Uncovering the structure of (super)conformal field theories

A Dissertation Presented

by

Pedro Liendo

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Physics

Stony Brook University

June 2013

Stony Brook University

The Graduate School

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Abstract of the Dissertation

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Conformal field theories (CFTs) are of central importance in modern theoretical physics, with applications that range from condensed matter physics to particle theory phenomenology. In this Ph.D. thesis we study CFTs from two somehow orthogonal (but complementary) points of view.

In the first approach we concentrate our efforts in two specific examples: the Veneziano limit of $\mathcal{N} = 2$ and $\mathcal{N} = 1$ superconformal QCD. The addition of supersymmetry makes these theories amenable to analytical analysis. In particular, we use the correspondence between single trace operators and states of a spin chain to study the integrability properties of each theory. Our results indicate that these theories are not completely integrable, but they do contain some subsectors in which integrability might hold.

In the second approach, we consider the so-called "bootstrap program", which is the ambitious idea that the restrictions imposed by conformal symmetry (crossing symmetry in particular) are so powerful that starting from a few basic assumptions one should be able to fix the form of a theory. In this thesis we apply bootstrap techniques to CFTs in the presence of a boundary. We study two-point functions using analytical and numerical methods. One-loop results were re-obtained from crossing symmetry alone and a variety of numerical bounds for conformal dimensions of operators were obtained. These bounds are quite general and valid for any CFT in the presence of a boundary, in contrast to our first approach where a specific set of theories was studied.

A natural continuation of this work is to apply bootstrap techniques to supersymmetric theories. Some preliminary results along these lines are presented.

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Acknowledgements

It is a pleasure to thank my advisor Leonardo Rastelli for his guidance and support. For teaching me how to do science and always pushing me to do my best. For being not only a mentor but also a friend.

From my first years as a grad student, I would like to acknowledge the amazing teaching skills of Konstatin Likharev, George Sterman, and Peter van Nieuwenhuizen. I hope one day to be able to teach with the same passion and clarity.

As the years passed by, courses became less frequent and more specialized, and textbooks just did not have all the answers. Here I would like to thank Martin Rocek and Warren Siegel for countless discussions and for sharing with me their knowledge and expertise.

Thanks are also due to my collaborators Abhijit Gadde, Elli Pomoni, and Wenbin Yan for the work we did on the first part of my Ph.D. (the "spin chain part" of this thesis). For the work done in the second part (the "bootstrap part"), I am indebted to Madalena Lemos, Wolfger Peelaers, and especially Balt van Rees.

A salute to the "2007 gang": Marcos Crichigno, Ozan Erdogan, Pilar Staig, and Savvas Zafeiropoulos. For the physics discussions, the coffee breaks, the movie nights, the soccer games, the endless complaining and the endless fights. For the good and the bad moments. In summary, for making grad school an even more engaging experience.

Not quite from the 2007 batch but certainly honorary members: I would like thank Melvin Irizarry for the study sessions and discussions, for the lunches at SAC, the wedding in Puerto Rico, and for encouraging me to play the MGS saga. Special thanks also go to Raul Santos, for being such a cheerful person and for the always enjoyable and illuminating discussions about physics, film, music, and Chilean culture.

Finally, allow me to thank Pilar Staig one more time, for being a wonderful wife. For always being there.

Chapter 1

Introduction

Conformal field theories (CFTs) are of central importance in modern theoretical physics, with applications in condensed matter physics and particle theory phenomenology, among others. On a more fundamental level, they play a deep role in our understanding of Quantum Field Theory (QFT) in general. QFTs are expected to flow to conformal fixed points in the extreme ultraviolet and infrared energy regimes. This being the case, a natural first step towards understanding general field theories would be to understand CFTs; a reduced set that however represents the asymptotic behavior of general QFTs at high and low energies. Once a strong understanding of CFTs is obtained, one can proceed to study more general field theories starting from a conformal fixed point and adding relevant operators.

CFTs play a crucial role in String Theory as well. In particular, in the worldsheet formulation, but also from the spacetime point of view, thanks to the AdS/CFT correspondence. In recent years, String Theory has become a major tool in understanding various aspects of QFT, especially for some selected supersymmetric models. Nowadays, the line dividing String Theory and QFT is somehow blurred, with both frameworks complementing each other. CFTs therefore allow us to understand QFT from several fronts.

Having said that, this thesis is devoted to the study of CFTs. The work presented here is naturally divided into two parts: the "spin chain part", in which we will enhance the four-dimensional conformal symmetry to *superconformal* symmetry; and the "bootstrap part", where we will consider CFTs in the presence of a boundary, and we will focus mostly on the three-dimensional case. These two lines of research are somehow orthogonal but complement each other and together give a coherent picture of the implications of conformal symmetry in field theory. Both subjects will be studied in detail in the remainder of this thesis, but let us start with a short description motivating each of them and highlighting the main results of our work.

1.1 Integrability in supersymmetric theories

The study of supersymmetric CFTs might help us understand one of the long-standing unsolved problems in theoretical physics: the strongly-coupled regime of QCD. Since the arrival of the gauge/string duality one can ask the following question: what is the string dual of QCD? This does not solve the problem of understanding QCD at strong coupling, but it does give a whole new angle to attack it. The vast majority of the results concerning the AdS/CFT correspondence rely heavily on supersymmetry. A logical line of development would be to systematically reduce the number of supersymmetries, until we reach the more realistic case of no supersymmetry at all. We can therefore ask: what is the string dual of $\mathcal{N}$=2 SYM? The scope of this question is certainly smaller than the one posed before, but it does point out towards the same ultimate goal.

One of the most remarkable aspects of the original AdS/CFT example (Type IIB String Theory/$\mathcal{N} = 4$ SYM) is the presence of *integrability* (see chapter 2). If integrability is present in less supersymmetric theories, it would certainly play a major role in helping uncover the structure of the putative string dual descriptions. It is then natural to ask whether this prominent feature is present in theories in which supersymmetry is not maximal. In the first part of this thesis we attempt to answer the question whether integrability is present in theories with reduced supersymmetry. In particular, the Veneziano limit of $\mathcal{N} = 2$ superconformal QCD and $\mathcal{N} = 1$ super QCD in the upper edge of the conformal window. In chapter 2 we will give an introduction to the "spin chain picture" in which single trace operators of a large-N gauge theory are associated with states on a spin chain. The Hamiltonian acting on the chain is identified with the dilation operator of the conformal theory. The approach we take is perturbative and in chapter 3 we obtain explicit expressions for the *complete* one-loop dilation operator of the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ theories. Its integrability properties are studied in chapter 4 by means of direct diagonalization of the Hamiltonian. This analysis is complemented with a higher loop calculation also in chapter 4 where we conclude that the $\mathcal{N} = 2$ theory is not completely integrable. However, we argue that integrability might still be present in a special subsector of the theory.

1.2 The modern bootstrap program

The second part of the thesis concerns the "bootstrap program". It is the ambitious idea that postulates that the restrictions imposed by conformal symmetry are so powerful, that starting with a few basic assumptions (in particular, crossing symmetry) one should be able fix the form of the theory. The bootstrap approach was mostly studied in two dimensions; however, there has been a revival of the program applied to conformal field theories in higher dimensions. The authors of [3] showed that it is possible to extract important information from the crossing symmetry equations using numerical methods. Among other things, they obtained a numerical bound for the conformal dimension of certain operators valid in any four-dimensional theory.

In this thesis we will concentrate on CFTs in the presence of a boundary. Boundary CFTs have diverse physical applications, in condensed matter they describe surface phenomena in systems near criticality, while in string theory two-dimensional world-sheet BCFTs are interpreted as D-branes. In particular, we will study the conformal block decomposition of two-point functions in the presence of a boundary. In chapters 5 and 6 we present various implications of these boundary bootstrap equations. Our results are obtained using analytic as well as numerical techniques. On the analytic side, in chapter 5 we set up the boundary bootstrap equations for scalars and reproduce the one-loop critical exponents of the Wilson-Fisher fixed point in the epsilon expansion. Additionally, we use the embedding space formalism to write the bootstrap equations for the stress tensor. On the numerical front, in chapter 6 we apply the numerical techniques of [3] to obtain a handful of bounds valid for arbitrary theories, but also more specific results concerning the $3d$ Ising model. In particular, we managed to bound bulk quantities using the boundary bootstrap equations. We also obtained bounds for the stress tensor, which is notoriously difficult to study using the standard bulk crossing symmetry equations (*i.e.* without a boundary) due to the proliferation of spacetime indices in the equations.

We conclude in chapter 7 with a summary of our results and discuss possible extensions. In particular, we describe the "supersymmetric bootstrap" which is a natural continuation of the two lines of research presented here, in which we enhance the bootstrap equations adding the constraints coming from supersymmetry.

Chapter 2

The Dilation Operator and Spin Chains

As we will review below, the dilation operator of a planar gauge theory can be identified with the "Hamiltonian" of a spin chain. In the case of $\mathcal{N} = 4$ super Yang-Mills (SYM), this spin chain turns out to be *integrable*: it has an infinite set of conserved charges and its spectrum is described by a set of algebraic equations (see *e.g.* [4–9] for a partial list of references and [10] for a recent comprehensive review). The spectrum of energies of the spin chain is identified with the spectrum of conformal dimensions in the gauge theory; if we solve the spectral problem of the spin chain, we also solve the spectral problem of the gauge theory. Perturbative field theory calculations of the dilation operator have played a crucial role in uncovering the integrability properties of $\mathcal{N} = 4$ SYM. As the integrability structure is common to the planar field theory and the dual string sigma model, one might even imagine an alternative history where the AdS/CFT correspondence is discovered following the hints of the field theory integrability.

In this thesis we present the calculation of the complete planar one-loop dilation operator (or Hamiltonian) of two paradigmatic superconformal theories: the $\mathcal{N} = 2$ $SU(N_c)$ super Yang-Mills theory with $N_f = 2N_c$ fundamental hypermultiplets, in the flavor singlet sector; and $\mathcal{N} = 1$ $SU(N_c)$ super QCD (SQCD) in the conformal window.

Let us start our discussion with the $\mathcal{N} = 2$ theory. This theory is perhaps the simplest $4d$ conformal field theory outside the "universality class" of $\mathcal{N} = 4$ SYM and is a very interesting case study. It admits a large N expansion in the Veneziano sense [11] of $N_f \sim N_c \to \infty$ with $\lambda = g_{YM}^2 N_c$ fixed, and a perturbative expansion in the exactly marginal 't Hooft coupling λ. Is the planar theory integrable? Does it have a dual string description? Some progress in answering these two questions, which are logically independent, was described in [12, 13]. In particular in [13] the planar one-loop dilation operator in the scalar subsector was obtained. As explained in [12, 13], it is illuminating to embed $\mathcal{N} = 2$ superconformal QCD (SCQCD) into the $\mathcal{N} = 2$ $SU(N_c) \times SU(N_{\check{c}})$ quiver theory (with $N_{\check{c}} \equiv N_c$) which has two independent marginal couplings g_{YM} and $\check{g}_{YM}$. The quiver theory interpolates between the standard $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM for $\check{g}_{YM} = g_{YM}$ and SCQCD for $\check{g}_{YM} = 0$. With minor extra work, we can keep our calculations more general and derive the full one-loop spin chain Hamiltonian for the whole interpolating quiver theory. In the closed subsector of scalar chiral fields the Hamiltonian of the quiver theory has been obtained to three loops [14].

The quiver theory is known to be integrable at the orbifold point $\check{g}_{YM} = g_{YM}$ [15], but it is definitely not integrable for generic values of the couplings, since the two-body magnon S-matrix does not obey the Yang-Baxter equation [13]. The question whether integrability is recovered in the (somewhat singular) SCQCD limit $\check{g}_{YM} \to 0$ is the subject of the first part of this thesis.

Before the work presented here, only the one-loop Hamiltonian in the scalar sector was known for SCQCD [13]. There are two natural ways in which this result can be improved. One is to add more fields and consider the *complete* one-loop Hamiltonian, where in addition to scalar fields we consider fermions and covariant derivatives. The other is to go to higher loops in some tractable subsectors where the integrability properties of the theory can be studied. In this thesis we pursue both directions, starting with the complete one-loop result in this chapter. The higher loop analysis of the $SU(2|1)$ subsector is presented in chapter 4.

We find that the full spin-chain Hamiltonian of $\mathcal{N} = 2$ SCQCD is completely fixed by symmetry, as is the case for $\mathcal{N} = 4$ SYM [16, 17]. This came as a surprise, because representation theory is less restrictive for the $\mathcal{N} = 2$ superconformal algebra. Unlike $\mathcal{N} = 4$ SYM, where each site of the spin chain hosts a single ultrashort irreducible representation, in our case single-site letters decompose into three distinct irreps, and the tensor product of two nearest-neighbor state spaces has a considerably more intricate decomposition. Nevertheless, by a non-trivial generalization of Beisert's approach [16, 17], we find that symmetry is sufficient to determine the Hamiltonian up to overall normalization. The generalization to the interpolating quiver is then as simple as one may hope: symmetry leaves a single undetermined parameter, which gets identified with the ratio of the two marginal gauge couplings.

For $\mathcal{N} = 1$ SQCD, we again consider the large N Veneziano limit of $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c fixed, near the upper edge of the superconformal window $N_f \lesssim 3N_c$. If one defines

$$\frac{N_f}{N_c} = 3 - \epsilon\,, \tag{2.1}$$

the large N theory flows for $\epsilon \ll 1$ to a weakly-coupled Banks-Zaks fixed point [18], with 't Hooft coupling $g^2_{YM} N_c \sim \epsilon$. The one-loop planar dilation operator captures the spectrum of the theory at this isolated fixed point, while higher-loop corrections (reorganized in powers of ϵ) correspond to moving down the conformal window. The dual "magnetic" theory admits a perturbative expansion starting from the lower edge of the conformal window $N_f \gtrsim \frac{3}{2} N_c$, with a Banks-Zaks fixed point that is weakly-coupled for $\tilde{\epsilon} \ll 1$, where

$$\frac{N_f}{N_c} = \frac{3}{2} + \tilde{\epsilon}\,. \tag{2.2}$$

A complete large N solution of SQCD would entail determining the dilation operator of the electric theory to all orders in ϵ, and that of the magnetic theory to all orders in $\tilde{\epsilon}$. The resummations of both expansions should then coincide – in the ultimate triumph of Seiberg duality. Needless to say, this is a tall order, and one cannot hope to fulfill this program unless integrability comes to the rescue.

The one-loop dilation operator of $\mathcal{N} = 1$ SQCD in the Veneziano limit has been determined in the scalar subsector [19], and shown to coincide with the Ising spin chain in a transverse magnetic field, one of the best known integrable models. This is a tantalizing hint, well-worth subjecting to more stringent tests. As in the $\mathcal{N} = 2$ case, we consider the evaluation of the complete one-loop Hamiltonian a natural continuation of the work of [19].

The calculation for $\mathcal{N} = 1$ SQCD proceeds along similar lines as $\mathcal{N} = 2$ SCQCD, and again we are able to fix the one-loop Hamiltonian from symmetry considerations alone. A preliminary investigation of its integrability properties is presented in chapter 4.

2.1 The $\mathcal{N} = 2$ Spin Chain

We begin by quickly reviewing $\mathcal{N} = 2$ superconformal QCD, the closely related $\mathbb{Z}_2$ quiver theory, and the structure of their spin chains. For more details, including the explicit Lagrangians, we refer to [13].

	$SU(N_c)$	$U(N_f)$	$SU(2)_R$	$U(1)_r$
${\mathcal{Q}_\alpha}^{\mathcal{I}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$+1/2$
${\mathcal{S}_{\mathcal{I}}}^{\alpha}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$
$A_{\alpha\dot{\alpha}}$	Adj	$\mathbf{1}$	$\mathbf{1}$	0
ϕ	Adj	$\mathbf{1}$	$\mathbf{1}$	-1
$\lambda_{\mathcal{I}\alpha}$	Adj	$\mathbf{1}$	$\mathbf{2}$	$-1/2$
$Q_{\mathcal{I}}$	$\square$	$\square$	$\mathbf{2}$	0
ψ_α	$\square$	$\square$	$\mathbf{1}$	$+1/2$
$\tilde{\psi}_\alpha$	$\bar{\square}$	$\bar{\square}$	$\mathbf{1}$	$+1/2$

Table 2.1: Field content and symmetries of $\mathcal{N} = 2$ SCQCD. We show the quantum numbers of the Poincaré supercharges ${\mathcal{Q}_\alpha}^{\mathcal{I}}$, of the conformal supercharges ${\mathcal{S}_{\mathcal{I}}}^{\alpha}$ and of the elementary component fields. Conjugate objects (such as $\bar{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}$ and $\bar{\phi}$) are not written explicitly.

2.1.1 Field Content and Symmetries

We summarize in table 2.1 the field content and quantum numbers of the $\mathcal{N} = 2$ SYM theory with gauge group $SU(N_c)$ and $N_f = 2N_c$ fundamental hypermultiplets, which we refer to as $\mathcal{N} = 2$ superconformal QCD. Its global symmetry group is $U(N_f) \times SU(2)_R \times U(1)_r$, where $SU(2)_R \times U(1)_r$ is the R-symmetry subgroup of the superconformal group. We use indices $\alpha, \beta = \pm$ and $\dot{\alpha}, \dot{\beta} = \pm$ for the Lorentz group, $\mathcal{I}, \mathcal{J} = 1, 2$ for $SU(2)_R$, $i, j = 1, \dots N_f$ for the flavor group $U(N_f)$ and $a, b = 1, \dots N_c$ for the color group $SU(N_c)$. The $\mathcal{N} = 2$ vector multiplet consists of a gauge field $A_{\alpha\dot{\alpha}}$, two Weyl spinors $\lambda_{\mathcal{I}\alpha}$, $\mathcal{I} = 1, 2$, which form a doublet under $SU(2)_R$, and one complex scalar ϕ, all in the adjoint representation of $SU(N_c)$. Each $\mathcal{N} = 2$ hypermultiplet consists of an $SU(2)_R$ doublet $Q_{\mathcal{I}}$ of complex scalars and of two Weyl spinors ψ_α and $\tilde{\psi}_\alpha$, $SU(2)_R$ singlets.

$\mathcal{N} = 2$ SCQCD, which has one exactly marginal coupling g_{YM}, can be viewed as a limit of the $\mathcal{N} = 2$ $\mathbb{Z}_2$ quiver theory with gauge group[1] $SU(N_c) \times$

[1]The gauge groups are identical, $N_{\check{c}} \equiv N_c$, but we find it useful to distinguish with a "check" all the quantities pertaining to the second gauge group.

	$SU(N_c)$	$SU(N_{\check{c}})$	$SU(2)_R$	$SU(2)_L$	$U(1)_R$
${\mathcal{Q}_\alpha}^{\mathcal{I}}$	**1**	**1**	**2**	**1**	+1/2
${\mathcal{S}_{\mathcal{I}}}^{\alpha}$	**1**	**1**	**2**	**1**	–1/2
$A_{\alpha\dot{\alpha}}$	Adj	**1**	**1**	**1**	0
$\check{A}_{\alpha\dot{\alpha}}$	**1**	Adj	**1**	**1**	0
ϕ	Adj	**1**	**1**	**1**	–1
$\check{\phi}$	**1**	Adj	**1**	**1**	–1
$\lambda_{\mathcal{I}\alpha}$	Adj	**1**	**2**	**1**	–1/2
$\check{\lambda}_{\mathcal{I}\alpha}$	**1**	Adj	**2**	**1**	–1/2
$Q_{\mathcal{I}\hat{\mathcal{I}}}$	$\square$	$\overline{\square}$	**2**	**2**	0
$\psi_{\hat{\mathcal{I}}\,\alpha}$	$\square$	$\overline{\square}$	**1**	**2**	+1/2
$\tilde{\psi}_{\hat{\mathcal{I}}\,\alpha}$	$\overline{\square}$	$\square$	**1**	**2**	+1/2

Table 2.2: Field content and symmetries of the quiver theory that interpolates between the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM and $\mathcal{N}=2$ SCQCD.

