is to work out four dimensional physics from such solutions. The obstacle involved in construction of effective four dimensional theory is to understand how massive KK modes decouple in time-dependent warped geometries. At this point, future work in this approach will be to understand how one can study the non-linear terms arising from such a reduction. Also it is important to analyze the stability of these solutions.

The usual approach to address physics of string compactification is through computing four dimensional effective action. This leads to interesting models of string cosmology, mainly with de Sitter vacua. With the presence of fluxes and localized sources, one has to treat warping carefully in constructing effective action. We study such effects of warping in de Sitter vacua, mainly around slowly varying limits to understand stability issues. It is important to address time-dependent warping effects in effective potential setup to make concrete connections with inflationary observations. Also, to understand flux solutions with complete 10D solutions, one has to extend their internal geometries beyond Calabi-Yau manifolds. With techniques of $SU(3)$-structure, toric manifolds can be used for AdS solutions. Similarly, one can study negatively curved manifolds to obtain de Sitter solutions.

Appendix A

Appendix

A.1 Definitions from GR

Following definitions are based on 4-dimensional Gravity theories which can be generalized to higher dimensions easily.

1. **Singularity:** Space-time has a singularity if it is geodesically incomplete. Singularities are usually classified mainly into 2 groups: a) **space-like**, b) **timelike**.

2. **Dominant Energy Condition:** At any point p, for every future pointing timelike vector v, vector $j(v) = -v^\mu T^\nu_\mu \partial_\nu$ is future pointing non-spacelike.

3. **Strong Energy Condition:** At any point p, for every non-spacelike vector v, $(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) v^\mu v^\nu \geq 0$.

4. **Weak Energy Condition:** At any point p, for every non-spacelike vector v, $T_{\mu\nu} v^\mu v^\nu \geq 0$.

5. **Null Energy Condition:** At any point p, for every future pointing null vector u, $T_{\mu\nu} u^\mu u^\nu \geq 0$.

6. **Cauchy Surface:** For 3-surface $\mathcal{S}$, $D^+(\mathcal{S})[D^-(\mathcal{S})]$ is a set of points $p \in M$ such that each past [future] directed inextensible non-spacelike curve through p passes through $\mathcal{S}$. The 3-surface $\mathcal{S}$ is called **Cauchy**

Surface if $D^+(\mathcal{S}) \bigcup D^-(\mathcal{S}) = M$.

7. **Homogeneity:** A spactime is (spatially) homogeneous if there is a one-parameter family of spacelike hypersurfaces Σ_t such that for each t and any two points p, q on Σ_t, there exists an isometry of spacetime metric, which takes p to q.

8. **Isotropy:** A spacetime is isotropic at each point if there is a congruence of timelike curves filling teh spacetime, such that given any point p and any 2 unit "spatial" tangent vectors, there is an isometry of spacetime metric which leaves p and tangents at p to those timelike curves fixed, but rotates 1 spatial vector into another.

9. **Causal Structure** If point $p \in M$, the causal future, $J^+(p)$, of point p is the set of points in M lying on future pointing timelike or null curves beginning at p. The future pointing timelike curves starting from p, we obtain the chronological future $I^+(p)$ of p. Similar definitions can be made for causal past and chronological past.

10. **de Sitter Universe:** Consider a 5-dimensional Minkowski space $\mathbb{M}_5$, the de Sitter (dS) Universe is the hypersurface described by,

$$dS_4 = \{-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = -R^2\}$$

R is called the de Sitter radius. The de Sitter metric is the induced metric from the standard flat metric on $\mathbb{M}_5$. Also, dS_4 is an Einstein manifold with positive scalar curvature $\Lambda = \frac{3}{R^2}$, and the Einstein tensor satisfies $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$. The coordinate system is obtained by setting $X_0 = R \sinh \tau$, $X_i = R\omega_i \cosh \tau$, $i = 1, ..., 4$, where $\infty < \tau < \infty$ and the ω_i are such that $\sum_i \omega_i^2 = 1$. The induced metric on dS_4 is given by,

$$ds^2 = R^2(-d\tau^2 + (\cosh^2 \tau) d\Omega_3^2)$$

In these coordinates dS_4 looks like a 3-sphere which starts out infinitely large at $\tau = -\infty$, then shrinks to a minimal finite size at $\tau = 0$, then grows again to infinite size as $\tau \to +\infty$.