$SU(N_{\check{c}})$, which has two exactly marginal couplings g_{YM} and $\check{g}_{YM}$, as $\check{g}_{YM} \to 0$. When $g_{YM} = \check{g}_{YM}$ the quiver theory is the familiar $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM. Table 2 summarizes the field content and symmetries of the quiver theory. Besides the R-symmetry group $SU(2)_R \times U(1)_r$, the theory has an additional $SU(2)_L$ global symmetry, whose indices we denote by $\hat{\mathcal{I}}, \hat{\mathcal{J}} = \hat{1}, \hat{2}$. Supersymmetry organizes the component fields into the $\mathcal{N}=2$ vector multiplets of each factor of the gauge group, $(\phi, \lambda_{\mathcal{I}}, A_{\alpha\dot{\alpha}})$ and $(\check{\phi}, \check{\lambda}_{\mathcal{I}}, \check{A}_{\alpha\dot{\alpha}})$, and into two bifundamental hypermultiplets, $(Q_{\mathcal{I},\hat{1}}, \psi_{\hat{1}}, \tilde{\psi}_{\hat{1}})$ and $(Q_{\mathcal{I},\hat{2}}, \psi_{\hat{2}}, \tilde{\psi}_{\hat{2}})$.

Setting $\check{g}_{YM} = 0$, the second vector multiplet $(\check{\phi}, \check{\lambda}_{\mathcal{I}}, \check{A}_{\alpha\dot{\alpha}})$ becomes free and completely decouples from the rest of the theory, which coincides with $\mathcal{N}=2$ SCQCD (the field content is the same and $\mathcal{N}=2$ susy does the rest). The $SU(N_{\check{c}})$ symmetry can now be interpreted as a global flavor symmetry. In fact there is a symmetry enhancement $SU(N_{\check{c}}) \times SU(2)_L \to U(N_f = 2N_c)$: the $SU(N_{\check{c}})$ index $\check{a}$ and the $SU(2)_L$ index $\hat{\mathcal{I}}$ can be combined into a single flavor index $i \equiv (\check{a}, \hat{I}) = 1, \ldots 2N_c$.

We work in the large $N_c \equiv N_{\check{c}}$ limit, keeping fixed the 't Hooft couplings

$$\lambda \equiv g_{YM}^2 N_c \equiv 8\pi^2 g^2 \,, \qquad \check{\lambda} \equiv \check{g}_{YM}^2 N_{\check{c}} \equiv 8\pi^2 \check{g}^2 \,. \tag{2.3}$$

We will often refer to the theory with arbitrary g and $\check{g}$ as the "interpolating SCFT", thinking of keeping g fixed as we vary $\check{g}$ from $\check{g} = g$ (orbifold theory) to $\check{g} = 0$ ($\mathcal{N} = 2$ SCQCD $\oplus$ extra $N_{\check{c}}^2 - 1$ free vector multiplets).

2.1.2 The Spin Chain

The planar dilation operator of a gauge theory can be represented as the Hamiltonian of a spin chain. Each site of the chain is occupied by a "letter" of the gauge theory: a letter $\mathcal{D}^k\mathcal{A}$ can be any of the elementary fields $\mathcal{A}$ acted on by an arbitrary number of gauge-covariant derivatives $\mathcal{D}$. A closed chain corresponds to a single-trace operator.

In the interpolating SCFT, letters belonging to the vector multiplets are in the adjoint representation of either gauge group (index structures ${}^{a}{}_{b}$ and ${}^{\check{a}}{}_{\check{b}}$), while letters belonging to the hypermultiplets are in a bifundamental representation (index structures ${}^{a}{}_{\check{b}}$ and ${}^{\check{a}}{}_{b}$). In SCQCD, vector letters have index structure ${}^{a}{}_{b}$, while hyper letters have stuctures ${}^{a}{}_{i}$ and ${}^{i}{}_{b}$. We restrict attention to the flavor-singlet sector of SCQCD. Then, as explained in [12, 13], in the Veneziano limit of $N_c \to \infty$, $N_f \to \infty$ with $N_f/N_c \equiv 2$ and $g_{YM}^2 N_c$ fixed, the basic building blocks are the "generalized single-trace operators", where consecutive letters have contracted color or flavor indices, for example

$$\mathrm{Tr}[\bar{\phi}\phi\phi Q_{\mathcal{I}} \bar{Q}^{\mathcal{J}} \bar{\phi}] = \bar{\phi}^{a}{}_{b} \phi^{b}{}_{c} \phi^{c}{}_{d} Q_{\mathcal{I}}{}^{d}{}_{i} \bar{Q}^{\mathcal{J}i}{}_{e} \bar{\phi}^{e}{}_{a} \,, \quad a,b,c,d,e = 1, \ldots N_c \,, \quad i = 1, \ldots N_f \,. \tag{2.4}$$

In the large N Veneziano limit the action of the dilation operator is well-defined on generalized single-traces, because mixing with multi-traces is suppressed. We write the planar dilation operator as

$$D = g^2 H \,, \tag{2.5}$$

where H is the spin-chain Hamiltonian. At one-loop, H is of nearest-neighbor form,

$$H = \sum_{\ell=1}^{L} H_{\ell,\ell+1} \,. \tag{2.6}$$

The one-loop Hamiltonian of the interpolating theory depends on the ratio of the couplings, $\kappa \equiv \check{g}/g$, while the one-loop Hamiltonian of SCQCD has no parameters. We can obtain H_{SCQCD} as the $\kappa \to 0$ limit of the interpolating

Hamiltonian, restricted to the $U(N_f)$ singlet subsector (consecutive $SU(2)_L$ indices are contracted).

2.2 Lifting the Full One-loop Hamiltonian from a Subsector

Computing the complete one-loop Hamiltonian appears to be a daunting combinatorial task, because of the sheer number of possible two-letter structures on which the Hamiltonian can act. For $\mathcal{N}=4$ SYM, Beisert [16] was able to determine the full one-loop Hamiltonian by making maximal use of the power of superconformal symmetry. The letters of $\mathcal{N}=4$ SYM belong to a single representation of the superconformal algebra, the ultrashort "singleton" representation V_F. The tensor product of two singletons has a simple decomposition into an infinite sum of irreducible representations,

$$V_F \times V_F = \sum_{j=0}^{\infty} V_j \,. \tag{2.7}$$

The one-loop Hamiltonian can then be written as

$$H_{12} = \sum_{j=0}^{\infty} f(j)\, \mathcal{P}_j \,, \tag{2.8}$$

where P_j is a projector on the V_j module for letters at sites 1 and 2. Beisert's strategy was to identify a simple closed subsector of the theory, such that each of the V_j modules contains a representative within the subsector. The coefficients $f(j)$ and thus the full Hamiltonian can be read off from the Hamiltonian of the closed subsector. A particularly clever choice [17] of subsector is the $SU(1,1)\times U(1|1)$ subsector comprising the letters $D^n_{+\dot{+}}\lambda_+$, where λ_α is one of the four Weyl fermions. To obtain the Hamiltonian in the subsector a Feynman diagram calculation is needed. However, as pointed out by Beisert [17], the algebraic constraints of superconformal symmetry are so powerful that they fix the Hamiltonian of this sector, up to the overall normalization which corresponds to a rescaling of the coupling.[2] All in all, the one-loop Hamiltonian of $\mathcal{N}=4$ SYM is determined by superconformal symmetry alone.

In adapting Beisert's strategy to our case, we are faced with the compli-

[2] In his first calculation [16], Beisert considered the $SU(1,1)$ subsector consisting of the letters $\mathcal{D}^n_{+\dot{+}}Z$, where Z is a complex scalar, and determined the $SU(1,1)$ one-loop Hamiltonian by direct evaluation of Feynman diagrams.

cation that the letters belong to three distinct representations of the $\mathcal{N}=2$ superconformal algebra, with their tensor products containing different copies of the same module. This leads to a rather intricate mixing problem. Nevertheless, the problem turns out to be tractable and we are able to identify a subsector from which the full Hamiltonian can be lifted.

2.2.1 $\mathcal{N}=2$ superconformal representations

Our notations for superconformal representations are borrowed from [20] and reviewed in Appendix A.1. The letters of SCQCD (as well as of the whole interpolating theory) belong to three superconformal representations, which we denote by $\mathcal{H}$, $\mathcal{V}$ and $\bar{\mathcal{V}}$. The hypermultiplet letters ($Q_{\mathcal{I}}$ and its descendants[3]) belong to the representation $\mathcal{H} \equiv \hat{\mathcal{B}}_{\frac{1}{2}}$, while the vector multiplet letters split into the two conjugate representations $\mathcal{V} \equiv \bar{\mathcal{E}}_{1(0,0)}$ (ϕ and its descendants) and $\bar{\mathcal{V}} \equiv \mathcal{E}_{1(0,0)}$ ($\bar{\phi}$ and its descendants). It is not difficult, using $\mathcal{N}=2$ superconformal characters[4], to evaluate the relevant tensor products[5]

$$\mathcal{H} \times \mathcal{H} = \sum_{q=-1}^{\infty} \hat{\mathcal{C}}_{0(\frac{q}{2},\frac{q}{2})}\,, \tag{2.9}$$

$$\mathcal{H} \times \mathcal{V} = \sum_{q=-1}^{\infty} \hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q}{2})} = \mathcal{V} \times \mathcal{H}\,, \tag{2.10}$$

$$\mathcal{H} \times \bar{\mathcal{V}} = \sum_{q=-1}^{\infty} \hat{\mathcal{C}}_{0(\frac{q}{2},\frac{q+1}{2})} = \bar{\mathcal{V}} \times \mathcal{H}\,, \tag{2.11}$$

[3] We are suppressing for now $SU(2)_L$ indices, since $SU(2)_L$ commutes with the superconformal algebra.

[4] See for example [21] for an illustration of superconformal character techniques in $\mathcal{N}=4$ case.

[5] Following [20], we extend the definition of the $\hat{\mathcal{C}}$ multiplets to $j_1, j_2 = -\frac{1}{2}$ according to the rules:
$\hat{\mathcal{C}}_{0(-\frac{1}{2},-\frac{1}{2})} \equiv \hat{\mathcal{B}}_1$, $\hat{\mathcal{C}}_{0(0,-\frac{1}{2})} \equiv \bar{\mathcal{D}}_{\frac{1}{2}(0,0)}$, $\hat{\mathcal{C}}_{0(-\frac{1}{2},0)} \equiv \mathcal{D}_{\frac{1}{2}(0,0)}$, $\hat{\mathcal{C}}_{0(\frac{1}{2},-\frac{1}{2})} \equiv \bar{\mathcal{D}}_{\frac{1}{2}(\frac{1}{2},0)}$ and $\hat{\mathcal{C}}_{0(-\frac{1}{2},\frac{1}{2})} \equiv \mathcal{D}_{\frac{1}{2}(0,\frac{1}{2})}$.

$$\mathcal{V} \times \mathcal{V} \;=\; \bar{\mathcal{E}}_{2(0,0)} + \sum_{q=0}^{\infty} \hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q-1}{2})}\,, \tag{2.12}$$

$$\bar{\mathcal{V}} \times \bar{\mathcal{V}} \;=\; \mathcal{E}_{2(0,0)} + \sum_{q=0}^{\infty} \hat{\mathcal{C}}_{0(\frac{q-1}{2},\frac{q+1}{2})}\,, \tag{2.13}$$

$$\mathcal{V} \times \bar{\mathcal{V}} \;=\; \sum_{q=0}^{\infty} \hat{\mathcal{C}}_{0(\frac{q}{2},\frac{q}{2})} = \bar{\mathcal{V}} \times \mathcal{V}\,. \tag{2.14}$$

The two-site Hamiltonian H_{12} can still be written as a sum of superconformal projectors, but we must take into account mixing between different sectors. For example, since the representation $\hat{\mathcal{C}}_{0(\frac{q}{2},\frac{q}{2})}$ appears in the tensor products $\mathcal{H} \times \mathcal{H}$, $\mathcal{V} \times \bar{\mathcal{V}}$ and $\bar{\mathcal{V}} \times \mathcal{V}$, these states will mix. The restriction of H_{12} to this subspace takes the form

$$H_{12} = A_{11}(-1)\,\mathcal{P}_{(-\frac{1}{2},-\frac{1}{2})} + \sum_{q=0}^{\infty} \begin{pmatrix} A_{11}(q) & A_{12}(q) & A_{13}(q) \\ A_{21}(q) & A_{22}(q) & A_{23}(q) \\ A_{31}(q) & A_{32}(q) & A_{33}(q) \end{pmatrix} \mathcal{P}_{(\frac{q}{2},\frac{q}{2})}\,, \tag{2.15}$$

where for each q the 3×3 matrix $A_{rs}(q)$ is the mixing matrix of $\mathcal{H}\times\mathcal{H}$, $\mathcal{V}\times\bar{\mathcal{V}}$ and $\bar{\mathcal{V}}\times\mathcal{V}$. Similarly, there is mixing between $\mathcal{H}\times\mathcal{V}$ and $\mathcal{V}\times\mathcal{H}$, and between $\mathcal{H}\times\bar{\mathcal{V}}$ and $\bar{\mathcal{V}}\times\mathcal{H}$, but no mixing for either $\mathcal{V}\times\mathcal{V}$ and $\bar{\mathcal{V}}\times\bar{\mathcal{V}}$, since these latter products decompose into representations that do not appear anywhere else.

2.2.2 A Convenient Subsector

A straightforward way to obtain the coefficients that multiply the superconformal projectors would be to evaluate the dilation operator on the superconformal primaries of each module. The projectors act trivially on these states and the mixing matrix could be read immediately. However, the primaries are complicated objects (see Appendix B.3) and it will be easier to consider certain descendants instead.

We have identified a closed subsector, somewhat analogous to the $SU(1,1)\times U(1|1)$ subsector [17] of $\mathcal{N}=4$ SYM. In SCQCD, our subsector consists of the letters λ_{2+}, $\bar{\lambda}_{2\dot{+}}$, Q_2 and $\bar{Q}_2$, acted upon by an arbitrary number of covariant derivatives $\mathcal{D}_{+\dot{+}}$. Note that all the $SU(2)_R$ indices are taken to be subscripts[6]

[6] If the natural position of the $SU(2)_R$ index is as a superscript, as in $\bar{\lambda}^{\mathcal{I}}_{\dot{\alpha}}$ and $\bar{Q}^{\mathcal{I}}$, we lower it using $\epsilon_{\mathcal{I}\mathcal{J}}$.

with the value $\mathcal{I}=2$. In the interpolating theory, we add $\check{\lambda}_{2+}$ and $\check{\bar{\lambda}}_{2\dot{+}}$ to the list. It will be convenient to define (with $\mathcal{D}\equiv\mathcal{D}_{+\dot{+}}$)

$$\lambda_k=\frac{\mathcal{D}^k}{k!}\lambda_{2+}\,,\qquad \bar{\lambda}_k=\frac{\mathcal{D}^k}{k!}\bar{\lambda}_{2\dot{+}}\,, \tag{2.16}$$

$$\check{\lambda}_k=\frac{\mathcal{D}^k}{k!}\check{\lambda}_{2+}\,,\qquad \check{\bar{\lambda}}_k=\frac{\mathcal{D}^k}{k!}\check{\bar{\lambda}}_{2\dot{+}}\,, \tag{2.17}$$

$$Q_{k\,\hat{\mathcal{I}}}=\frac{\mathcal{D}^k}{k!}Q_{2\,\hat{\mathcal{I}}}\,,\qquad \bar{Q}_k^{\hat{\mathcal{I}}}=\frac{\mathcal{D}^k}{k!}\bar{Q}^{\hat{\mathcal{I}}}{}_2\,. \tag{2.18}$$

The $SU(2)_L$ indices $\hat{\mathcal{I}}=\hat{1},\hat{2}$ will often be suppressed to avoid cluttering.

The sector (2.16)-(2.18) is closed to all loops, as one easily checks by using conservation of the engineering dimension and of the Lorentz and the R-symmetry quantum numbers. The subgroup of the superconformal group acting on the sector is $SU(1,1)\times SU(1|1)\times SU(1|1)\times U(1)$. The $SU(1,1)$ generators are

$$\mathcal{J}'_+(g) \;=\; \mathcal{P}_{+\dot{+}}(g)\,, \tag{2.19}$$

$$\mathcal{J}'_-(g) \;=\; \mathcal{K}^{+\dot{+}}(g)\,, \tag{2.20}$$

$$\mathcal{J}'_3(g) \;=\; \frac{1}{2}D_0+\frac{1}{2}\delta D(g)+\frac{1}{2}\mathcal{L}_+{}^+ +\frac{1}{2}\dot{\mathcal{L}}_{\dot{+}}{}^{\dot{+}}\,, \tag{2.21}$$

where $\delta D(g)\equiv D(g)-D_0$ is the difference between the quantum dilation operator and its classical limit $D_0=D(0)$. The states $Q_{k=0}$ and $\bar{Q}_{k=0}$ are primaries of spin $-\frac{1}{2}$ representations of $SU(1,1)$, while the states $\lambda_{k=0}$, $\check{\lambda}_{k=0}$, $\bar{\lambda}_{k=0}$, $\check{\bar{\lambda}}_{k=0}$ are primaries of spin -1 representations of $SU(1,1)$. The $SU(1|1)\times SU(1|1)\times U(1)$ generators will be presented in section 3.1, and they play a key role in the evaluation the complete one-loop Hamiltonian.

Each of the modules appearing on the right hand side of the tensor products (2.9)-(2.14) contains a representative in this subsector. The representatives are primaries of $SU(1,1)$, and descendants with respect to the full $SU(2,2|2)$. This is sufficient to uplift the Hamiltonian of the subsector to the full Hamiltonian.

2.3 The $\mathcal{N}=1$ Spin Chain

Let us now review $\mathcal{N}=1$ SQCD, the $\mathcal{N}=1$ supersymmetric Yang-Mills theory with gauge group $SU(N_c)$ and N_f flavors of fundamental quarks. In table 2.3 we recall the familiar symmetries of the theory and set our notations. Besides the $\mathcal{N}=1$ vector multiplet ($A_{\alpha\dot{\alpha}}$, λ_α), in the adjoint representation of the gauge group, there are two sets of N_f chiral multiplets, (Q,ψ_α) and

$(\tilde{Q}, \tilde{\psi}_\alpha)$, respectively in the fundamental and antifundamental representations of $SU(N_c)$. The color and flavor structure is then

$$(A^a_{\ b}, \lambda^a_{\ b})\,, \qquad (Q^{ai}, \psi^{ai})\,, \qquad (\tilde{Q}_{a\tilde{\imath}}, \tilde{\psi}_{a\tilde{\imath}})\,, \tag{2.22}$$

where $a = 1, \dots N_c$ are color indices, and $i = 1, \dots N_f$ and $\tilde{\imath} = 1, \dots N_f$ two independent sets of flavor indices, corresponding to the independent flavor symmetries of the gauge-fundamental and of the gauge-antifundamental chiral multiplets.

	$SU(N_c)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_r$
$\mathcal{Q}_\alpha$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	-1
$\mathcal{S}^\alpha$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	1
λ_α	Adj	$\mathbf{1}$	$\mathbf{1}$	0	1
$A_{\alpha\dot{\alpha}}$	Adj	$\mathbf{1}$	$\mathbf{1}$	0	0
Q	$\square$	$\square$	$\mathbf{1}$	1	$1 - \frac{N_c}{N_f}$
ψ_α	$\square$	$\square$	$\mathbf{1}$	1	$-\frac{N_c}{N_f}$
$\tilde{Q}$	$\overline{\square}$	$\mathbf{1}$	$\overline{\square}$	-1	$1 - \frac{N_c}{N_f}$
$\tilde{\psi}_\alpha$	$\overline{\square}$	$\mathbf{1}$	$\overline{\square}$	-1	$-\frac{N_c}{N_f}$

Table 2.3: Field content and symmetries of $\mathcal{N} = 1$ SQCD. We use $\alpha = \pm$ and $\dot{\alpha} = \dot{\pm}$ for Lorentz spinor indices. $\mathcal{Q}_\alpha$ and $\mathcal{S}^\alpha$ denote respectively the Poincaré and conformal supercharges. Conjugate objects such as $\bar{\lambda}_{\dot{\alpha}}$ are not written explicitly.

In the large N Veneziano limit, the basic flavor-singlet local gauge-invariant operators are again "generalized single-traces" of the schematic form

$$\mathrm{Tr}\left(\phi^{k_1} \mathcal{M}^{k_2} \phi^{k_3} \mathcal{M}^{k_4} \dots\right)\,. \tag{2.23}$$

Here ϕ denotes any of the color-adjoint "letters", for example $\phi^a_{\ b} = (\mathcal{D}^n \lambda)^a_{\ b}$, where $\mathcal{D}$ is a gauge-covariant derivative, while $\mathcal{M}^a_{\ b}$ is any of the gauge-adjoint composite objects obtained by the flavor contraction of a fundamental and an antifundamental letter, for example $\mathcal{M}^a_{\ b} = Q^{ai} \bar{Q}_{bi}$ or $\mathcal{M}^a_{\ b} = \bar{\tilde{\psi}}^{a\tilde{\imath}} \tilde{Q}_{b\tilde{\imath}}$.