11. **Anti de Sitter Universe:** Consider a 5-dimensional Minkowski space $\mathbb{M}_5$, the Anti de Sitter (AdS) Universe is the hypersurface described by,

$$AdS_4 = \{-x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2 = -R^2\}$$

R is called the AdS radius. The de Sitter metric is the induced metric from the standard flat metric on $\mathbb{M}_5$. Also, AdS_4 is an Einstein manifold with negative scalar curvature $\Lambda = -\frac{3}{R^2}$, and the Einstein tensor satisfies $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$.The coordinate system is obtained by setting $X_0 = R\cosh\rho\cos\tau$, $X_4 = R\cosh\rho\sin\tau$, $X_i = \omega_i \sinh\rho$, $i = 1, 2, 3$, where the ω_i are such that $\sum_i \omega_i^2 = 1$. The induced metric on AdS_4 is given by

$$ds^2 = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega^2)$$

A.2 Solutions in Seven Dimensions

In this section, we present 7D solutions that are related to the trivial solution in (2.3.1) by $O(2,2)$ transformations. First, let us look at the solution that can be obtained from (2.3.1) by using Buscher rules along z direction. We write down the Buscher rules here for convenience:

$$g'_{zz} = \frac{1}{g_{zz}} \qquad g'_{az} = \frac{B_{az}}{g_{zz}} \qquad g'_{ab} = g_{ab} - \frac{g_{az}g_{zb} + B_{az}B_{zb}}{g_{zz}} \tag{A.2.1}$$

$$\varphi' = \varphi - \frac{1}{2}\ln g_{zz} \qquad B'_{az} = \frac{g_{ay}}{g_{zz}} \qquad B'_{ab} = B_{ab} - \frac{g_{az}B_{zb} + B_{az}g_{zb}}{g_{zz}} \tag{A.2.2}$$

The 7D solution we get using Buscher transformations is given by

$$ds^2 = e^{-\sigma/2}\hat{g}_{tt}dt^2 + \boldsymbol{dx}^2 + 2e^{-\sigma/2}\hat{g}_{t\theta}dtd\theta + e^{-\sigma/2}\hat{g}_{\theta\theta}d\theta^2 + \beta^2 e^{-2\sigma}\left(d\phi + A_a^{(2)}dx^a\right)^2 + e^{-2\sigma}dz^2, \tag{A.2.3}$$

$$B = \hat{A}_a dx^a \wedge dz + \hat{A}_\phi d\phi \wedge dz, \ \varphi = \varphi_0 - \sigma \tag{A.2.4}$$

where $a \in \{t, \boldsymbol{x}\}$ and $A_a^{(2)} = e^2\sigma g_{\phi a}$. We can verify that this solution also reduces to (2.3.4). Solutions generated using a general $O(2,2)$ duality transformation on (2.3.4) need not be equivalent to the above solution. Under general $O(2,2)$ transformations the two dimensional part of the internal manifold and the B field transforms as described in (2.2.7). As an example, let us study the action of the following $O(2,2)$ matrix on the 6D solution in (2.3.4)

$$\Omega = \frac{1}{2}\begin{bmatrix} 1+c & s & c-1 & -s \\ -s & 1-c & -s & 1+c \\ c-1 & s & 1+c & -s \\ s & 1+c & s & 1-c \end{bmatrix}$$

where $c = \cosh\mu$ and $s = \sinh\mu$ (following the notations in [27]). The internal manifold and B field transforms as follows:

$$\tilde{g}_{\phi\phi} = \frac{(1+c)^2 + g_{\phi\phi}^2 + (1-c+s\alpha)^2\beta^2 + g_{\phi\phi}\left(-2(1+c)s\alpha + (1+c)^2\alpha^2 + s^2(1+\beta^4)\right)}{4g_{\phi\phi}\beta^2}$$