2.3.1 $\mathcal{N} = 1$ superconformal representations

As in the $\mathcal{N} = 2$ case we will make crucial use of superconformal symmetry to constraint the form of the the spin chain Hamiltonian. The letters that occupy each site of the $\mathcal{N} = 1$ SQCD chain belong to four distinct irreducible representations of the $SU(2,2|1)$ superconformal algebra. We denote them by $\mathcal{X}$ (chiral multiplet), $\bar{\mathcal{X}}$ (antichiral multiplet), $\mathcal{V}$ (vector multiplet) and $\bar{\mathcal{V}}$ (conjugate vector multiplet).

The setup is now analogous to $\mathcal{N} = 2$ SCQCD. The Hamiltonian density acts on two adjacent sites and can be written as a sum of projectors onto the irreducible representations that span the two-site state space. Because of the index structure of the spin chain, not all orderings of two single-site representations are allowed in the two-site state space. For example, it is not possible to have two Qs adjacent to each other because there is no way in which to contract the indices, so two adjacent $\mathcal{X}$ representations are not allowed. On the other hand, Q and $\bar{Q}$ can be placed together and, in fact, there are two ways in which this can be done, we can contract either adjacent *gauge* indices or adjacent *flavor* indices. The gauge-contracted combinations are (the order matters):

$$\mathcal{V}\times\mathcal{V} \qquad \bar{\mathcal{V}}\times\mathcal{V} \qquad \bar{\mathcal{V}}\times\bar{\mathcal{V}} \qquad \mathcal{V}\times\bar{\mathcal{V}} \tag{2.24}$$

$$\mathcal{V}\times\mathcal{X} \qquad \tilde{\mathcal{X}}\times\mathcal{V} \qquad \bar{\mathcal{V}}\times\mathcal{X} \qquad \tilde{\mathcal{X}}\times\bar{\mathcal{V}} \tag{2.25}$$

$$\mathcal{V}\times\bar{\tilde{\mathcal{X}}} \qquad \bar{\mathcal{X}}\times\mathcal{V} \qquad \bar{\mathcal{V}}\times\bar{\tilde{\mathcal{X}}} \qquad \bar{\mathcal{X}}\times\bar{\mathcal{V}} \tag{2.26}$$

$$\bar{\mathcal{X}}\times\mathcal{X} \qquad \bar{\mathcal{X}}\times\bar{\tilde{\mathcal{X}}} \qquad \tilde{\mathcal{X}}\times\mathcal{X} \qquad \tilde{\mathcal{X}}\times\bar{\tilde{\mathcal{X}}} \qquad , \tag{2.27}$$

while the flavor-contracted combinations are:

$$\mathcal{X}\times\bar{\mathcal{X}} \qquad\qquad \bar{\tilde{\mathcal{X}}}\times\tilde{\mathcal{X}}\,. \tag{2.28}$$

For clarity we have added a "tilde" to distinguish the fundamental from the antifundamental chiral multiplets, though of course this is a distinction that pertains to the color and flavor structure, not the superconformal structure ($\mathcal{X}$ and $\tilde{\mathcal{X}}$ are isomorphic as superconformal representations).

The classification of multiplets of the $\mathcal{N} = 1$ superconformal algebra is reviewed in Appendix B.1. We follow the notations of [22], according to which the multiplets that span the single-site state space are given by

$$\mathcal{X} = \tilde{\mathcal{X}} = \bar{\mathcal{D}}_{(0,0)}\,, \qquad \bar{\mathcal{X}} = \bar{\tilde{\mathcal{X}}} = \mathcal{D}_{(0,0)}, \qquad \mathcal{V} = \bar{\mathcal{D}}_{(\frac{1}{2},0)}\,, \qquad \bar{\mathcal{V}} = \mathcal{D}_{(0,\frac{1}{2})}\,. \tag{2.29}$$

Using superconformal characters it is not difficult to decompose the tensor products of any two such multiplets into irreducible representations. We find

$$\tilde{\mathcal{X}} \times \mathcal{X} = \bar{\mathcal{B}}_{\frac{4}{3}(0,0)} + \sum_{q=0}^{\infty} \hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q}{2})}\,, \tag{2.30}$$

$$\bar{\mathcal{X}} \times \bar{\tilde{\mathcal{X}}} = \mathcal{B}_{-\frac{4}{3}(0,0)} + \sum_{q=0}^{\infty} \hat{\mathcal{C}}_{(\frac{q}{2},\frac{q+1}{2})}\,, \tag{2.31}$$

$$\mathcal{X} \times \bar{\mathcal{X}} = \bar{\mathcal{X}} \times \mathcal{X} = \sum_{q=0}^{\infty} \hat{\mathcal{C}}_{(\frac{q}{2},\frac{q}{2})} = \tilde{\mathcal{X}} \times \bar{\tilde{\mathcal{X}}} = \bar{\tilde{\mathcal{X}}} \times \tilde{\mathcal{X}}\,, \tag{2.32}$$

$$\mathcal{V} \times \mathcal{X} = \bar{\mathcal{B}}_{\frac{5}{3}(\frac{1}{2},0)} + \sum_{q=1}^{\infty} \hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q-1}{2})} = \tilde{\mathcal{X}} \times \mathcal{V}\,, \tag{2.33}$$

$$\bar{\mathcal{V}} \times \mathcal{X} = \sum_{q=0}^{\infty} \hat{\mathcal{C}}_{(\frac{q}{2},\frac{q+1}{2})} = \tilde{\mathcal{X}} \times \bar{\mathcal{V}}\,, \tag{2.34}$$

$$\bar{\mathcal{X}} \times \mathcal{V} = \sum_{q=0}^{\infty} \hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q}{2})} = \mathcal{V} \times \bar{\tilde{\mathcal{X}}}\,, \tag{2.35}$$

$$\bar{\mathcal{X}} \times \bar{\mathcal{V}} = \mathcal{B}_{-\frac{5}{3}(0,\frac{1}{2})} + \sum_{q=1}^{\infty} \hat{\mathcal{C}}_{(\frac{q-1}{2},\frac{q+1}{2})} = \bar{\mathcal{V}} \times \bar{\tilde{\mathcal{X}}}\,, \tag{2.36}$$

$$\mathcal{V} \times \mathcal{V} = \bar{\mathcal{B}}_{2(0,0)} + \bar{\mathcal{B}}_{2(1,0)} + \sum_{q=2}^{\infty} \hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q-2}{2})}\,, \tag{2.37}$$

$$\bar{\mathcal{V}} \times \bar{\mathcal{V}} = \mathcal{B}_{-2(0,0)} + \mathcal{B}_{-2(0,1)} + \sum_{q=2}^{\infty} \hat{\mathcal{C}}_{(\frac{q-2}{2},\frac{q+1}{2})}\,, \tag{2.38}$$

$$\mathcal{V} \times \bar{\mathcal{V}} = \sum_{q=1}^{\infty} \hat{\mathcal{C}}_{(\frac{q}{2},\frac{q}{2})} = \bar{\mathcal{V}} \times \mathcal{V}\,. \tag{2.39}$$

2.3.2 Another convenient subsector

Consider the subsector generated by the letters

$$\lambda_k = \frac{\mathcal{D}^k}{k!}\lambda_+\,, \qquad \bar{\mathcal{F}}_k = \frac{\mathcal{D}^k}{k!}\bar{\mathcal{F}}_{\dot{+}\dot{+}}\,, \tag{2.40}$$

$$Q_k = \frac{\mathcal{D}^k}{k!}Q\,, \qquad \bar{\psi}_k = \frac{\mathcal{D}^k}{k!}\bar{\psi}_{\dot{+}}\,, \tag{2.41}$$

$$\tilde{Q}_k = \frac{\mathcal{D}^k}{k!}\tilde{Q}\,, \qquad \bar{\tilde{\psi}}_k = \frac{\mathcal{D}^k}{k!}\bar{\tilde{\psi}}_{\dot{+}}\,. \tag{2.42}$$

with $\mathcal{D} \equiv \mathcal{D}_{+\dot{+}}$. By using conservation of the engineering dimension, of the Lorentz spins and of the R-charge it is easy to see that this sector is closed to all loops under the action of the dilation operator. Moreoever, the one-loop Hamiltonian restricted to this subsector can be uplifted to the full one-loop Hamiltonian, as each of the modules appearing on the right hand side of the tensor products (2.30–2.39) contains a representative within the subsector. The representatives are primaries of $SU(1,1)$, and descendants with respect to the full $SU(2,2|1)$ algebra.

The subgroup of the superconformal group acting on the sector is $SU(1,1)\times U(1|1)$. The $SU(1,1)$ generators are the same as in the $\mathcal{N}=2$ case. The $U(1|1)$ generators will be given in chapter 3 and will be essential for our calculation of the complete one-loop Hamiltonian.

Chapter 3

Algebraic Evaluation of the Dilation Operator

As described in the previous chapter, the evaluation of the complete one-loop Hamiltonian can be simplified dramatically by restricting its action to a special subsector with $SU(1,1)$ symmetry. However, even with this simplification, it remains to calculate the $SU(1,1)$ Hamiltonian using Feynman diagrams. The diagrammatic calculation was done in [16] for $\mathcal{N}=4$ SYM and [23] for $\mathcal{N}=2$ SCQCD. In this chapter we will use the more elegant algebraic approach developed in [17], avoiding Feynman diagrams altogether.

3.1 The $\mathcal{N}=2$ Hamiltonian

In this section we describe the algebraic evaluation of the one-loop Hamiltonian in the $SU(1,1)\times SU(1|1)\times SU(1|1)\times U(1)$ subsector, and its uplifting to the full Hamiltonian. We present the result for the interpolating theory, as a function of $\kappa = \check{g}/g$. The result for SCQCD is obtained by taking the limit $\kappa \to 0$ and focussing on the relevant subspace (that is, discarding the "checked" fields and contracting adjacent $SU(2)_L$ indices). We can focus on evaluating the Hamiltonian on two-site states with open indices ${}^{a}{}_{b}$ and ${}^{a}{}_{\check{b}}$, since the Hamiltonian acting on the structures ${}^{\check{a}}{}_{\check{b}}$ and ${}^{\check{a}}{}_{b}$ is immediately obtained by interchanging $g \leftrightarrow \check{g}$.

In addition to the $SU(1,1)$, our closed subsector has an extra $SU(1|1)\times$

$SU(1|1) \times U(1)$ symmetry. The generators of the two $SU(1|1)$s are[1]

$$B = \frac{1}{2}\mathcal{L}_{-}{}^{-} + \frac{1}{2}\dot{\mathcal{L}}_{\dot{-}}{}^{\dot{-}} + \frac{1}{2}D_0 + r\,, \quad \mathcal{S}(g) = S_1{}^{-}(g)\,, \quad \mathcal{Q}(g) = Q_{-}{}^{1}(g)\,, \quad (3.1)$$

$$\bar{B} = \frac{1}{2}\mathcal{L}_{-}{}^{-} + \frac{1}{2}\dot{\mathcal{L}}_{\dot{-}}{}^{\dot{-}} + \frac{1}{2}D_0 - r\,, \quad \bar{\mathcal{S}}(g) = \bar{S}^{\dot{-}2}(g)\,, \quad \bar{\mathcal{Q}}(g) = \bar{Q}_{\dot{-}2}(g)\,, \quad (3.2)$$

and can be checked to commute with the $SU(1,1)$ generators (2.19). The $U(1)$ is a central element corresponding to the quantum part of the dilatation operator, $\delta D(g)$.

The (anti)commutators are

$$[B, \mathcal{Q}(g)] = \mathcal{Q}(g)\,, \qquad [\bar{B}, \bar{\mathcal{Q}}(g)] = \bar{\mathcal{Q}}(g)\,, \quad (3.3)$$
$$[B, \mathcal{S}(g)] = -\mathcal{S}(g)\,, \qquad [\bar{B}, \bar{\mathcal{S}}(g)] = -\bar{\mathcal{S}}(g)\,, \quad (3.4)$$
$$\{\mathcal{S}(g), \mathcal{Q}(g)\} = \frac{1}{2}\delta D(g)\,, \qquad \{\bar{\mathcal{S}}(g), \bar{\mathcal{Q}}(g)\} = \frac{1}{2}\delta D(g)\,. \quad (3.5)$$

The operator $L = B + \bar{B}$ evaluates to 1 on each of the elementary letters of the subsector, and thus measures the "length" of a state. Since

$$[L, \mathcal{Q}(g)] = \mathcal{Q}(g)\,, \qquad [L, \bar{\mathcal{Q}}(g)] = \bar{\mathcal{Q}}(g)\,, \quad (3.6)$$
$$[L, \mathcal{S}(g)] = -\mathcal{S}(g)\,, \qquad [L, \bar{\mathcal{S}}(g)] = -\bar{\mathcal{S}}(g)\,. \quad (3.7)$$

we learn that $\mathcal{Q}(g)$ and $\bar{\mathcal{Q}}(g)$ increase the length of a state by one unit while $\mathcal{S}(g)$ and $\bar{\mathcal{S}}(g)$ decrease it.

3.1.1 First order expressions for $\mathcal{Q}(g)$ and $\mathcal{S}(g)$

In the classical limit $g \to 0$ one easily checks that the $SU(1|1)$ generators annihilate all the states of the subsector, consistent with the fact that they must change the length of a state. As in [17], we know that there must be quantum corrections to $\mathcal{Q}(g)$ and $\mathcal{S}(g)$, because their anticommutator must yield a non-vanishing quantum dilation operator. Writing $\mathcal{Q}(g) = g\mathcal{Q} + O(g^2)$, the most general ansatz for the action of $\mathcal{Q}$ on λ compatible with Lorentz and

[1] The bar in $\bar{B}$, $\bar{\mathcal{Q}}$, and $\bar{\mathcal{S}}$ does not denote complex conjugation, we are going to impose the appropriate hermiticity condition below.

R-charge conservation is

$$
\begin{aligned}
\mathcal{Q}\lambda_n \;\; &= \;\; \sum_{k'=0}^{n} a_{n,k'} Q_{k'} \bar{Q}_{n-k'} \\
&\quad + \sum_{k'=0}^{n-1} b_{n,k'} \lambda_{k'} \bar{\lambda}_{n-k'-1} + \sum_{k'=0}^{n-1} c_{n,k'} \bar{\lambda}_{k'} \lambda_{n-k'-1}
\end{aligned}
\tag{3.8}
$$

for arbitrary coefficients $a_{n,k'}$, $b_{n,k'}$ and $c_{n,k'}$. The coefficients can be constrained by requiring that $\mathcal{Q}$ commutes with the $SU(1,1)$ algebra. Requiring $[\mathcal{J}', \mathcal{Q}]\lambda_n = 0$ fixes $a_{n,k'}$ to be a constant $a_{n,k'} = \alpha'$, and $b_{n,k'} = c_{n,k'} = 0$. This is however too restrictive, and as in $\mathcal{N} = 4$ SYM [17], one should only require that $[\mathcal{J}', \mathcal{Q}]$ annihilates all gauge invariant states (closed spin chains). We should demand $[\mathcal{J}', \mathcal{Q}]\lambda_n \sim 0$, where $\sim$ stands for equivalence up to a gauge transformation. There are two independent gauge transformations, corresponding to adding an extra $\bar{\lambda}$ or $\bar{\bar{\lambda}}$ to the chain, so we impose

$$
[\mathcal{Q}, \mathcal{J}'_+]\lambda_n \;=\; \alpha\left(\lambda_n \bar{\lambda} + \bar{\lambda}\lambda_n\right) , \tag{3.9}
$$

$$
[\mathcal{Q}, \mathcal{J}'_+]\bar{\lambda}_n \;=\; \alpha\left(\bar{\lambda}_n \bar{\lambda} + \bar{\lambda}\bar{\lambda}_n\right) , \tag{3.10}
$$

$$
[\mathcal{Q}, \mathcal{J}'_+]Q_n \;=\; \alpha\left(\bar{\lambda} Q_n - \gamma Q_n \bar{\bar{\lambda}}\right) , \tag{3.11}
$$

$$
[\mathcal{Q}, \mathcal{J}'_+]\bar{Q}_n \;=\; \alpha\left(\gamma \bar{\bar{\lambda}} \bar{Q}_n - \bar{Q}_n \bar{\lambda}\right) , \tag{3.12}
$$

$$
[\mathcal{Q}, \mathcal{J}'_+]\check{\lambda}_n \;=\; \alpha\gamma\left(\check{\lambda}_n \bar{\bar{\lambda}} + \bar{\bar{\lambda}}\check{\lambda}_n\right) , \tag{3.13}
$$

$$
[\mathcal{Q}, \mathcal{J}'_+]\bar{\bar{\lambda}}_n \;=\; \alpha\gamma\left(\bar{\bar{\lambda}}_n \bar{\bar{\lambda}} + \bar{\bar{\lambda}}\bar{\bar{\lambda}}_n\right) , \tag{3.14}
$$

where we have labelled by α and $\alpha\gamma$ the two independent gauge parameters. We now find

$$
a_{n,k'} \;=\; \alpha' , \tag{3.15}
$$

$$
b_{n,k'} \;=\; \frac{\alpha}{n-k'} , \tag{3.16}
$$

$$
c_{n,k'} \;=\; \frac{\alpha}{k'+1} , \tag{3.17}
$$

where at this stage α and α' are arbitrary constants. Similarly, for the action on the other states of the sector,

$$
\begin{aligned}
\mathcal{Q}\check{\lambda}_n &= \sum_{k'=0}^{n} \alpha'' \bar{Q}_{k'} Q_{n-k'} \\
&\quad + \alpha\gamma \left(\sum_{k'=0}^{n-1} \frac{1}{n-k'} \check{\lambda}_{k'} \bar{\check{\lambda}}_{n-k'-1} + \sum_{k'=0}^{n-1} \frac{1}{k'+1} \bar{\check{\lambda}}_{k'} \check{\lambda}_{n-k'-1} \right) , \qquad (3.18) \\
\mathcal{Q}\bar{\lambda}_n &= \alpha \sum_{k'=0}^{n-1} \frac{n+1}{(k'+1)(n-k')} \bar{\lambda}_{k'} \bar{\lambda}_{n-k'-1} \,, \qquad (3.19) \\
\mathcal{Q}\bar{\check{\lambda}}_n &= \alpha\gamma \sum_{k'=0}^{n-1} \frac{n+1}{(k'+1)(n-k')} \bar{\check{\lambda}}_{k'} \bar{\check{\lambda}}_{n-k'-1} \,, \qquad (3.20) \\
\mathcal{Q}Q_n &= \alpha \sum_{k'=0}^{n-1} \left(\frac{1}{k'+1} \bar{\lambda}_{k'} Q_{n-k'-1} - \frac{\gamma}{n-k'} Q_{k'} \bar{\check{\lambda}}_{n-k'-1} \right) , \qquad (3.21) \\
\mathcal{Q}\bar{Q}_n &= \alpha \sum_{k'=0}^{n-1} \left(\frac{\gamma}{k'+1} \bar{\check{\lambda}}_{k'} \bar{Q}_{n-k'-1} - \frac{1}{n-k'} \bar{Q}_{k'} \bar{\lambda}_{n-k'-1} \right) . \qquad (3.22)
\end{aligned}
$$

One can check that the commutators $[\mathcal{J}'_-, \mathcal{Q}] = 0$ and $[\mathcal{J}'_3, \mathcal{Q}] = 0$ are then identically satisfied with the action of $\mathcal{Q}$ given by the above expressions. An analogous analysis can be performed for $\mathcal{S}$. Now the relevant gauge transformations are

$$
[\mathcal{S}, \mathcal{J}'_-] \bar{\lambda}_k \bar{\lambda}_{n-k} = \beta \left(\delta_{k=0} + \delta_{n=k} \right) \bar{\lambda}_n \,, \quad [\mathcal{S}, \mathcal{J}'_-] \bar{\check{\lambda}}_k \bar{\check{\lambda}}_{n-k} = \beta\gamma' \left(\delta_{k=0} + \delta_{n=k} \right) \bar{\check{\lambda}}_n \,, \qquad (3.23)
$$

$$
\begin{aligned}
&[\mathcal{S}, \mathcal{J}'_-] \lambda_k \bar{\lambda}_{n-k} = \beta \delta_{n=k} \lambda_n \,, && [\mathcal{S}, \mathcal{J}'_-] \check{\lambda}_k \bar{\check{\lambda}}_{n-k} = \beta\gamma' \delta_{n=k} \check{\lambda}_n \,, \qquad (3.24) \\
&[\mathcal{S}, \mathcal{J}'_-] \bar{\lambda}_k \lambda_{n-k} = \beta \delta_{k=0} \lambda_n \,, && [\mathcal{S}, \mathcal{J}'_-] \bar{\check{\lambda}}_k \check{\lambda}_{n-k} = \beta\gamma' \delta_{k=0} \check{\lambda}_n \,, \qquad (3.25) \\
&[\mathcal{S}, \mathcal{J}'_-] \bar{\lambda}_k Q_{n-k} = \beta \delta_{k=0} Q_n \,, && [\mathcal{S}, \mathcal{J}'_-] \bar{\check{\lambda}}_k \bar{Q}_{n-k} = \beta\gamma' \delta_{k=0} \bar{Q}_n \,, \qquad (3.26) \\
&[\mathcal{S}, \mathcal{J}'_-] \bar{Q}_k \bar{\lambda}_{n-k} = -\beta \delta_{n=k} \bar{Q}_n \,, && [\mathcal{S}, \mathcal{J}'_-] Q_k \bar{\check{\lambda}}_{n-k} = -\beta\gamma' \delta_{n=k} Q_n \,, \qquad (3.27)
\end{aligned}
$$

and the action of $\mathcal{S}$ consistent with them is

$$
\begin{aligned}
\mathcal{S}Q_{k\hat{\mathcal{I}}}\bar{Q}^{\hat{\mathcal{J}}}_{n-k} &= \frac{\beta'}{n+1}\lambda_n\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\,, & \mathcal{S}\bar{Q}^{\hat{\mathcal{J}}}_{k}Q_{n-k\hat{\mathcal{I}}} &= \frac{\beta''}{n+1}\check{\lambda}_n\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\,, && (3.28)\\
\mathcal{S}\bar{\lambda}_k\bar{\lambda}_{n-k} &= \beta\bar{\lambda}_{n+1}\,, & \mathcal{S}\bar{\check{\lambda}}_k\bar{\check{\lambda}}_{n-k} &= \gamma'\beta\bar{\check{\lambda}}_{n+1}\,, && (3.29)\\
\mathcal{S}\lambda_k\bar{\lambda}_{n-k} &= \beta\frac{k+1}{n+2}\lambda_{n+1}\,, & \mathcal{S}\check{\lambda}_k\bar{\check{\lambda}}_{n-k} &= \gamma'\beta\frac{k+1}{n+2}\check{\lambda}_{n+1}\,, && (3.30)\\
\mathcal{S}\bar{\lambda}_k\lambda_{n-k} &= \beta\frac{n-k+1}{n+2}\lambda_{n+1}\,, & \mathcal{S}\bar{\check{\lambda}}_k\check{\lambda}_{n-k} &= \gamma'\beta\frac{n-k+1}{n+2}\check{\lambda}_{n+1}\,, && (3.31)\\
\mathcal{S}\bar{\lambda}_kQ_{n-k} &= \beta Q_{n+1}\,, & \mathcal{S}Q_k\bar{\check{\lambda}}_{n-k} &= -\gamma'\beta Q_{n+1}\,, && (3.32)\\
\mathcal{S}\bar{Q}_k\bar{\lambda}_{n-k} &= -\beta\bar{Q}_{n+1}\,, & \mathcal{S}Q_k\bar{\check{\lambda}}_{n-k} &= -\gamma'\beta Q_{n+1}\,. && (3.33)
\end{aligned}
$$

With these expressions, the remaining commutators $[\mathcal{J}'_+,\mathcal{S}]=0$ and $[\mathcal{J}'_3,\mathcal{S}]=0$ are automatically satisfied.

As we are interested in unitary representations of the superconformal algebra, we impose the hermiticity condition[2]

$$\mathcal{Q}^\dagger = \mathcal{S}\,, \tag{3.34}$$

which implies the following reality constraints for the undetermined coefficients:

$$
\begin{aligned}
\alpha &= \beta^*\,, && (3.35)\\
\alpha' &= \beta'^*\,, && (3.36)\\
\alpha'' &= \beta''^*\,, && (3.37)\\
\gamma &= \gamma'^*\,. && (3.38)
\end{aligned}
$$

Having determined the $O(g)$ action of $\mathcal{Q}(g)$ and $\mathcal{S}(g)$, we are now in the position to evaluate the one-loop Hamiltonian, since the algebra (3.5) implies

$$H' = 2\{\mathcal{S},\mathcal{Q}\}\,. \tag{3.39}$$

Let us proceed to find H' in the different subspaces:

[2]To exhibit hermiticity explicitly one needs to rescale the fermion letters as $\chi_n \to \frac{\chi_n}{\sqrt{n+1}}$, where χ_n stands for $\lambda_n\,,\check{\lambda}_n\,,\bar{\lambda}_n$ or $\bar{\check{\lambda}}_n$.

3.1.2 $\mathcal{V} \times \mathcal{V}$ and $\bar{\mathcal{V}} \times \bar{\mathcal{V}}$

The $\bar{\mathcal{V}} \times \bar{\mathcal{V}}$ case is identical with $\mathcal{N} = 4$, we refer the interested reader to [17] for details of the calculation. The result is

$$H'_{12}\bar{\lambda}_k\bar{\lambda}_{n-k} = 2|\alpha|^2 \sum_{k'=0}^{n} c_{n,k,k'}\bar{\lambda}_{k'}\bar{\lambda}_{n-k'} \, , \tag{3.40}$$

with

$$c_{n,k,k'} = \delta_{k=k'}\left(h(k+1) + h(n-k+1)\right) - \frac{\delta_{k\neq k'}}{|k-k'|} + \frac{\delta_{k>k'}}{n-k'+1} + \frac{\delta_{k<k'}}{k'+1} \, ,$$

where $h(k)$ are the harmonic numbers, $h(k) = \sum_{j=1}^{k}\frac{1}{j}$ and $h(0) \equiv 0$.

For $\mathcal{V} \times \mathcal{V}$ the calculation is very similar, and the result is

$$H'_{12}\lambda_k\lambda_{n-k} = 2|\alpha|^2 \sum_{k'=0}^{n} c_{n,k,k'}\lambda_{k'}\lambda_{n-k'} \, , \tag{3.41}$$

with

$$\begin{aligned} c_{n,k,k'} &= \delta_{k=k'}\left(h(k+1) + h(n-k+1) + \frac{|\alpha'|^2}{|\alpha|^2} - 1\right) \\ &\quad - \frac{\delta_{k\neq k'}}{|k-k'|} + \frac{\delta_{k>k'}}{n-k'+1} + \frac{\delta_{k<k'}}{k'+1} \, . \end{aligned} \tag{3.42}$$

We now impose the physical requirement that the action of the Hamiltonian on $\lambda\lambda$ is identical to the action on $\bar{\lambda}\bar{\lambda}$ (this is CPT invariance in the field theory). This fixes $|\alpha'|^2 = |\alpha|^2$, which implies $\alpha' = e^{i\theta_1}\alpha$, where θ_1 is an arbitrary phase.

Using the oscillator representation (see Appendix A.2) it is easy to check that H'_{12} is invariant under $SU(1,1)$. We can then write the Hamiltonian density as

$$H'_{12} = \sum_{j=0}^{\infty} A(j)\mathcal{P}'_{-1-j} \, , \tag{3.43}$$

where $\mathcal{P}'_{-1-j}$ is a projector on the $SU(1,1)$ module of spin $-1-j$. To obtain the coefficients $A(j)$ we act on the $SU(1,1)$ highest weights,

$$\mathcal{J}(j) = -\frac{(j+2)}{(j+1)} \sum_{k=0}^{j} \frac{(-1)^k}{k+1} \binom{j}{k} \binom{j+1}{k} \mathcal{D}^{j-k}\lambda_{2+}\mathcal{D}^k\lambda_{2+} \, . \tag{3.44}$$

The result is

$$H'_{12}\mathcal{J}(j) = 4|\alpha|^2 h(j+1)\mathcal{J}(j)\,, \tag{3.45}$$

which implies $A(j) = 4|\alpha|^2 h(j+1)$. The lifting procedure is now straightforward: $\mathcal{J}(j)$ is not only an $SU(1,1)$ highest weight but also a superconformal descendant, it can be obtained by applying $-\frac{1}{2}\mathcal{R}_2{}^1\mathcal{Q}_+{}^2$ to (A.29) for $j=0$ and $\mathcal{Q}_+{}^1\bar{\mathcal{Q}}_{\dot{+}2}$ to (A.30) for $j>0$. The $SU(1,1)$ modules are sub-modules of the the superconformal modules with $j=q$. The only module not present in this sub-sector is $\bar{\mathcal{E}}_{2(0,0)}$, but we know that this is a protected multiplet so its coefficient is just zero. All in all, the Hamiltonian density in $\mathcal{V}\times\mathcal{V}$ is

$$H_{12} = 0\times\mathcal{P}_{\bar{\mathcal{E}}} + |\alpha|^2\sum_{q=0}^{\infty} 4h(q+1)\mathcal{P}_{(\frac{q+1}{2},\frac{q-1}{2})}\,. \tag{3.46}$$

The overall constant $|\alpha|^2$ cannot be fixed algebraically and is related to a rescaling of the coupling. To match with the Feynman diagram calculation of [23] we need to set $|\alpha|^2 = 1$.

3.1.3 $\bar{\mathcal{V}}\times\mathcal{H}\leftrightarrow\mathcal{H}\times\bar{\mathcal{V}}$

Since this case is somewhat different from $\mathcal{N}=4$ SYM because of multiplet mixing, let us give a few more details of the calculation. We need to evaluate

$$H'_{12}\lambda_k Q_{n-k} = 2(\mathcal{S}\mathcal{Q}+\mathcal{Q}\mathcal{S})\lambda_k Q_{n-k}\,. \tag{3.47}$$

In the first term inside the parenthesis we can act with $\mathcal{Q}$ in either the first or the second site, we will denote this contributions by $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Both choices will increase the length of the chain by one, which implies that $\mathcal{S}$ can act in either sites 1-2 or 2-3, we will denote this by $\mathcal{S}_{12}$ and S_{23}. Taking into account all possible combinations the action of the Hamiltonian is

$$H'_{12} = 2\left(\mathcal{S}_{12}\mathcal{Q}_1 + \mathcal{S}_{23}\mathcal{Q}_1 + \mathcal{S}_{12}\mathcal{Q}_2 + \mathcal{S}_{23}\mathcal{Q}_2 + \mathcal{Q}_1\mathcal{S}_{12}\right). \tag{3.48}$$

Each individual contribution can be calculated by straightforward application of the action of the supercharges given in the previous section,

$$\mathcal{S}_{12}\mathcal{Q}_1\bar{\lambda}_k Q_{n-k} = 4h(k)\bar{\lambda}_k Q_{n-k}\,, \tag{3.49}$$

$$\mathcal{S}_{23}\mathcal{Q}_1\bar{\lambda}_k Q_{n-k} = -2\sum_{k'=0}^{k-1}\left(\frac{1}{k'+1}+\frac{1}{k-k'}\right)\bar{\lambda}_{k'}Q_{n-k'}\,, \tag{3.50}$$

$$\mathcal{S}_{12}\mathcal{Q}_2\bar{\lambda}_k Q_{n-k} = -2\sum_{k'=k+1}^{n}\left(\frac{1}{k'-k}\bar{\lambda}_{k'}Q_{n-k'}+\frac{\gamma}{n-k'+1}Q_{k'}\bar{\check{\lambda}}_{n-k'}\right) \tag{3.51}$$

$$\mathcal{S}_{23}\mathcal{Q}_2\bar{\lambda}_k Q_{n-k} = 2(1+|\gamma|^2)h(n-k)\bar{\lambda}_k Q_{n-k}\,, \tag{3.52}$$

$$\mathcal{Q}_1\mathcal{S}_{12}\bar{\lambda}_k Q_{n-k} = 2\sum_{k'=0}^{n}\left(\frac{1}{k'+1}\bar{\lambda}_{k'}Q_{n-k'}-\frac{\gamma}{n-k'+1}Q_{k'}\bar{\check{\lambda}}_{n-k'}\right)\,. \tag{3.53}$$

Now, since $\mathcal{S}_{12}\mathcal{Q}_1$ and $\mathcal{S}_{23}\mathcal{Q}_2$ act at the single site level, they are analogous to the self-energy contributions in a field theory calculation. As usual for spin chains, we distribute them evenly in two adjacent sites by adding an extra factor of one half. An analogous calculation can be done for $2\{\mathcal{S},\mathcal{Q}\}Q_k\bar{\check{\lambda}}_{n-k}$, the action of the Hamiltonian in this subspace is

$$H'_{12}\bar{\lambda}_k Q_{n-k} = 2\sum_{k'=0}^{n}a_{n,k,k'}\bar{\lambda}_{k'}Q_{n-k'}+2\sum_{k'=0}^{n}b_{n,k,k'}Q_{k'}\bar{\check{\lambda}}_{n-k'}\,, \tag{3.54}$$

$$H'_{12}Q_k\bar{\check{\lambda}}_{n-k} = 2\sum_{k'=0}^{n}\check{a}_{n,k,k'}Q_{k'}\bar{\check{\lambda}}_{n-k'}+2\sum_{k'=0}^{n}\check{b}_{n,k,k'}\bar{\lambda}_{k'}Q_{n-k'}\,, \tag{3.55}$$

where

$$a_{n,k,k'} = \delta_{k=k'}\left(h(k+1)+\frac{1+|\gamma|^2}{2}h(n-k)\right)-\frac{\delta_{k\neq k'}}{|k-k'|}+\frac{\delta_{k<k'}}{k'+1}\,, \tag{3.56}$$

$$b_{n,k,k'} = -\gamma\frac{\delta_{k\geq k'}}{n-k'+1}\,, \tag{3.57}$$

$$\check{a}_{n,k,k'} = \frac{1+|\gamma|^2}{2}h(k)\delta_{k=k'}+|\gamma|^2\left(h(n-k+1)\delta_{k=k'}-\frac{\delta_{k\neq k'}}{|k-k'|}+\frac{\delta_{k>k'}}{n-k'+1}\right)\,, \tag{3.58}$$

$$\check{b}_{n,k,k'} = -\gamma^*\frac{\delta_{k\leq k'}}{k'+1}\,. \tag{3.59}$$

In this case, the Hamiltonian density H'_{12} is *not* an $SU(1,1)$ invariant. However, conformal symmetry only dictates that the total Hamiltonian $\sum_\ell H'_{\ell,\ell+1}$ acting on a *closed* spin chain must be invariant. A redefinition of the two-site

Hamiltonian of the form

$$H'_{\ell,\ell+1} \to H'_{\ell,\ell+1} - K_\ell + K_{\ell+1}\,, \tag{3.60}$$

where K_ℓ is a local operator at site ℓ, leaves the total Hamiltonian invariant. So what we must really check is whether we can make the two-site Hamiltonian invariant by an appropriate choice of K_ℓ. The choice of K_ℓ that makes H'_{12} invariant for the whole $SU(1,1) \times SU(1|1) \times SU(1|1) \times U(1)$ subsector is

$$K_\ell = \sum_{k=0}^{\infty} \left(f(k) P^\ell_{Q_k} - f(k) P^\ell_{\bar{Q}_k} \right)\,, \quad f(k) = (1 - |\gamma|^2) h(k)\,, \tag{3.61}$$

where $P^\ell_{Q_k}$ is the projector on the state Q_k at site ℓ, and similarly for $P^\ell_{\bar{Q}_k}$. We have verified this claim for the restriction of H'_{12} to each of the tensor products. For the tensor products $\mathcal{V} \times \mathcal{H} \leftrightarrow \mathcal{H} \times \mathcal{V}$, the transformation (3.60, 3.61) amounts to redefining the coefficients (3.56, 3.58) as

$$a_{n,k,k'} \to a_{n,k,k'} + \frac{1}{2} f(n-k)\,, \quad \check{a}_{n,k,k'} \to \check{a}_{n,k,k'} - \frac{1}{2} f(k)\,. \tag{3.62}$$

The new coefficients read

$$a_{n,k,k'} \;=\; \delta_{k=k'}(h(k+1) + h(n-k)) - \frac{\delta_{k\neq k'}}{|k-k'|} + \frac{\delta_{k<k'}}{k'+1}\,, \tag{3.63}$$

$$\check{a}_{n,k,k'} \;=\; |\gamma|^2 \left(\delta_{k=k'}(h(k) + h(n-k+1)) - \frac{\delta_{k\neq k'}}{|k-k'|} + \frac{\delta_{k>k'}}{n-k'+1} \right), \tag{3.64}$$

and these combinations *are* $SU(1,1)$ invariant as can be easily checked with the oscillator representation. (The coefficients $b_{n,k,k'}$ and $\check{b}_{n,k,k'}$ were never problematic). Now we can write H'_{12} in (3.54, 3.55) as a sum of projectors

$$H'_{12} = \sum_{j=0}^{\infty} \begin{pmatrix} A_{11}(j) & A_{12}(j) \\ A_{21}(j) & A_{22}(j) \end{pmatrix} \mathcal{P}'_{-\frac{3}{2}-j}\,. \tag{3.65}$$

To obtain the undetermined coefficients we act on the $SU(1,1)$ highest weights (of spin $-\frac{3}{2} - j$),

$$\mathcal{J}(j) = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \binom{j+1}{k} \mathcal{D}^{j-k} \bar{\lambda}_{2+} \mathcal{D}^k Q_2 \,, \tag{3.66}$$

$$\mathcal{K}(j) = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \binom{j+1}{k+1} \mathcal{D}^{j-k} Q_2 \mathcal{D}^k \bar{\bar{\lambda}}_{2+} \,. \tag{3.67}$$

As before, these are also superconformal descendants. They can be obtained by applying $-\frac{1}{2}\mathcal{R}_2{}^1\mathcal{R}_2{}^1\mathcal{Q}_+{}^2$ to (A.33) and (A.36) for $j = 0$ and $\mathcal{Q}_+{}^1\bar{\mathcal{Q}}_{\dot{+}2}$ to (A.34) and (A.37) for $j > 0$.

$$H'_{12}\mathcal{J}(j) = 2\left(h(j+1) + h(j)\right)\mathcal{J}(j) - \frac{2\gamma}{j+1}\mathcal{K}(j)\,, \tag{3.68}$$

$$H'_{12}\mathcal{K}(j) = 2|\gamma|^2\left(h(j+1) + h(j)\right)\mathcal{K}(j) - \frac{2\gamma^*}{j+1}\mathcal{J}(j)\,. \tag{3.69}$$

The lifting procedure works as before: there is a one-to-one relationship between $SU(1,1)$ modules and superconformal modules, now with $q + 1 = j$. Defining $\gamma \equiv \eta e^{i\theta_2}$, where η and θ_2 are real parameters, we find

$$H_{12} = 2\sum_{q=-1}^{\infty} \begin{pmatrix} h(q+2) + h(q+1) & -\frac{\eta}{q+2}e^{i\theta_2} \\ -\frac{\eta}{q+2}e^{-i\theta_2} & \eta^2(h(q+2) + h(q+1)) \end{pmatrix} \mathcal{P}_{(\frac{q+1}{2},\frac{q}{2})} \,. \tag{3.70}$$

The phase θ_2 does not enter in any physical anomalous dimension, and can in fact be set to zero by a similarity transformation.

A quick check: Let's consider the action of the Hamiltonian on the two dimensional vector space formed by $\bar{\phi}Q$ and $Q\bar{\bar{\phi}}$. These are the superconformal primaries of the $q = -1$ modules. The mixing matrix is just (3.70) evaluated at $q = -1$. The result is

$$H_{12} = \begin{pmatrix} 2 & -2\eta \\ -2\eta & 2\eta^2 \end{pmatrix} \,, \tag{3.71}$$

in perfect agreement with [13], if we identify $\eta \equiv \kappa$. This is a nice check because in the above calculation we never considered $\bar{\phi}$ and $\bar{\bar{\phi}}$.