$$\tilde{g}_{z\phi} = \frac{s\left(-(-1+c-2s\alpha+\alpha^2+s)g_{\phi\phi} - (1+c)\beta^2\right)\left(1+g_{\phi\phi}\beta^2\right)}{4g_{\phi\phi}\beta^2}$$

$$\tilde{g}_{zz} = \frac{s^2\beta^2 + g_{\phi\phi}^2 + (s-(1+c)\alpha)^2\beta^2 + g_{\phi\phi}\left(-2c(1+s\alpha-\beta^4) + 1 + 2s\alpha + s^2\alpha^2 + c^2(1+\beta^4)\right)}{4g_{\phi\phi}\beta^2}$$

$$\tilde{B}_{z\phi} = \frac{((g_{\phi\phi}(-1+c-2s\alpha+(1+c)\alpha^2)+(1+c)\beta^2)(-g_{\phi\phi}(s-2c\alpha+s\alpha^2)-s\beta^2))}{4g_{\phi\phi}\beta^2}$$

We can verify that this solution reduces to a solution that is not equivalent to (2.3.4). Note that the above solution and the solution in (A.2.3) can be uplifted to solutions of type II supergravity trivially.

A.3 Generating Black Hole solutions

Let's follow Sen's trick of generating BH solutions [94] (Explained in A Peet's TASI lectures). Consider a neutral Black hole in (d-1) dimensions.

$$ds^2 = -(1-K(\rho))dt^2 + (1-K(\rho))^{-1}d\rho^2 + \rho^2 d\Omega_{d-3}^2 \tag{A.3.1}$$

The mass of Black hole is . This solution can be trivially lift this solution to d dimensions.

$$d\hat{s}^2 = -(1-K(\rho))dt^2 + (1-K(\rho))^{-1}d\rho^2 + \rho^2 d\Omega_{d-3}^2 + dy^2 \tag{A.3.2}$$

If we boost in t-y coordinates, $\begin{bmatrix} dt \\ dy \end{bmatrix} \to \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} dt \\ dy \end{bmatrix}$

Now the metric takes the form,

$$\begin{aligned} d\hat{s}'^2 &= (-dt^2+dy^2) + K(\rho)dt^2 + (1-K(\rho))^{-1}d\rho^2 + \rho^2 d\Omega_{d-3}^2 + dy^2 \\ &= -(1-K(\rho)\cosh^2\theta)dt^2 + (1+K(\rho)\sinh^2\theta)dy^2 + \sinh 2\theta dt dy \\ &\quad +(1-K(\rho))^{-1}d\rho^2 + \rho^2 d\Omega_{d-3}^2 \end{aligned} \tag{A.3.3}$$

Now performing KK reduction along y-coordinate gives us a new BH solution in (d-1) dimensions.

$$d\hat{s}^2 = ds_{(d-1)}^2 + e^{2\chi}(dy+\mathcal{A})^2 \tag{A.3.4}$$

One can extract useful KK quantities.

$$g_{tt} = -\frac{1-K(\rho)}{1+K(\rho)\sinh^2\theta} \tag{A.3.5}$$

$$\mathcal{A}_t = \frac{K\sinh 2\theta}{2(1+K(\rho)\sinh^2\theta)} \tag{A.3.6}$$

$$e^{2\chi} = (1+K(\rho)\sinh^2\theta) \tag{A.3.7}$$

This leads to Kerr Black hole solution with Mass and Angular momentum.

The $\theta \to 0$ gives the original black hole. Usually finding concrete solutions of Supergravity are hard. This approach as explained in chapter 2 can be used for various solutions to obtain various full solutions with possible string theory embeddings.

A.4 Chern Classes

Here we will discuss Chern classes of vector bundles and their properties in brief[87, 93].

Definition: Let $E \to M$ be a complex vector bundle whose fiber is $\mathbb{C}^k$. The structure group G ($\subset GL(k, \mathbb{C})$) with a connection $\mathcal{A}$ and its strength $\mathcal{F}$, then the Chern class is defined as

$$c(E) = det\left(1 + \frac{i\mathcal{F}}{2\pi}\right). \tag{A.4.1}$$

It can be decomposed as

$$c(E) = 1 + c_1(E) + c_2(E) + ... \tag{A.4.2}$$

such that i-th Chern class $c_i(E) \in \Omega^{2i}(M)$. Hence, it is clear that if n is the rank of a vector bundle E, then $c_i(E) = 0$ for $i > n$.