3.1.4 $\mathcal{H}\times\mathcal{H}\leftrightarrow\mathcal{V}\times\bar{\mathcal{V}}\leftrightarrow\bar{\mathcal{V}}\times\mathcal{V}$

Following similar steps as in the previous subsection, we obtain for this subspace

$$\begin{aligned} H'_{12}Q_{k\,\hat{\mathcal{I}}}\bar{Q}^{\hat{\mathcal{J}}}_{n-k} &= 2\sum_{k'=0}^{n}(a_{n,k,k'})^{\hat{\mathcal{J}}\hat{\mathcal{K}}}_{\hat{\mathcal{I}}\hat{\mathcal{L}}}\ Q_{k'\,\hat{\mathcal{K}}}\bar{Q}^{\hat{\mathcal{L}}}_{n-k'} \\ &+\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}2\sum_{k'=0}^{n-1}\left(b_{n,k,k'}\lambda_{k'}\bar{\lambda}_{n-k'-1}+c_{n,k,k'}\bar{\lambda}_{k'}\lambda_{n-k'-1}\right)\ , \end{aligned} \tag{3.72}$$

where (with $\hat{\mathbb{I}}\equiv\delta^{\hat{\mathcal{K}}}_{\hat{\mathcal{I}}}\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{L}}}$, $\hat{\mathbb{K}}\equiv\delta^{\hat{\mathcal{J}}}_{\hat{\mathcal{I}}}\delta^{\hat{\mathcal{K}}}_{\hat{\mathcal{L}}}$)

$$a_{n,k,k'} = \frac{\hat{\mathbb{K}}}{(n+1)}+\kappa^2\hat{\mathbb{I}}\left(\delta_{k=k'}(h(k)+h(n-k))-\frac{\delta_{k\neq k'}}{|k-k'|}\right)\ , \tag{3.73}$$

$$b_{n,k,k'} = \frac{e^{-i\theta_1}}{n+1}\left(-\frac{\delta_{k>k'}}{n-k'}+\frac{\delta_{k\leq k'}}{k'+1}\right)\ , \tag{3.74}$$

$$c_{n,k,k'} = -\frac{e^{-i\theta_1}}{(n+1)}\left(-\frac{\delta_{k>k'}}{n-k'}+\frac{\delta_{k\leq k'}}{k'+1}\right)\ . \tag{3.75}$$

For the action on the fermions, we get

$$H_{12}\lambda_k\bar{\lambda}_{n-k} = 2\sum_{k'=0}^{n}\left(a_{n,k,k'}\lambda_{k'}\bar{\lambda}_{n-k'}+b_{n,k,k'}\bar{\lambda}_{k'}\lambda_{n-k'}\right)+2\sum_{k'=0}^{n+1}c_{n,k,k'}Q_{k'\,\hat{\mathcal{I}}}\bar{Q}^{\hat{\mathcal{I}}}_{n+1-k'}\ , \tag{3.76}$$

$$H_{12}\bar{\lambda}_k\lambda_{n-k} = 2\sum_{k'=0}^{n}\left(a_{n,k,k'}\bar{\lambda}_{k'}\lambda_{n-k'}+b_{n,k,k'}\lambda_{k'}\bar{\lambda}_{n-k'}\right)-2\sum_{k'=0}^{n+1}c_{n,k,k'}Q_{k'\,\hat{\mathcal{I}}}\bar{Q}^{\hat{\mathcal{I}}}_{n+1-k'}\ , \tag{3.77}$$

where

$$\begin{aligned} a_{n,k,k'} &= \delta_{k=k'}\left(h(k+1)+h(n-k+1)-\frac{1}{n+2}\right)-\frac{\delta_{k\neq k'}}{|k-k'|} \\ &+\delta_{k>k'}\frac{k+1}{(n+2)(n-k'+1)}+\delta_{k<k'}\frac{n-k+1}{(n+2)(k'+1)}\ , \end{aligned} \tag{3.78}$$

$$b_{n,k,k'} = \frac{e^{i\theta_1}}{n+2}\left(\delta_{k=k'}+\delta_{k>k'}\frac{n-k+1}{n-k'+1}+\delta_{k<k'}\frac{k+1}{k'+1}\right)\ , \tag{3.79}$$

$$c_{n,k,k'} = -e^{i\theta_1}\left(-\delta_{k\geq k'}+\frac{k+1}{n+2}\right)\ . \tag{3.80}$$

Let us now distinguish the two possible combinations of $SU(2)_L$ indices:

$SU(2)_L$ singlet

The Hamiltonian density can be written as

$$H'_{12} = A_{11}(0)\mathcal{P}'_{-1} + \sum_{j=1}^{\infty} \begin{pmatrix} A_{11}(j) & A_{12}(j) & A_{13}(j) \\ A_{21}(j) & A_{22}(j) & A_{23}(j) \\ A_{31}(j) & A_{32}(j) & A_{33}(j) \end{pmatrix} \mathcal{P}'_{-1-j}\,, \tag{3.81}$$

To fix the undetermined constants we consider the $SU(1,1)$ highest weights (of spin $-1-j$),

$$\mathcal{J}(j) = -\sum_{k=0}^{j}(-1)^k \binom{j}{k}\binom{j}{k}\, \mathcal{D}^{j-k}Q_2\mathcal{D}^k\bar{Q}_2\,, \tag{3.82}$$

$$\mathcal{K}(j) = \sqrt{2j(j+1)}\sum_{k=0}^{j-1}(-1)^k\frac{1}{k+1}\binom{j}{k}\binom{j-1}{i}\mathcal{D}^{j-k-1}\lambda_{2+}\mathcal{D}^k\bar{\lambda}_{2\dot{+}}\,, \tag{3.83}$$

$$\bar{\mathcal{K}}(j) = -\sqrt{2j(j+1)}\sum_{k=0}^{j-1}(-1)^k\frac{1}{k+1}\binom{j}{k}\binom{j-1}{k}\mathcal{D}^{j-k-1}\bar{\lambda}_{2\dot{+}}\mathcal{D}^k\lambda_{2+}\,. \tag{3.84}$$

These states are superconformal descendants obtained by acting with $-\frac{1}{2}\mathcal{R}_2{}^1\mathcal{R}_2{}^1$ on (A.38) for $j=0$, and with $\mathcal{Q}_+{}^1\bar{\mathcal{Q}}_{\dot{+}2}$ on (A.39) and (A.41) for $j>0$. The action of the Hamiltonian is, for $j>0$,

$$\begin{aligned}
H'_{12}\mathcal{J}(j) &= 4\kappa^2 h(j)\mathcal{J}(j) + \frac{2\sqrt{2}e^{-i\theta_1}}{\sqrt{j(j+1)}}\mathcal{K}(j) + \frac{2\sqrt{2}e^{-i\theta_1}}{\sqrt{j(j+1)}}\bar{\mathcal{K}}(j)\,, \\
H'_{12}\mathcal{K}(j) &= 2(h(j+1)+h(j-1))\mathcal{K}(j) - \frac{2}{j(j+1)}\bar{\mathcal{K}}(j) + \frac{2\sqrt{2}e^{i\theta_1}}{\sqrt{j(j+1)}}\mathcal{J}(j)\,, \\
H'_{12}\bar{\mathcal{K}}(j) &= 2(h(j+1)+h(j-1))\bar{\mathcal{K}}(j) - \frac{2}{j(j+1)}\mathcal{K}(j) + \frac{2\sqrt{2}e^{i\theta_1}}{\sqrt{j(j+1)}}\mathcal{J}(j)\,,
\end{aligned}$$

and for $j=0$,

$$H'_{12}\mathcal{J}(0) = 4\mathcal{J}(0)\,. \tag{3.85}$$

We can immediately read off the full one-loop Hamiltonian density in the $\mathcal{H}\times\mathcal{H}\leftrightarrow\mathcal{V}\times\bar{\mathcal{V}}\leftrightarrow\bar{\mathcal{V}}\times\mathcal{V}$ subspace,

$$\begin{aligned} H_{12} &= 4\mathcal{P}_{(-\frac{1}{2},-\frac{1}{2})} \\ &\quad +2\sum_{q=0}^{\infty}\begin{pmatrix} 2\kappa^2 h(q+1) & \frac{\sqrt{2}}{\sqrt{(q+1)(q+2)}}e^{-i\theta_1} & \frac{\sqrt{2}}{\sqrt{(q+1)(q+2)}}e^{-i\theta_1} \\ \frac{\sqrt{2}}{\sqrt{(q+1)(q+2)}}e^{i\theta_1} & h(q+2)+h(q) & -\frac{1}{(q+1)(q+2)} \\ \frac{\sqrt{2}}{\sqrt{(q+1)(q+2)}}e^{i\theta_1} & -\frac{1}{(q+1)(q+2)} & h(q+2)+h(q) \end{pmatrix}\mathcal{P}_{(\frac{q}{2},\frac{q}{2})}\,. \end{aligned} \tag{3.86}$$

The phase θ_1 can be set to zero by a similarity transformation, and we find perfect agreement with the field theory calculation of [23].

A quick check: Let's consider the action of the Hamiltonian on the three-dimensional vector space spanned by $2\phi\bar{\phi}$, $2\bar{\phi}\phi$ and $Q_{\mathcal{I}\hat{\mathcal{I}}}\bar{Q}^{\hat{\mathcal{I}}\mathcal{I}}$. These are the superconformal primaries of the $q=0$ modules. The mixing matrix is the one given in (3.86) evaluated at $q=0$,

$$H_{12}=\begin{pmatrix} 4\kappa^2 & 2 & 2 \\ 2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\,, \tag{3.87}$$

again in agreement with [13].

$SU(2)_L$ triplet

In this case $\mathcal{H}\times\mathcal{H}$ does not mix with $\mathcal{V}\times\bar{\mathcal{V}}$ and $\bar{\mathcal{V}}\times\mathcal{V}$, and the Hamiltonian on $\mathcal{H}\times\mathcal{H}$ is simply

$$H_{12}=\sum_{q=0}^{\infty}4\kappa^2 h(q+1)\mathcal{P}_{(\frac{q}{2},\frac{q}{2})}\,. \tag{3.88}$$

3.2 The $\mathcal{N}=2$ Harmonic Action

While we have obtained an explicit expression for the full one-loop Hamiltonian in terms of superconformal projectors, evaluating this expression on concrete states is still a rather cumbersome procedure. For $\mathcal{N}=4$ SYM Beisert [16] was able to find a very explicit and elegant formula for the the action of the Hamiltonian on any state, using the oscillator representation, which he called the "harmonic action". Beisert's approach easily generalizes to our case and allows to write a harmonic action for the interpolating SCFT.

3.2.1 $\mathcal{V} \times \mathcal{V}$

For a state in $\mathcal{V} \times \mathcal{V}$ we found that the action of the Hamiltonian is identical with that of $\mathcal{N} = 4$ SYM. Let's review then how the harmonic action works in this case. As pointed out in [16] a general state in $\mathcal{V} \times \mathcal{V}$ can be written as

$$|s_1, ..., s_n; A\rangle_{\mathcal{V}\times\mathcal{V}} = A^{\dagger}_{s_1 A_1} ... A^{\dagger}_{s_n A_n} |0\rangle \otimes |0\rangle \,, \tag{3.89}$$

where $A^{\dagger}_A = (\mathbf{a}^{\dagger}_{\alpha}, \mathbf{b}^{\dagger}_{\dot{\alpha}}, \mathbf{c}^{\dagger}_{\mathcal{I}})$ and $s_i = 1, 2$ indicates in which site the oscillator sits. The action of the Hamiltonian on this state does not change the number of oscillators but merely shifts them from site 1 to site 2 (or vice versa) in all possible combinations. This can be written as

$$H_{12}|s_1, ..., s_n; A\rangle_{\mathcal{V}\times\mathcal{V}} = \sum_{s'_1,...,s'_n} c_{n,n_{12},n_{21}} \delta_{C_1,0}\delta_{C_2,0} |s'_1, ..., s'_n; A\rangle_{\mathcal{V}\times\mathcal{V}} \,, \tag{3.90}$$

where the delta functions project onto states with zero central charge and n_{ij} counts the number of oscillators moving from site i to site j. The explicit formula for the function $c_{n,n_{12},n_{21}}$ is

$$c_{n,n_{12},n_{21}} = 2(-1)^{1+n_{12}n_{21}} \frac{\Gamma(\frac{1}{2}(n_{12}+n_{21}))\Gamma(1+\frac{1}{2}(n-n_{12}-n_{21}))}{\Gamma(1+\frac{1}{2}n)} \,, \tag{3.91}$$

with $c_{n,0,0} = 2h(\frac{n}{2})$. In [16] it was proven that this function is a superconformal invariant and that it has the appropriate eigenvalues when acting on the $\hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q-1}{2})}$ modules, namely

$$H_{12}\hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q-1}{2})} = 4h(q+1)\hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q-1}{2})} \,. \tag{3.92}$$

3.2.2 $\mathcal{V} \times \mathcal{H} \leftrightarrow \mathcal{H} \times \mathcal{V}$

General states in $\mathcal{V} \times \mathcal{H}$ and $\mathcal{H} \times \mathcal{V}$ can be written as

$$\begin{aligned} |s_1, ..., s_n; A\rangle_{\mathcal{V}\times\mathcal{H}} &= A^{\dagger}_{s_1 A_1} ... A^{\dagger}_{s_n A_n} |0\rangle \otimes |\mathbf{d}\rangle \,, && (3.93)\\ |s_1, ..., s_n; A\rangle_{\mathcal{H}\times\check{\mathcal{V}}} &= A^{\dagger}_{s_1 A_1} ... A^{\dagger}_{s_n A_n} |\mathbf{d}\rangle \otimes |\check{0}\rangle \,, && (3.94) \end{aligned}$$

where $|\mathbf{d}\rangle = \mathbf{d}^\dagger|0\rangle$. We claim that the action of H_{12} is given by [3]

$$
\begin{aligned}
H_{12}|s_1, ..., s_n; A\rangle_{\mathcal{V}\times\mathcal{H}} &= \sum_{s'_1,...,s'_n} c_{n+1,n_{12},n_{21}}|s_1, ..., s_n; A\rangle_{\mathcal{V}\times\mathcal{H}} \\
&\quad +\kappa \sum_{s'_1,...,s'_n} c_{n+1,n_{12},n_{21}+1}|s_1, ..., s_n; A\rangle_{\mathcal{H}\times\check{\mathcal{V}}} \quad (3.95)
\end{aligned}
$$

and

$$
\begin{aligned}
H_{12}|s_1, ..., s_n; A\rangle_{\mathcal{H}\times\check{\mathcal{V}}} &= \kappa^2 \sum_{s'_1,...,s'_n} c_{n+1,n_{12},n_{21}}|s_1, ..., s_n; A\rangle_{\mathcal{H}\times\check{\mathcal{V}}} \\
&\quad -\kappa \sum_{s'_1,...,s'_n} c_{n+1,n_{12},n_{21}+1}|s_1, ..., s_n; A\rangle_{\mathcal{V}\times\mathcal{H}} \quad (3.96)
\end{aligned}
$$

Invariance under the superconformal group is guaranteed by the same arguments given in [16]. The only thing we need to check is that this expression correctly reproduces the 2×2 matrix given in (3.70). This can be easily done with an algebra software like *Mathematica*..

Let us work out an example. Consider the action of the Hamiltonian on $\lambda_{\mathcal{I}} Q_{\mathcal{J}}$ (Lorentz and $SU(2)_L$ indices are open and go along for the ride). First, we need to write the state in a "canonical order" to make sure all our signs are correct,

$$\lambda_{\mathcal{I}} Q_{\mathcal{J}} = \mathbf{a}^\dagger_{(1)}\mathbf{c}^\dagger_{(1)\mathcal{I}}|0\rangle \otimes \mathbf{c}^\dagger_{(2)\mathcal{J}}|\mathbf{d}\rangle = \mathbf{a}^\dagger_{(1)}\mathbf{c}^\dagger_{(1)\mathcal{I}}\mathbf{c}^\dagger_{(2)\mathcal{J}}|0\rangle \otimes |\mathbf{d}\rangle\,, \quad (3.97)$$

$$Q_{\mathcal{I}}\check{\lambda}_{\mathcal{J}} = \mathbf{c}^\dagger_{(1)\mathcal{I}}|\mathbf{d}\rangle \otimes \mathbf{a}^\dagger_{(2)}\mathbf{c}^\dagger_{(2)\mathcal{J}}|\check{0}\rangle = -\mathbf{c}^\dagger_{(1)\mathcal{I}}\mathbf{a}^\dagger_{(2)}\mathbf{c}^\dagger_{(2)\mathcal{J}}|\mathbf{d}\rangle \otimes |\check{0}\rangle\,. \quad (3.98)$$

For $\lambda_1 Q_1$, the action of the Hamiltonian is

$$
\begin{aligned}
H_{12}\lambda_1 Q_1 &= c_{4,0,0}\mathbf{a}^\dagger_{(1)}\mathbf{c}^\dagger_{(1)1}\mathbf{c}^\dagger_{(2)1}|0\rangle \otimes |\mathbf{d}\rangle + c_{4,1,1}\mathbf{a}^\dagger_{(1)}\mathbf{c}^\dagger_{(2)1}\mathbf{c}^\dagger_{(1)1}|0\rangle \otimes |\mathbf{d}\rangle \\
&\quad +\kappa\left(c_{4,1,1}\mathbf{a}^\dagger_{(2)}\mathbf{c}^\dagger_{(1)1}\mathbf{c}^\dagger_{(2)1}|\mathbf{d}\rangle \otimes |\check{0}\rangle + c_{4,2,2}\mathbf{a}^\dagger_{(2)}\mathbf{c}^\dagger_{(2)1}\mathbf{c}^\dagger_{(1)1}|\mathbf{d}\rangle \otimes |\check{0}\rangle\right) \\
&= 2\lambda_1 Q_1 - 2\kappa Q_1\check{\lambda}_1\,, \quad (3.99)
\end{aligned}
$$

[3]To simplify the notation we will omit the delta functions $\delta_{C_1,0}\delta_{C_2,0}$.

while for $\lambda_1 Q_2$,

$$\begin{aligned}
H_{12}\lambda_1 Q_2 &= c_{4,0,0}\mathbf{a}^\dagger_{(1)}\mathbf{c}^\dagger_{(1)1}\mathbf{c}^\dagger_{(2)2}|0\rangle\otimes|\mathbf{d}\rangle + c_{4,1,1}\mathbf{a}^\dagger_{(1)}\mathbf{c}^\dagger_{(2)1}\mathbf{c}^\dagger_{(1)1}|0\rangle\otimes|\mathbf{d}\rangle \\
&\quad +\kappa c_{4,1,1}\left(\mathbf{a}^\dagger_{(1)}\mathbf{c}^\dagger_{(2)1}\mathbf{c}^\dagger_{(2)2}|\mathbf{d}\rangle\otimes|\check{0}\rangle + \mathbf{a}^\dagger_{(2)}\mathbf{c}^\dagger_{(1)1}\mathbf{c}^\dagger_{(2)2}|\mathbf{d}\rangle\otimes|\check{0}\rangle\right) \\
&\quad +c_{4,1,1}\mathbf{a}^\dagger_{(2)}\mathbf{c}^\dagger_{(1)1}\mathbf{c}^\dagger_{(1)1}|0\rangle\otimes|\mathbf{d}\rangle + \kappa c_{4,2,2}\mathbf{a}^\dagger_{(2)}\mathbf{c}^\dagger_{(2)1}\mathbf{c}^\dagger_{(1)2}|\mathbf{d}\rangle\otimes|\check{0}\rangle \\
&= 3\lambda_1 Q_2 - \lambda_2 Q_1 + \phi\psi - \kappa\left(Q_1\check{\lambda}_2 + Q_2\check{\lambda}_1 - \psi\check{\phi}\right) . \qquad (3.100)
\end{aligned}$$

Similar calculations can be done for $\lambda_2 Q_1$ and $\lambda_2 Q_2$. The final result is

$$\begin{aligned}
H_{12}\lambda_{\mathcal{I}} Q_{\mathcal{J}} &= 3\lambda_{\mathcal{I}} Q_{\mathcal{J}} - \lambda_{\mathcal{J}} Q_{\mathcal{I}} - \epsilon_{\mathcal{I}\mathcal{J}}\phi\tilde{\psi} \\
&\quad -\kappa Q_{\mathcal{I}}\check{\lambda}_{\mathcal{J}} - \kappa Q_{\mathcal{J}}\check{\lambda}_{\mathcal{I}} - \kappa\epsilon_{\mathcal{I}\mathcal{J}}\tilde{\psi}\check{\phi} . \qquad (3.101)
\end{aligned}$$

3.2.3 $\mathcal{H}\times\mathcal{H}\leftrightarrow\mathcal{V}\times\bar{\mathcal{V}}\leftrightarrow\bar{\mathcal{V}}\times\mathcal{V}$

For these multiplets we have the following states

$$\begin{aligned}
|s_1,...,s_n;A\rangle_{\mathcal{H}\times\mathcal{H}} &= A^\dagger_{s_1A_1}...A^\dagger_{s_nA_n}|\mathbf{d}\rangle\otimes|\tilde{\mathbf{d}}\rangle , \qquad (3.102)\\
|s_1,...,s_n;A\rangle_{\mathcal{V}\times\bar{\mathcal{V}}} &= A^\dagger_{s_1A_1}...A^\dagger_{s_nA_n}|\mathbf{d}\tilde{\mathbf{d}}\rangle\otimes|0\rangle , \qquad (3.103)\\
|s_1,...,s_n;A\rangle_{\bar{\mathcal{V}}\times\mathcal{V}} &= A^\dagger_{s_1A_1}...A^\dagger_{s_nA_n}|0\rangle\otimes|\mathbf{d}\tilde{\mathbf{d}}\rangle . \qquad (3.104)
\end{aligned}$$

Let us consider the $SU(2)_L$ triplet and singlet cases separately. We have found:

$SU(2)_L$ **singlet**

$$\begin{aligned}
H_{12}|s_1,...,s_n;A\rangle_{\mathcal{H}\times\mathcal{H}} &= \sum_{s'_1,...,s'_n}\left(\kappa^2 c_{n,n_{12},n_{21}} - 2c_{n+2,n_{12}+2,n_{21}}\right)|s_1,...,s_n;A\rangle_{\mathcal{H}\times\mathcal{H}} \\
&\quad +2\sum_{s'_1,...,s'_n} c_{n+2,n_{12},n_{21}+1}|s_1,...,s_n;A\rangle_{\mathcal{V}\times\bar{\mathcal{V}}} \\
&\quad +2\sum_{s'_1,...,s'_n} c_{n+2,n_{12}+1,n_{21}}|s_1,...,s_n;A\rangle_{\bar{\mathcal{V}}\times\mathcal{V}} , \qquad (3.105)
\end{aligned}$$

$$
\begin{aligned}
H_{12}|s_1, ..., s_n; A\rangle_{\mathcal{V}\times\bar{\mathcal{V}}} &= \sum_{s'_1,\dots,s'_n} c_{n+2,n_{12},n_{21}}|s_1, ..., s_n; A\rangle_{\mathcal{V}\times\bar{\mathcal{V}}} \\
&\quad + \sum_{s'_1,\dots,s'_n} c_{n+2,n_{12}+2,n_{21}}|s_1, ..., s_n; A\rangle_{\bar{\mathcal{V}}\times\mathcal{V}} \\
&\quad + \sum_{s'_1,\dots,s'_n} c_{n+2,n_{12}+1,n_{21}}|s_1, ..., s_n; A\rangle_{\mathcal{H}\times\mathcal{H}} \,,
\end{aligned} \tag{3.106}
$$

$$
\begin{aligned}
H_{12}|s_1, ..., s_n; A\rangle_{\bar{\mathcal{V}}\times\mathcal{V}} &= \sum_{s'_1,\dots,s'_n} c_{n+2,n_{12},n_{21}}|s_1, ..., s_n; A\rangle_{\bar{\mathcal{V}}\times\mathcal{V}} \\
&\quad + \sum_{s'_1,\dots,s'_n} c_{n+2,n_{12},n_{21}+2}|s_1, ..., s_n; A\rangle_{\mathcal{V}\times\bar{\mathcal{V}}} \\
&\quad + \sum_{s'_1,\dots,s'_n} c_{n+2,n_{12},n_{21}+1}|s_1, ..., s_n; A\rangle_{\mathcal{H}\times\mathcal{H}} \,.
\end{aligned} \tag{3.107}
$$

$SU(2)_L$ **triplet**

$$
H_{12}|s_1, ..., s_n; A\rangle_{\mathcal{H}\times\mathcal{H}} = \kappa^2 \sum_{s'_1,\dots,s'_n} c_{n,n_{12},n_{21}}|s_1, ..., s_n; A\rangle_{\mathcal{H}\times\mathcal{H}} \,. \tag{3.108}
$$

3.3 The $\mathcal{N} = 1$ Hamiltonian

The procedure here is the same as in the previous section. We consider the $SU(1,1)$ subsector and determine the Hamiltonian using the constraints of superconformal symmetry. We then uplift the result to the complete theory and obtain the full Hamiltonian as a sum of superconformal projectors. Finally we rewrite the Hamiltonian in an "harmonic action" form.

In addition to the $SU(1,1)$ symmetry, the subsector (2.40–2.42) has an additional $U(1|1)$ symmetry generated by,

$$
L = \mathcal{L}_-{}^- + \dot{\mathcal{L}}_{\dot{-}}{}^{\dot{-}} + D_0 \,, \quad \bar{\mathcal{Q}}(g) = \bar{Q}_{\dot{-}}(g) \,, \quad \bar{\mathcal{S}}(g) = \bar{\mathcal{S}}_{\dot{-}}(g) \,, \quad \delta D(g) \tag{3.109}
$$

and can be checked to commute with the $SU(1,1)$ generators (2.19). Their (anti)commutators are

$$
[L, \bar{\mathcal{Q}}(g)] = \bar{\mathcal{Q}}(g) \,, \tag{3.110}
$$

$$[L,\bar{\mathcal{S}}(g)] = -\bar{\mathcal{S}}(g)\,, \tag{3.111}$$

$$\{\bar{\mathcal{S}}(g),\bar{\mathcal{Q}}(g)\} = \frac{1}{2}\delta D(g)\,. \tag{3.112}$$

The generator L can be identified with the length operator, this implies that $\bar{\mathcal{Q}}$ increases the length of a chain while $\bar{\mathcal{S}}$ decreases it.

3.3.1 First order expressions for $\mathcal{Q}(g)$ and $\mathcal{S}(g)$

The procedure is now very similar to the one before and we shall be brief. Writing $\mathcal{Q}(g) = g\mathcal{Q} + O(g^2)$, we formulate an ansatz for the action of the supercharges on the states of the sector compatible with the quantum numbers of the fields and impose invariance under the $SU(1,1)$ algebra, $[\mathcal{J}',\bar{\mathcal{Q}}] = 0$ and $[\mathcal{J}',\bar{\mathcal{S}}] = 0$. As before, strict invariance is too restrictive and one needs only to impose vanishing of these commutators up to local gauge transformations on the chain. It can be easily checked that the following transformations evaluate to zero on any closed chain,

$$\begin{aligned}
[\mathcal{J}'_+,\bar{\mathcal{Q}}]\lambda_n &= \alpha\left(\lambda_n\lambda + \lambda\lambda_n\right)\,, && (3.113)\\
[\mathcal{J}'_+,\bar{\mathcal{Q}}]\bar{\mathcal{F}}_n &= \alpha\left(-\bar{\mathcal{F}}_n\lambda + \lambda\bar{\mathcal{F}}_n\right)\,, && (3.114)\\
[\mathcal{J}'_+,\bar{\mathcal{Q}}]Q_n &= \alpha\lambda Q_n\,, && (3.115)\\
[\mathcal{J}'_+,\bar{\mathcal{Q}}]\bar{\psi}_n &= \alpha\bar{\psi}_n\lambda\,, && (3.116)\\
[\mathcal{J}'_+,\bar{\mathcal{Q}}]\tilde{Q}_n &= -\alpha\tilde{Q}_n\lambda\,, && (3.117)\\
[\mathcal{J}'_+,\bar{\mathcal{Q}}]\bar{\tilde{\psi}}_n &= \alpha\lambda\bar{\tilde{\psi}}_n\,, && (3.118)
\end{aligned}$$

where α is an arbitrary gauge parameter. The action of $\bar{\mathcal{Q}}$ consistent with these transformations is

$$\bar{\mathcal{Q}}\lambda_n = \alpha\sum_{k'=0}^{n-1}\frac{n+1}{(k'+1)(n-k')}\lambda_{k'}\lambda_{n-k'-1}\,, \tag{3.119}$$

$$\begin{aligned}\bar{\mathcal{Q}}\bar{\mathcal{F}}_n &= \alpha\sum_{k'=0}^{n-1}\left(-\frac{1}{n-k'}\bar{\mathcal{F}}_{k'}\lambda_{n-k'-1}+\frac{1}{k'+1}\lambda_{k'}\bar{\mathcal{F}}_{n-k'+1}\right)\\ &\quad +\alpha'\sum_{k'=0}^{n}Q^i_{k'}\bar{\psi}_{n-k'\,i}+\alpha''\sum_{k'=0}^{n}\bar{\bar{\psi}}^{\tilde{\imath}}_{k'}\tilde{Q}_{n-k'\,\tilde{\imath}}\,,\end{aligned} \tag{3.120}$$

$$\bar{\mathcal{Q}}Q_n = \alpha\sum_{k'=0}^{n-1}\frac{1}{k'+1}\lambda_{k'}Q_{n-k'-1}\,, \tag{3.121}$$

$$\bar{\mathcal{Q}}\bar{\psi}_n = \alpha\sum_{k'=0}^{n-1}\frac{1}{n-k'}\bar{\psi}_{k'}\lambda_{n-k'-1}\,, \tag{3.122}$$

$$\bar{\mathcal{Q}}\tilde{Q}_n = -\alpha\sum_{k'=0}^{n-1}\frac{1}{n-k'}\tilde{Q}_{k'}\lambda_{n-k'+1}\,, \tag{3.123}$$

$$\bar{\mathcal{Q}}\bar{\bar{\psi}}_n = \alpha\sum_{k'=0}^{n-1}\frac{1}{k'+1}\lambda_{k'}\bar{\psi}_{n-k'+1}\,. \tag{3.124}$$

The terms with α' and α'' are invariant on their own and that's why we assigned them independent gauge parameters. Similarly, the gauge transformations associated with the $\bar{\mathcal{S}}$ supercharge are

$$[\mathcal{J}'_-,\bar{\mathcal{S}}]\lambda_k\lambda_{n-k} = \beta\left(\delta_{k=0}+\delta_{n=k}\right)\lambda_n\,, \tag{3.125}$$

$$[\mathcal{J}'_-,\bar{\mathcal{S}}]\lambda_k\bar{\mathcal{F}}_{n-k} = \beta\delta_{k=0}\bar{\mathcal{F}}_n\,, \qquad [\mathcal{J}'_-,\bar{\mathcal{S}}]\bar{\mathcal{F}}_k\lambda_{n-k} = -\beta\delta_{n=k}\bar{\mathcal{F}}_n\,, \tag{3.126}$$

$$[\mathcal{J}'_-,\bar{\mathcal{S}}]\lambda_kQ_{n-k} = \beta\delta_{k=0}Q_n\,, \qquad [\mathcal{J}'_-,\bar{\mathcal{S}}]\tilde{Q}_k\lambda_{n-k} = -\beta\delta_{n=k}\tilde{Q}_n\,, \tag{3.127}$$

$$[\mathcal{J}'_-,\bar{\mathcal{S}}]\bar{\psi}_k\lambda_{n-k} = \beta\delta_{n=k}\bar{\psi}_n\,, \qquad [\mathcal{J}'_-,\bar{\mathcal{S}}]\lambda_k\bar{\bar{\psi}}_{n-k} = \beta\delta_{k=0}\bar{\bar{\psi}}_n\,, \tag{3.128}$$

and the action of $\bar{\mathcal{S}}$ consistent with them is

$$
\begin{aligned}
\bar{\mathcal{S}}\lambda_k\lambda_{n-k} &= \beta\lambda_{n+1}\,, &(3.129)\\
\mathcal{S}\lambda_k\bar{\mathcal{F}}_{n-k} &= \beta\frac{(n-k+2)(n-k+1)}{(n+3)(n+2)}\bar{\mathcal{F}}_{n+1}\,, &(3.130)\\
\mathcal{S}\bar{\mathcal{F}}_k\lambda_{n-k} &= -\beta\frac{(k+2)(k+1)}{(n+3)(n+2)}\bar{\mathcal{F}}_{n+1}\,, &(3.131)\\
\bar{\mathcal{S}}\lambda_k Q_{n-k} &= \beta Q_{n+1}\,, &(3.132)\\
\bar{\mathcal{S}}\tilde{Q}_k\lambda_{n-k} &= -\beta\tilde{Q}_{n+1}\,, &(3.133)\\
\bar{\mathcal{S}}\bar{\psi}_k\lambda_{n-k} &= \beta\frac{k+1}{n+2}\bar{\psi}_{n+1}\,, &(3.134)\\
\bar{\mathcal{S}}\lambda_k\bar{\bar{\psi}}_{n-k} &= \beta\frac{n-k+1}{n+2}\bar{\bar{\psi}}_{n+1}\,, &(3.135)\\
\mathcal{S}Q^i_k\bar{\psi}_{n-k\,i} &= \beta'\frac{n-k+1}{(n+2)(n+1)}\bar{\mathcal{F}}_n\,, &(3.136)\\
\mathcal{S}\bar{\bar{\psi}}^{\tilde{\imath}}_k\tilde{Q}_{n-k\,\tilde{\imath}} &= \beta''\frac{k+1}{(n+2)(n+1)}\bar{\mathcal{F}}_n\,. &(3.137)
\end{aligned}
$$

As before, the terms with β' and β'' are invariant on their own. Note that the action of $\bar{\mathcal{S}}$ on $Q_k\bar{\psi}_{n-k}$ and $\bar{\bar{\psi}}_k\tilde{Q}_{n-k}$ is non-zero only for the flavor-contracted combinations. Indeed the action on the gauge-contracted combinations would have to give a single letter with open flavor indices which is impossible. Now we impose the hermiticity condition

$$\bar{\mathcal{Q}}^\dagger = \bar{\mathcal{S}}\,, \tag{3.138}$$

which implies the following reality constraints for the undetermined coefficients:

$$\alpha = \beta^*\,, \quad \alpha' = \beta'^*\,, \quad \alpha'' = \beta''^*\,. \tag{3.139}$$

3.3.2 The Hamiltonian as a sum of projectors

Having determined the $O(g)$ action of $\bar{\mathcal{Q}}(g)$ and $\bar{\mathcal{S}}(g)$, the one-loop Hamiltonian is easily obtained from

$$H' = 2\{\bar{\mathcal{S}},\bar{\mathcal{Q}}\}\,. \tag{3.140}$$

This result suffers from a certain gauge ambiguity analogous to the one encounter in the $\mathcal{N}=2$ calculation, we fix it by demanding the stronger condition that the Hamiltonian density be $SU(1,1)$ invariant. We can then write H'_{12} as a sum of projectors onto the $SU(1,1)$ irreps of the two-site state space, with coefficients determined by explicit evaluation on the primary of each module. The uplifting procedure is straightforward: one writes the full Hamiltonian as a sum over $SU(2,2|1)$ projectors and fixes the coefficients by comparison with the $SU(1,1)$ subsector, as each $SU(1,1)$ primary is also a $SU(2,2|1)$ descendant. We simply quote the results in the various subspaces.

$\mathcal{V} \times \mathcal{V}$ and $\bar{\mathcal{V}} \times \bar{\mathcal{V}}$

We find

$$H_{12} = 0 \times \mathcal{P}_{\bar{\mathcal{B}}_{2(0,0)}} + 2|\alpha|^2 \sum_{q=1}^{\infty} 2h(q) \mathcal{P}_{(\frac{q+1}{2}, \frac{q-2}{2})} \,, \tag{3.141}$$

for $\mathcal{V} \times \mathcal{V}$ and

$$H_{12} = 0 \times \mathcal{P}_{\mathcal{B}_{-2(0,0)}} + 2(-2|\alpha|^2 + \frac{|\alpha'|^2}{2} + \frac{|\alpha''|^2}{2}) \sum_{q=1}^{\infty} 2h(q) \mathcal{P}_{(\frac{q-2}{2}, \frac{q+1}{2})} \,, \tag{3.142}$$

for $\bar{\mathcal{V}} \times \bar{\mathcal{V}}$. CPT invariance implies that these expressions should be identical, which imposes an extra restriction on $|\alpha'|^2$ and $|\alpha''|^2$ namely,

$$|\alpha'|^2 + |\alpha''|^2 = 6|\alpha|^2 \,. \tag{3.143}$$

Now, α' and α'' are parameters associated with $(\mathcal{X}, \bar{\mathcal{X}})$ and $(\tilde{\mathcal{X}}, \bar{\tilde{\mathcal{X}}})$ repectively. Parity (which is a symmetry of the theory, see section 4.1) interchanges the two, in order to have parity invariant Hamiltonian we need to set

$$\alpha' = \sqrt{3} e^{i\theta} \alpha \,, \tag{3.144}$$

$$\alpha'' = -\sqrt{3} e^{i\theta} \alpha \,, \tag{3.145}$$

where θ is an arbitrary phase, which can be set to zero by a similarity transformation.

$\mathcal{V}\times\mathcal{X}$, $\bar{\mathcal{X}}\times\bar{\mathcal{V}}$, $\bar{\mathcal{V}}\times\mathcal{X}$ and $\bar{\mathcal{X}}\times\mathcal{V}$

The Hamiltonian in these subspaces is[4]

$$\mathcal{V}\times\mathcal{X} = 2|\alpha|^2\sum_{q=0}^{\infty}(h(q+1)+h(q)-\frac{1}{2})\mathcal{P}_{(\frac{q+1}{2},\frac{q-1}{2})} = \tilde{\mathcal{X}}\times\mathcal{V}\,, \tag{3.146}$$

$$\bar{\mathcal{X}}\times\bar{\mathcal{V}} = 2|\alpha|^2\sum_{q=0}^{\infty}(h(q+1)+h(q)-\frac{1}{2})\mathcal{P}_{(\frac{q-1}{2},\frac{q+1}{2})} = \bar{\mathcal{V}}\times\bar{\tilde{\mathcal{X}}}\,, \tag{3.147}$$

$$\bar{\mathcal{V}}\times\mathcal{X} = 2|\alpha|^2\sum_{q=0}^{\infty}(h(q+2)+h(q)-\frac{1}{2})\mathcal{P}_{(\frac{q}{2},\frac{q+1}{2})} = \tilde{\mathcal{X}}\times\bar{\mathcal{V}}\,, \tag{3.148}$$

$$\bar{\mathcal{X}}\times\mathcal{V} = 2|\alpha|^2\sum_{q=0}^{\infty}(h(q+2)+h(q)-\frac{1}{2})\mathcal{P}_{(\frac{q+1}{2},\frac{q}{2})} = \mathcal{V}\times\bar{\tilde{\mathcal{X}}}\,. \tag{3.149}$$

$\bar{\mathcal{X}}\times\mathcal{X}$, $\tilde{\mathcal{X}}\times\mathcal{X}$, $\tilde{\mathcal{X}}\times\bar{\tilde{\mathcal{X}}}$ and $\bar{\mathcal{X}}\times\bar{\tilde{\mathcal{X}}}$ (gauge contracted)

Here we find

$$\bar{\mathcal{X}}\times\mathcal{X} = 2|\alpha|^2\sum_{q=0}^{\infty}(h(q+1)+h(q)-1)\mathcal{P}_{(\frac{q}{2},\frac{q}{2})} = \tilde{\mathcal{X}}\times\bar{\tilde{\mathcal{X}}}\,, \tag{3.150}$$

$$\tilde{\mathcal{X}}\times\mathcal{X} = 2|\alpha|^2\sum_{q=-1}^{\infty}(2h(q+1)-1)\mathcal{P}_{(\frac{q+1}{2},\frac{q}{2})}\,, \tag{3.151}$$

$$\bar{\mathcal{X}}\times\bar{\tilde{\mathcal{X}}} = 2|\alpha|^2\sum_{q=-1}^{\infty}(2h(q+1)-1)\mathcal{P}_{(\frac{q}{2},\frac{q+1}{2})}\,. \tag{3.152}$$

$\mathcal{X}\times\bar{\mathcal{X}}$ (flavor contracted), $\bar{\tilde{\mathcal{X}}}\times\tilde{\mathcal{X}}$ (flavor contracted), $\mathcal{V}\times\bar{\mathcal{V}}$ and $\bar{\mathcal{V}}\times\mathcal{V}$.