Properties of Chern Class: For vector bundles E and F, tangent bundle T and cotangent bundle T^* and line bundle L,

$$c(E \oplus F) = c(E) \wedge c(F), \tag{A.4.3}$$

$$c(L) = (1 + x), \tag{A.4.4}$$

$$c(E \otimes L) = \sum_{i=1}^{n} c_i(E)(1 + x)^{n-i}, \tag{A.4.5}$$

$$c(T^*) = \sum_k (-1)^k c_k(T). \tag{A.4.6}$$

A.5 Toric Geometry

We have considered the symplectic description of toric varieties for this work. Toric geometry can be described using simple combinatorial data. Interested readers can follow [86, 88].

Consider a rank-d integer lattice $N \cong Z^d$ and the real extension of N, $N_{\mathbb{R}} = \mathbb{R} \otimes N$. A subset $\sigma \subset N_{\mathbb{R}}$ is a called a strongly convex rational polyhedral cone with apex 0 if $\sigma \cap (-\sigma) = 0$ and there exist elements v_1,...,v_r of N such that

$$\sigma = \{a_1 v_1 + ...a_r v_r; 0 \leq a_1, ...a_r \in \mathbb{R}\}. \tag{A.5.1}$$

The set v_1,...,v_r is usually called generators of cone σ. τ is called a face of σ if its generators are a subset of the generators of σ.

A fan Σ is a collection of cones $\{\sigma_1, ..., \sigma_k\}$ such that

1. $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.

2. If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$.

3. If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face of both cones.

The support $|\Sigma|$ of a fan Σ is the union of all cones in the fan.

The toric variety X(Σ) can be constructed corresponding to a fan Σ by taking the union of affine toric varieties. [86] One can obtain lot of information about toric manifolds using the fan description.

A Weil divisor is a finite sum of irreducible hypersurfaces with a co-dimension one. $D = \sum n_i V_i$ such that $n_i \in \mathbb{Z}$ and V_i are irreducible sub-varieties. There is a one-to-one mapping from generators of $\Sigma(1)$ and T-Weil divisors. If $\{v_1, ..., v_k\}$ are rays in a fan, Weil divisor is

$$D = \sum_{i=1}^{k} a_i V_i$$

where a_i are integers. For toric varieties, there is a correspondence between divisors and line bundles.

One of the interesting property of smooth toric manifolds for flux compactifications is all odd betti numbers vanish and even betti numbers are given by

$$\beta_{2k} = \sum_{i=k}^{n} (-1)^{(i-k)} \binom{i}{k} d_{n-i}. \tag{A.5.2}$$

Here d_k is the number of k-dimensional cones in Σ.

A.6 String and Einstein Frame

We saw in Chapter 2, how various frames affect String actions, let's understand 2 of these frames in this section. While studying Bosonic strings, we wrote the action as

$$S_{string} = \frac{1}{2\kappa^2} \int d^D X \sqrt{-G} e^{-2\phi} \left[R - \frac{1}{2 \cdot 3!} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\partial_\mu \phi \partial^\mu \phi \right] \tag{A.6.1}$$

It is important to notice 2 things here. The Einstein-Hilbert action doesn't come out naturally as there is $e^{-2\phi}$ factor in front of R and the kinetic term of ϕ comes with wrong sign. The way to get out of this problem is by using metric rescaling.

$$\tilde{G}_{\mu\nu} = e^{-4\phi/(D-2)} G_{\mu\nu} \tag{A.6.2}$$

Now the action takes the familiar form with correct sign for kinetic term of ϕ,

$$S_{Einstein} = \frac{1}{2\kappa^2} \int d^D X \sqrt{-\tilde{G}} \left[\tilde{R} - e^{-\phi/3} \frac{1}{2 \cdot 3!} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{6} \partial_\mu \phi \partial^\mu \phi \right] \tag{A.6.3}$$

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