This is the most involved subspace since we have mixing between different copies of the same superconformal multiplets. The Hamiltonian acting on this

[4] Some of the modules with low q are not present in the subsector so the corresponding coefficients are not determined by the algebraic constraints. However, these coefficients can be fixed by invoking CPT.

subspace is a 4×4 matrix,

$$
\begin{aligned}
H_{12} = 3|\alpha|^2 & \begin{array}{cc} & \begin{array}{cccc} \mathcal{X}\bar{\mathcal{X}} & \bar{\tilde{\mathcal{X}}}\tilde{\mathcal{X}} & \mathcal{V}\bar{\mathcal{V}} & \bar{\mathcal{V}}\mathcal{V} \end{array} \\ \begin{array}{c} \bar{\mathcal{X}}\mathcal{X} \\ \tilde{\mathcal{X}}\bar{\tilde{\mathcal{X}}} \\ \bar{\mathcal{V}}\mathcal{V} \\ \mathcal{V}\bar{\mathcal{V}} \end{array} & \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \mathcal{P}_{(0,0)} \\
+ 2|\alpha|^2 \sum_{q=1}^{\infty} & \begin{array}{cc} & \begin{array}{cccc} \mathcal{X}\bar{\mathcal{X}} & \bar{\tilde{\mathcal{X}}}\tilde{\mathcal{X}} & \mathcal{V}\bar{\mathcal{V}} & \bar{\mathcal{V}}\mathcal{V} \end{array} \\ \begin{array}{c} \bar{\mathcal{X}}\mathcal{X} \\ \tilde{\mathcal{X}}\bar{\tilde{\mathcal{X}}} \\ \bar{\mathcal{V}}\mathcal{V} \\ \mathcal{V}\bar{\mathcal{V}} \end{array} & \begin{pmatrix} 0 & 0 & \sqrt{3}e^{-i\theta} & -\sqrt{3}\frac{e^{-i\theta}}{q+1} \\ 0 & 0 & -\frac{\sqrt{3}e^{-i\theta}}{q+1} & \sqrt{3}e^{-i\theta} \\ \sqrt{3}\frac{e^{i\theta}}{q(q+2)} & -\frac{\sqrt{3}e^{i\theta}}{q(q+1)(q+2)} & h(q+2)+h(q-1) & \frac{2}{q(q+1)(q+2)} \\ -\frac{\sqrt{3}e^{i\theta}}{q(q+1)(q+2)} & \frac{\sqrt{3}e^{i\theta}}{q(q+2)} & \frac{2}{q(q+1)(q+2)} & h(q+2)+h(q-1) \end{pmatrix} \end{array} \mathcal{P}_{(\frac{q}{2},\frac{q}{2})} \,.
\end{aligned}
\tag{3.153}
$$

3.3.3 Scalar Sector

Let us compare our results with the scalar sector computation of [19]. Apart from providing a check of our procedure, this comparison allows us to fix the overall normalization of the Hamiltonian in terms of the gauge theory 't Hooft coupling. The action of the Hamiltonian for the gauge contracted Q and $\tilde{Q}$ pairs can be obtained from (3.150) at $q = 0$ and (3.151,3.152) at $q = -1$,

$$
H_{12} = 2|\alpha|^2 \begin{array}{cc} & \begin{array}{cccc} \bar{Q}Q & \bar{Q}\bar{\tilde{Q}} & \tilde{Q}Q & \tilde{Q}\bar{\tilde{Q}} \end{array} \\ \begin{array}{c} Q\bar{Q} \\ Q\tilde{Q} \\ \bar{\tilde{Q}}\bar{Q} \\ \bar{\tilde{Q}}\tilde{Q} \end{array} & \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix} \end{array} , \tag{3.154}
$$

in perfect agreement with equation (3.6) of [19] provided we identify $2|\alpha|^2 = \lambda$. For the flavor contracted pairs we obtain from the first matrix in (3.153),

$$H_{12} = 3|\alpha|^2 \left(\begin{array}{c|cc} & Q\bar{Q} & \bar{\tilde{Q}}\tilde{Q} \\ \hline \bar{Q}Q & 1 & -1 \\ \tilde{Q}\bar{\tilde{Q}} & -1 & 1 \end{array} \right), \tag{3.155}$$

again in agreement with [19]. It is interesting that the value of the transverse magnetic field for the Ising spin chain in the scalar sector, namely $h_{\text{Ising}} = N_f/N_c = 3$ [19], turns out to be determined by superconformal symmetry alone.

3.3.4 The $\mathcal{N} = 1$ Harmonic Action

In this section we present the explicit oscillator form of the Hamiltonian.

$\mathcal{V} \times \mathcal{V}$ and $\bar{\mathcal{V}} \times \bar{\mathcal{V}}$.

For states in these two subspaces the action of the Hamiltonian is identical with that of $\mathcal{N} = 4$ SYM and of $\mathcal{N} = 2$ SCQCD. General states in $\mathcal{V} \times \mathcal{V}$ and $\bar{\mathcal{V}} \times \bar{\mathcal{V}}$ can be written as

$$\begin{aligned} |s_1, ..., s_n; A\rangle_{\mathcal{V}\times\mathcal{V}} &= A^\dagger_{s_1 A_1} ... A^\dagger_{s_n A_n} |0\rangle \otimes |0\rangle \,, &(3.156)\\ |s_1, ..., s_n; A\rangle_{\bar{\mathcal{V}}\times\bar{\mathcal{V}}} &= A^\dagger_{s_1 A_1} ... A^\dagger_{s_n A_n} |\mathbf{d}_3\rangle \otimes |\mathbf{d}_3\rangle \,, &(3.157)\end{aligned}$$

where $A^\dagger_A = (\mathbf{a}^\dagger_\alpha, \mathbf{b}^\dagger_{\dot\alpha}, \mathbf{c}^\dagger)$ and $s_i = 1, 2$ indicates in which site the oscillator sits. The action of the Hamiltonian on this state does not change the number of oscillators but merely shifts them from site 1 to site 2 (or vice versa) in all possible combinations. This can be written as

$$\begin{aligned} H_{12}|s_1, ..., s_n; A\rangle_{\mathcal{V}\times\mathcal{V}} &= \sum_{s'_1,...,s'_n} c_{n,n_{12},n_{21}} \delta_{C_1,0}\delta_{C_2,0} |s'_1, ..., s'_n; A\rangle_{\mathcal{V}\times\mathcal{V}}, &(3.158)\\ H_{12}|s_1, ..., s_n; A\rangle_{\bar{\mathcal{V}}\times\bar{\mathcal{V}}} &= \sum_{s'_1,...,s'_n} c_{n,n_{12},n_{21}} \delta_{C_1,0}\delta_{C_2,0} |s'_1, ..., s'_n; A\rangle_{\bar{\mathcal{V}}\times\bar{\mathcal{V}}}, &(3.159)\end{aligned}$$

where the Kronecker deltas project onto states with zero central charge and n_{ij} counts the number of oscillators moving from site i to site j. The explicit formula for the function $c_{n,n_{12},n_{21}}$ is

$$c_{n,n_{12},n_{21}} = (-1)^{1+n_{12}n_{21}} \frac{\Gamma(\frac{1}{2}(n_{12}+n_{21}))\Gamma(1+\frac{1}{2}(n-n_{12}-n_{21}))}{\Gamma(1+\frac{1}{2}n)}\,, \tag{3.160}$$

with $c_{n,0,0} = h(\frac{n}{2})$. As before, this function is a superconformal invariant and has the appropriate eigenvalues when acting on the $\hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q-2}{2})}$ and $\hat{\mathcal{C}}_{(\frac{q-2}{2},\frac{q+1}{2})}$ modules, namely

$$\begin{aligned} H_{12}\hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q-2}{2})} &= 2h(q)\hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q-2}{2})}\,, & (3.161)\\ H_{12}\hat{\mathcal{C}}_{(\frac{q-2}{2},\frac{q+1}{2})} &= 2h(q)\hat{\mathcal{C}}_{(\frac{q-2}{2},\frac{q+1}{2})}\,. & (3.162)\end{aligned}$$

$\mathcal{V}\times\mathcal{X}$, $\tilde{\mathcal{X}}\times\mathcal{V}$, $\bar{\mathcal{X}}\times\bar{\mathcal{V}}$ and $\bar{\mathcal{V}}\times\bar{\tilde{\mathcal{X}}}$.

General states in these four subspaces can be written as

$$\begin{aligned} |s_1,...,s_n;A\rangle_{\mathcal{V}\times\mathcal{X}} &= A^\dagger_{s_1A_1}...A^\dagger_{s_nA_n}|0\rangle\otimes|\mathbf{d}_1\rangle\,, & (3.163)\\ |s_1,...,s_n;A\rangle_{\tilde{\mathcal{X}}\times\mathcal{V}} &= A^\dagger_{s_1A_1}...A^\dagger_{s_nA_n}|\tilde{\mathbf{d}}_1\rangle\otimes|0\rangle\,, & (3.164)\\ |s_1,...,s_n;A\rangle_{\bar{\mathcal{X}}\times\bar{\mathcal{V}}} &= A^\dagger_{s_1A_1}...A^\dagger_{s_nA_n}|\mathbf{d}_1\mathbf{d}_2\rangle\otimes|\mathbf{d}_3\rangle\,, & (3.165)\\ |s_1,...,s_n;A\rangle_{\bar{\mathcal{V}}\times\bar{\tilde{\mathcal{X}}}} &= A^\dagger_{s_1A_1}...A^\dagger_{s_nA_n}|\mathbf{d}_3\rangle\otimes|\tilde{\mathbf{d}}_1\tilde{\mathbf{d}}_2\rangle\,, & (3.166)\end{aligned}$$

where $|\mathbf{d}_i\rangle = \mathbf{d}_i^\dagger|0\rangle$ [5]. The action of H_{12} for all these four subspaces is given by [6]

$$H_{12}|s_1,...,s_n;A\rangle = \sum_{s'_1,...,s'_n} c_{n+1,n_{12},n_{21}}|s_1,...,s_n;A\rangle - \frac{1}{2}|s_1,...,s_n;A\rangle\,. \tag{3.167}$$

Invariance under the superconformal group is guaranteed because we are using the $c_{n,n_{12},n_{21}}$ constants. We need only to check that this expression has the correct eigenvalues when acting on the corresponding $\mathcal{N}=1$ primaries (see Appendix B.3), which can be easily done using *Mathematica*.

[5] The tilde in some of the $\mathbf{d}$ oscillators is just a reminder that we are looking at the $\tilde{\mathcal{X}}$ multiplet or its conjugate.

[6] To simplify the notation we will omit the Kronecker deltas $\delta_{C_1,0}\delta_{C_2,0}$.

$\bar{\mathcal{V}} \times \mathcal{X}$, $\tilde{\mathcal{X}} \times \bar{\mathcal{V}}$, $\bar{\mathcal{X}} \times \mathcal{V}$ and $\mathcal{V} \times \bar{\tilde{\mathcal{X}}}$.

In this case we have

$$\begin{aligned} |s_1, ..., s_n; A\rangle_{\bar{\mathcal{V}} \times \mathcal{X}} &= A^\dagger_{s_1 A_1} ... A^\dagger_{s_n A_n} |\mathbf{d}_3\rangle \otimes |\mathbf{d}_1\rangle \,, & (3.168) \\ |s_1, ..., s_n; A\rangle_{\tilde{\mathcal{X}} \times \bar{\mathcal{V}}} &= A^\dagger_{s_1 A_1} ... A^\dagger_{s_n A_n} |\tilde{\mathbf{d}}_1\rangle \otimes |\mathbf{d}_3\rangle \,, & (3.169) \\ |s_1, ..., s_n; A\rangle_{\bar{\mathcal{X}} \times \mathcal{V}} &= A^\dagger_{s_1 A_1} ... A^\dagger_{s_n A_n} |\mathbf{d}_1 \mathbf{d}_2\rangle \otimes |0\rangle \,, & (3.170) \\ |s_1, ..., s_n; A\rangle_{\mathcal{V} \times \bar{\tilde{\mathcal{X}}}} &= A^\dagger_{s_1 A_1} ... A^\dagger_{s_n A_n} |0\rangle \otimes |\tilde{\mathbf{d}}_1 \tilde{\mathbf{d}}_2\rangle \,, & (3.171) \end{aligned}$$

and the action of H_{12} reads

$$H_{12} |s_1, ..., s_n; A\rangle = \sum_{s'_1, ..., s'_n} c_{n+2, n_{12}, n_{21}} |s_1, ..., s_n; A\rangle - \frac{1}{2} |s_1, ..., s_n; A\rangle \,. \quad (3.172)$$

$\bar{\mathcal{X}} \times \mathcal{X}$ and $\tilde{\mathcal{X}} \times \bar{\tilde{\mathcal{X}}}$ (gauge contracted).

The states are given by

$$\begin{aligned} |s_1, ..., s_n; A\rangle_{\bar{\mathcal{X}} \times \mathcal{X}} &= A^\dagger_{s_1 A_1} ... A^\dagger_{s_n A_n} |\mathbf{d}_1 \mathbf{d}_2\rangle \otimes |\mathbf{d}_1\rangle \,, & (3.173) \\ |s_1, ..., s_n; A\rangle_{\tilde{\mathcal{X}} \times \bar{\tilde{\mathcal{X}}}} &= A^\dagger_{s_1 A_1} ... A^\dagger_{s_n A_n} |\tilde{\mathbf{d}}_1\rangle \otimes |\tilde{\mathbf{d}}_1 \tilde{\mathbf{d}}_2\rangle \,, & (3.174) \end{aligned}$$

and the action of H_{12} is

$$H_{12} |s_1, ..., s_n; A\rangle = \sum_{s'_1, ..., s'_n} c_{n+1, n_{12}, n_{21}} |s_1, ..., s_n; A\rangle - |s_1, ..., s_n; A\rangle \,. \quad (3.175)$$

$\tilde{\mathcal{X}} \times \mathcal{X}$ and $\bar{\mathcal{X}} \times \bar{\tilde{\mathcal{X}}}$.

The states are given by

$$\begin{aligned} |s_1, ..., s_n; A\rangle_{\tilde{\mathcal{X}} \times \mathcal{X}} &= A^\dagger_{s_1 A_1} ... A^\dagger_{s_n A_n} |\tilde{\mathbf{d}}_1\rangle \otimes |\mathbf{d}_1\rangle \,, & (3.176) \\ |s_1, ..., s_n; A\rangle_{\bar{\mathcal{X}} \times \bar{\tilde{\mathcal{X}}}} &= A^\dagger_{s_1 A_1} ... A^\dagger_{s_n A_n} |\mathbf{d}_1 \mathbf{d}_2\rangle \otimes |\tilde{\mathbf{d}}_1 \tilde{\mathbf{d}}_2\rangle \,, & (3.177) \end{aligned}$$

and the action of H_{12} is

$$H_{12} |s_1, ..., s_n; A\rangle = \sum_{s'_1, ..., s'_n} \left(c_{n+2, n_{12}, n_{21}} - c_{n+2, n_{12}+1, n_{21}+1} \right) |s_1, ..., s_n; A\rangle - |s_1, ..., s_n; A\rangle \,. \quad (3.178)$$

$\mathcal{X}\times\bar{\mathcal{X}}$ (flavor contracted), $\bar{\tilde{\mathcal{X}}}\times\tilde{\mathcal{X}}$ (flavor contracted), $\mathcal{V}\times\bar{\mathcal{V}}$ and $\bar{\mathcal{V}}\times\mathcal{V}$.

The states are

$$
\begin{aligned}
|s_1,...,s_n;A\rangle_{\mathcal{X}\times\bar{\mathcal{X}}} &= A^\dagger_{s_1A_1}...A^\dagger_{s_nA_n}|\mathbf{d}_1\rangle\otimes|\mathbf{d}_1\mathbf{d}_2\rangle\,, &(3.179)\\
|s_1,...,s_n;A\rangle_{\bar{\tilde{\mathcal{X}}}\times\tilde{\mathcal{X}}} &= A^\dagger_{s_1A_1}...A^\dagger_{s_nA_n}|\tilde{\mathbf{d}}_1\tilde{\mathbf{d}}_2\rangle\otimes|\tilde{\mathbf{d}}_1\rangle\,, &(3.180)\\
|s_1,...,s_n;A\rangle_{\mathcal{V}\times\bar{\mathcal{V}}} &= A^\dagger_{s_1A_1}...A^\dagger_{s_nA_n}|0\rangle\otimes|\mathbf{d}_3\rangle\,, &(3.181)\\
|s_1,...,s_n;A\rangle_{\bar{\mathcal{V}}\times\mathcal{V}} &= A^\dagger_{s_1A_1}...A^\dagger_{s_nA_n}|\mathbf{d}_3\rangle\otimes|0\rangle\,. &(3.182)
\end{aligned}
$$

The action of H_{12} is given by

$$
\begin{aligned}
H_{12}|s_1,...,s_n;A\rangle_{\mathcal{X}\times\bar{\mathcal{X}}} &= \sum_{s'_1,...,s'_n} c_{n+3,n_{12}+1,n_{21}+1}|s_1,...,s_n;A\rangle_{\mathcal{X}\times\bar{\mathcal{X}}} &(3.183)\\
&+3\sum_{s'_1,...,s'_n} c_{n+3,n_{12}+1,n_{21}+2}|s_1,...,s_n;A\rangle_{\bar{\tilde{\mathcal{X}}}\times\tilde{\mathcal{X}}}\\
&+\sqrt{3}e^{-i\theta}\sum_{s'_1,...,s'_n} c_{n+3,n_{12},n_{21}+1}|s_1,...,s_n;A\rangle_{\mathcal{V}\times\bar{\mathcal{V}}}\\
&+\sqrt{3}e^{-i\theta}\sum_{s'_1,...,s'_n} c_{n+3,n_{12}+1,n_{21}+1}|s_1,...,s_n;A\rangle_{\bar{\mathcal{V}}\times\mathcal{V}}\,,
\end{aligned}
$$

$$
\begin{aligned}
H_{12}|s_1,...,s_n;A\rangle_{\bar{\tilde{\mathcal{X}}}\times\tilde{\mathcal{X}}} &= \sum_{s'_1,...,s'_n} c_{n+3,n_{12}+1,n_{21}+1}|s_1,...,s_n;A\rangle_{\bar{\tilde{\mathcal{X}}}\times\tilde{\mathcal{X}}} &(3.184)\\
&-3\sum_{s'_1,...,s'_n} c_{n+3,n_{12}+1,n_{21}+2}|s_1,...,s_n;A\rangle_{\mathcal{X}\times\bar{\mathcal{X}}}\\
&+\sqrt{3}e^{-i\theta}\sum_{s'_1,...,s'_n} c_{n+3,n_{12}+1,n_{21}+1}|s_1,...,s_n;A\rangle_{\mathcal{V}\times\bar{\mathcal{V}}}\\
&+\sqrt{3}e^{-i\theta}\sum_{s'_1,...,s'_n} c_{n+3,n_{12}+1,n_{21}}|s_1,...,s_n;A\rangle_{\bar{\mathcal{V}}\times\mathcal{V}}\,,
\end{aligned}
$$

$$
\begin{aligned}
H_{12}|s_1, ..., s_n; A\rangle_{\mathcal{V}\times\bar{\mathcal{V}}} \;=\; & \sum_{s'_1,...,s'_n} c_{n+3,n_{12},n_{21}}|s_1, ..., s_n; A\rangle_{\mathcal{V}\times\bar{\mathcal{V}}} \\
& + \sum_{s'_1,...,s'_n} c_{n+3,n_{12}+2,n_{21}+1}|s_1, ..., s_n; A\rangle_{\bar{\mathcal{V}}\times\mathcal{V}} \\
& + \sqrt{3}e^{i\theta} \sum_{s'_1,...,s'_n} c_{n+3,n_{12}+1,n_{21}}|s_1, ..., s_n; A\rangle_{\mathcal{X}\times\bar{\mathcal{X}}} \\
& - \sqrt{3}e^{i\theta} \sum_{s'_1,...,s'_n} c_{n+3,n_{12},n_{21}+2}|s_1, ..., s_n; A\rangle_{\bar{\tilde{\mathcal{X}}}\times\tilde{\mathcal{X}}} \,,
\end{aligned} \tag{3.185}
$$

and

$$
\begin{aligned}
H_{12}|s_1, ..., s_n; A\rangle_{\bar{\mathcal{V}}\times\mathcal{V}} \;=\; & \sum_{s'_1,...,s'_n} c_{n+3,n_{12},n_{21}}|s_1, ..., s_n; A\rangle_{\bar{\mathcal{V}}\times\mathcal{V}} \\
& - \sum_{s'_1,...,s'_n} c_{n+3,n_{12}+2,n_{21}+1}|s_1, ..., s_n; A\rangle_{\mathcal{V}\times\bar{\mathcal{V}}} \\
& - \sqrt{3}e^{i\theta} \sum_{s'_1,...,s'_n} c_{n+3,n_{12}+2,n_{21}}|s_1, ..., s_n; A\rangle_{\mathcal{X}\times\bar{\mathcal{X}}} \\
& + \sqrt{3}e^{i\theta} \sum_{s'_1,...,s'_n} c_{n+3,n_{12},n_{21}+1}|s_1, ..., s_n; A\rangle_{\bar{\tilde{\mathcal{X}}}\times\tilde{\mathcal{X}}} \,.
\end{aligned} \tag{3.186}
$$

Chapter 4

Integrability Analysis

In the first part of this chapter we will use the complete one-loop Hamiltonians obtained above for a preliminary analysis of the integrability properties of the $\mathcal{N}=2$ and $\mathcal{N}=1$ spin chains. The analysis consists in the systematic search of "parity pairs". These pairs correspond to degeneracies in the spectrum of the theory and they are associated with extra conserved charges, one of the consequences of integrability. In the famous $\mathcal{N}=4$ SYM example, the presence of parity pairs was an early hint for the full itegrability of the theory. The results of our search are inconclusive, but they seem to indicate that the $\mathcal{N}=2$ and $\mathcal{N}=1$ theories do not retain the integrability properties present in $\mathcal{N}=4$ SYM. In the second part we make this conclusion stronger for the case of $\mathcal{N}=2$ SCQCD, by studying the Hamiltonian at higher loops in a special subsector with $SU(2|1)$ symmetry. We conclude the chapter with some comments on the $SU(2,1|2)$ sector of the theory, and suggest that it may be integrable to all loops.

4.1 Spectral analysis

Spectral studies in planar $\mathcal{N}=4$ SYM have shown the systematic presence of degenerate pairs of states of opposite "parity", where parity is the $\mathbb{Z}_2$ symmetry associated with complex conjugation of the $SU(N)$ gauge group [17, 24–26]. These degeneracies persist at higher loops, but are lifted by non-planar corrections. This phenomenon is naturally explained by the integrable structures of planar $\mathcal{N}=4$ SYM: the theory admits higher conserved charges that are parity-odd and map the degenerate eigenstates. In some models it is even possible to prove that the existence of parity pairs is a sufficient condition for integrability [17].

The upshot is that in $\mathcal{N}=4$ SYM the existence of parity pairs is one of the many pieces of evidence for the complete integrability of the theory. With this precedent in mind, we can look forward to a similar analysis in $\mathcal{N}=2$ SCQCD and in $\mathcal{N}=1$ SQCD. In this section we determine the low-lying spectrum of the one-loop dilation operator of both theories, in the closed non-compact subsectors that were used to uplift the full Hamiltonian.

4.1.1 $\mathcal{N}=2$ SCQCD

We start our analysis with $\mathcal{N}=2$ SCQCD and with the more general quiver theory that interpolates between the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM and $\mathcal{N}=2$ SCQCD.

Parity

The first thing we need to do is define a meaningful parity operation. We will take $\mathcal{N}=4$ SYM as our starting point where parity amounts to conjugation of the $SU(N)$ gauge group. Under parity, the Lie algebra generators transform as

$$T^a{}_b \to -(T^a{}_b)^* = -T^b{}_a\,, \tag{4.1}$$

where we have used hermiticity to trade conjugation by transposition.

Now, as reviewed in Chapter 2, $\mathcal{N}=2$ SCQCD can be thought of as a limit of a two-parameter $(g,\check{g})$ quiver theory with gauge group $SU(N_c)\times SU(N_{\check{c}})$ (with $N_{\check{c}}\equiv N_c$): one has $\mathcal{N}=2$ SCQCD at $\check{g}=0$ and the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM at $g=\check{g}$. Starting from $\mathcal{N}=4$ SYM with gauge group $SU(2N_c)$ the $\mathbb{Z}_2$ orbifold theory is obtained by the projection

$$A_{\alpha\dot{\alpha}} = \begin{pmatrix} A^a_{\alpha\dot{\alpha}\,b} & 0 \\ 0 & \check{A}^{\check{a}}_{\alpha\dot{\alpha}\,\check{b}} \end{pmatrix}, \quad Z = \begin{pmatrix} \phi^a{}_b & 0 \\ 0 & \check{\phi}^{\check{a}}{}_{\check{b}} \end{pmatrix}, \tag{4.2}$$

$$\lambda_{\mathcal{I}} = \begin{pmatrix} \lambda^a_{\mathcal{I}b} & 0 \\ 0 & \check{\lambda}^{\check{a}}_{\mathcal{I}\check{b}} \end{pmatrix}, \quad \lambda_{\hat{\mathcal{I}}} = \begin{pmatrix} 0 & \epsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}\psi^{a\hat{\mathcal{J}}}{}_{\check{a}} \\ \epsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}\tilde{\psi}^{\check{b}\hat{\mathcal{J}}}{}_b & 0 \end{pmatrix}, \tag{4.3}$$

$$\mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}} = \begin{pmatrix} 0 & Q_{\mathcal{I}\hat{\mathcal{I}}\check{a}}{}^{a} \\ -\epsilon_{\mathcal{I}\mathcal{J}}\epsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}\bar{Q}^{\check{b}\hat{\mathcal{J}}\mathcal{J}}{}_b & 0 \end{pmatrix}, \tag{4.4}$$

where $\mathcal{I},\hat{\mathcal{I}}=1,2$. The parity operation described above implies the following transformations. For the fields in the vector multiplets,

$$
\begin{aligned}
&A^{a}_{\alpha\dot\alpha\, b} \leftrightarrow -A^{b}_{\alpha\dot\alpha\, a} \quad \lambda^{a}_{\mathcal{I}b} \leftrightarrow -\lambda^{b}_{\mathcal{I}a} \quad \phi^{a}{}_{b} \leftrightarrow -\phi^{b}{}_{a}\,,\\
&\check A^{\check a}_{\alpha\dot\alpha\, \check b} \leftrightarrow -\check A^{\check b}_{\alpha\dot\alpha\, \check a} \quad \check\lambda^{\check a}_{\mathcal{I}\check b} \leftrightarrow -\check\lambda^{\check b}_{\mathcal{I}\check a} \quad \check\phi^{\check a}{}_{\check b} \leftrightarrow -\check\phi^{\check b}{}_{\check a}\,,
\end{aligned}
\tag{4.5}
$$

and analogous expressions for the conjugate fields. For the fields in the hypermultiplets,

$$
\psi^{a}_{\hat{\mathcal{I}}\check b} \leftrightarrow -\tilde\psi^{\check b}_{\hat{\mathcal{I}}a} \quad Q^{a}_{\mathcal{I}\hat{\mathcal{I}}\check b} \leftrightarrow \bar Q^{\check b}_{\mathcal{I}\hat{\mathcal{I}}a}\,,
\tag{4.6}
$$

and analogous expressions for the conjugate fields. These transformations remain a symmetry also away from the orbifold point (that is, for arbitrary $(g, \check g)$), as can be easily checked by inspection of the Lagrangian (see *e.g.* [13] for the explicit expression of the Lagrangian). This implies that the parity operation commutes with the dilation operator to all loops. Its action on single-trace states is then given by

$$
P|A_1 \dots A_L\rangle = (-1)^{L+k(k+1)/2}|A_L \dots A_1\rangle\,,
\tag{4.7}
$$

where k is the number of fermions and we replace $\psi \leftrightarrow \tilde\psi$, $\bar\psi \leftrightarrow \bar{\tilde{\psi}}$, $Q \leftrightarrow -\bar Q$.

Diagonalization

We consider the $SU(1,1)\times SU(1|1)\times SU(1|1)\times U(1)$ used to uplift the Hamiltonian. We focus on $SU(1,1)$ primaries (since descendants have the same anomalous dimensions) and take with no loss of generality $r \geq 0$. Table 4.1 corresponds to states with maximal r-charge. This subsector is made exclusively out of $\{\mathcal{D}^k\bar\lambda\}$ and is therefore identical to the $SU(1,1)$ subsector used in [17] to obtain the complete dilation operator of $\mathcal{N}=4$ SYM. Being a subsector of $\mathcal{N}=4$, it is integrable and, indeed, our results indicate that degenerate states with opposite parity show up consistently at each stage of the diagonalization. The notation $P_n(x)$ denotes the roots of a polynomial of order n, we will not write the polynomial explicitly because we are really interested in the amount of parity pairs and not in the actual values of the energies. We will denote by $P_n(x)$ all the roots of polynomials of order n we encounter, even if they are different from each other.

Being identical to the analogous $\mathcal{N}=4$ SYM sector, we cannot use the results of table 4.1 as a test for integrability. The true dynamics of $\mathcal{N}=2$ SCQCD is encoded in subspaces where the r-charge is not maximal. For this we need states with Q and $\bar Q$. Our results are presented in table 4.2. As opposed to the results of table 4.1 the presence of parity pairs here is less

L	r	Δ_0	$\delta\Delta^P$ $[2g_{YM}^2 N/\pi^2]$
3	$\frac{3}{2}$	7.5	$\frac{5}{4}^{\pm}$
		9.5	$\frac{133}{96}^{\pm}$
		10.5	$\frac{761}{480}^{\pm}$, $\frac{761}{560}^{-}$
		11.5	$\frac{179}{120}^{\pm}$
4	2	8	$\frac{5}{4}^{\pm}$
		9	$\frac{1}{48}(73 \pm \sqrt{37})^{-}$
		10	$\frac{19}{12}^{\pm}$, $\frac{133}{96}^{\pm}$
		11	$P_3(x)^{-}$, $\frac{761}{480}^{\pm}$
5	$\frac{5}{2}$	9.5	$\frac{1}{48}(73 \pm \sqrt{37})^{+}$
		10.5	$\frac{7}{4}^{\pm}$, $\frac{19}{12}^{\pm}$
		11.5	$P_3(x)^{-}$, $\frac{1}{24}(43 \pm \sqrt{5})^{\pm}$
Paired eigenvalues: ~ 69 %			

Table 4.1: $SU(1,1)$ primaries with maximal r-charge ($r = \frac{L}{2}$) in the $SU(1,1) \times SU(1|1) \times SU(1|1) \times U(1)$ sector of $\mathcal{N} = 2$ SCQCD. We have omitted the one-dimensional subspaces where there is no room for a parity pair.

systematic.

More insight is obtained if we also look at the $\mathbb{Z}_2$ orbifold ($\check{g} = g$). For the the orbifold theory (and for the whole interpolating theory with general $\check{g}$, g) we have an $SU(2)_L$ symmetry not present in $\mathcal{N} = 2$ SCQCD, so to make the analysis more transparent we restrict the diagonalization to $SU(2)_L$ singlets. Our results for the $\mathbb{Z}_2$ orbifold are shown in table 4.3. As in the case with maximal r-charge, parity pairs show up consistently. This is again expected because this theory is known to be integrable [15].

Finally we look at how some sample parity pairs of the orbifold theory evolve when we move away from the orbifold point. Our results are shown in table 4.4. We see that for arbitrary values of $\kappa \equiv \check{g}/g$ the pairs are lifted and they are not in general recovered in the SCQCD limit $\kappa \to 0$. (Note that not all $SU(2)_L$ gauge singlets evolve to legitimate states of $\mathcal{N} = 2$ SCQCD, which must obey the stronger condition of being $SU(N_f)$ singlets. In the last column of table 4.4 we indicate whether the states belong or not to $\mathcal{N} = 2$ SCQCD.)

L	r	Δ_0	$\delta\Delta^P\,[2g_{YM}^2 N/\pi^2]$
3	$\frac{1}{2}$	4.5	$\frac{3}{4}^-,\ \frac{3}{4}^-,\ \frac{3}{8}^+$
		5.5	$\frac{15}{16}^-,\ \frac{1}{24}(16\pm\sqrt{31})^-,\ \frac{1}{32}(21\pm\sqrt{57})^+$
		6.5	$\frac{25}{24}^-,\ \frac{25}{24}^-,\ \frac{25}{48}^+$
			$\frac{1}{96}(81\pm\sqrt{561})^-,\ \frac{1}{96}(83\pm\sqrt{409})^+$
4	0	5	$\frac{3}{4}^\pm,\ \frac{3}{8}^-$
		6	$1^\pm,\ \frac{15}{16}^+,\ \frac{15}{16}^+$
			$\frac{1}{32}(21\pm\sqrt{57})^-,\ \frac{1}{32}(21\pm\sqrt{57})^-$
			$\frac{1}{24}(16\pm\sqrt{31})^+,\ \frac{5}{8},\ 0^+$
	1	6	$1^-,\ \frac{15}{16}^+,\ \frac{1}{32}(21\pm\sqrt{57})^-$
		7	$\frac{5}{4}^\pm,\ \frac{9}{8}^-,\ \frac{25}{24}^+$
			$\frac{1}{16}(16\pm\sqrt{6})^+,\ \frac{1}{96}(81\pm\sqrt{561})^+$
			$\frac{1}{96}(83\pm\sqrt{409})^-$
5	$\frac{1}{2}$	6.5	$1^\pm,\ 1^+,\ \frac{15}{16}^-$
			$\frac{1}{32}(21\pm\sqrt{57})^+$
Paired eigenvalues: 16 %			

Table 4.2: $SU(1,1)$ primaries with $0 \leq r < \frac{L}{2}$ in the $SU(1,1)\times SU(1|1)\times SU(1|1)\times U(1)$ sector of $\mathcal{N}=2$ SCQCD.

4.1.2 $\mathcal{N}=1$ SQCD

We now repeat the same analysis for $\mathcal{N}=1$ SQCD. Inspired by the transformation used in the $\mathcal{N}=2$ theory, we define the following parity operation. For the fields in the vector multiplet,

$$A^a_{\mu b} \leftrightarrow -A^b_{\mu a} \quad \lambda^a{}_b \leftrightarrow -\lambda^b{}_a\,, \tag{4.8}$$

and analogous expressions for the conjugate fields. For the chiral multipets,

$$Q^{ai} \leftrightarrow -\tilde{Q}_{\tilde{\imath}a}, \quad \psi^{ai} \leftrightarrow -\tilde{\psi}_{\tilde{\imath}a}\,, \tag{4.9}$$

L	r	Δ_0	$\delta\Delta^P\ [2g_{YM}^2 N/\pi^2]$
3	$\frac{1}{2}$	4.5	$\frac{1}{2}^{+}, \frac{3}{4}^{-}, \frac{3}{4}^{-}, \frac{3}{4}^{-}$
			$\frac{3}{4}^{\pm}$
		5.5	$\frac{3}{4}^{-}, \frac{7}{8}^{-}, \frac{25}{24}^{-}, \frac{1}{2}^{+}$
			$\frac{15}{16}^{\pm}, \frac{1}{32}(27 \pm \sqrt{57})^{\pm}$
		6.5	$\frac{3}{4}^{+}, \frac{25}{24}^{-}, \frac{25}{24}^{-}, \frac{25}{24}^{-}$
			$\frac{5}{4}^{\pm}, \frac{15}{16}^{\pm}, \frac{25}{24}^{\pm}$
			$\frac{1}{96}(93 \pm \sqrt{249})^{\pm}$
4	0	5	$\frac{1}{2}^{-}, \frac{1}{2}^{-}, \frac{1}{8}(5 \pm \sqrt{13})^{-}$
			$\frac{3}{4}^{\pm}, \frac{3}{4}^{\pm}$
		6	$\frac{7}{8}^{+}, \frac{25}{24}^{+}, \frac{1}{2}^{-}, \frac{1}{2}^{-}$
			$\frac{1}{8}(5 \pm \sqrt{5})^{+}$
			$\frac{3}{4}^{\pm}, \frac{5}{4}^{\pm}, \frac{5}{8}^{\pm}, \frac{7}{8}^{\pm}$
			$\frac{15}{16}^{\pm}, \frac{15}{16}^{\pm}, \frac{1}{4}(3 \pm \sqrt{2})^{\pm}$
			$\frac{1}{32}(27 \pm \sqrt{57})^{\pm}, \frac{1}{32}(27 \pm \sqrt{57})^{\pm}$
Paired eigenvalues: ~ 68 %			

Table 4.3: $SU(1,1)$ primaries with $0 \leq r < \frac{L}{2}$ in the $SU(1,1) \times SU(1|1) \times SU(1|1) \times U(1)$ sector of the orbifold theory ($\breve{g} = g$). We have restricted the diagonalization to $SU(2)_L$ singlets.

and analogous expressions for the conjugate fields. Again, these transformations are a symmetry of the Lagrangian and therefore commute with the dilation operator to all loops. The action on single-trace states is given by

$$P|A_1 \dots A_L\rangle = (-1)^{L+k(k+1)/2}|A_L \dots A_1\rangle \,, \tag{4.10}$$

where k is the number of fermions and we make the replacements $\psi \leftrightarrow \tilde{\psi}$, $\bar{\psi} \leftrightarrow \bar{\tilde{\psi}}$, $Q \leftrightarrow \tilde{Q}$, $\bar{Q} \leftrightarrow \bar{\tilde{Q}}$. Our results for the diagonalization of generalized single-trace operators of length $L \leq 5$ are shown in table 4.5. We restrict to states with r-charge $0 < r < L$. (We omit the sectors with $r = L$ and $r = 0$, which are spanned respectively by $\{\lambda_k\}$ and $\{\bar{\mathcal{F}}_k\}$. These sectors are

L	r	Δ_0	$\kappa = 1$	$\kappa = 0.7$	$\kappa = 0.3$	$\kappa = 0$	SCQCD
3	$\frac{1}{2}$	5.5	$\frac{1}{32}(27+\sqrt{57})$	0.97	0.94	$\frac{15}{16}$	Yes
			$\frac{1}{32}(27+\sqrt{57})$	0.95	0.90	$\frac{1}{32}(27+\sqrt{57})$	Yes
3	$\frac{1}{2}$	5.5	$\frac{1}{32}(27-\sqrt{57})$	0.39	0.08	0	No
			$\frac{1}{32}(27-\sqrt{57})$	0.47	0.30	$\frac{1}{4}$	No
3	$\frac{1}{2}$	6.5	$\frac{5}{4}$	1.12	1.09	$\frac{1}{96}(81+\sqrt{561})$	Yes
			$\frac{5}{4}$	1.11	1.08	$\frac{1}{96}(83+\sqrt{409})$	Yes
3	$\frac{1}{2}$	6.5	$\frac{1}{96}(93+\sqrt{249})$	0.81	0.63	$\frac{1}{96}(81-\sqrt{561})$	Yes
			$\frac{1}{96}(93+\sqrt{249})$	0.82	0.68	$\frac{1}{96}(83-\sqrt{409})$	Yes
3	$\frac{1}{2}$	6.5	$\frac{1}{96}(93-\sqrt{249})$	0.48	0.09	0	No
			$\frac{1}{96}(93-\sqrt{249})$	0.57	0.31	$\frac{1}{4}$	No
4	0	6	$\frac{5}{4}$	1.08	1.01	1	Yes
			$\frac{5}{4}$	1.06	1.01	1	Yes
4	0	6	$\frac{5}{8}$	0.45	0.29	$\frac{1}{4}$	No
			$\frac{5}{8}$	0.49	0.28	$\frac{1}{4}$	No
4	0	6	$\frac{1}{4}(3+\sqrt{2})$	0.74	0.54	$\frac{1}{2}$	No
			$\frac{1}{4}(3+\sqrt{2})$	0.81	0.66	$\frac{5}{8}$	Yes
4	0	6	$\frac{1}{4}(3-\sqrt{2})$	0.26	0.06	0	No
			$\frac{1}{4}(3-\sqrt{2})$	0.30	0.08	0	No

Table 4.4: Examples of evolution of $\mathbb{Z}_2$ orbifold pairs for different values of the parameter $\kappa = \frac{\check{g}}{g}$.

isomorphic to the analogous sectors in $\mathcal{N} = 4$ SYM and thus inherit their integrability.)

The results are qualitatively similar to the ones for $\mathcal{N} = 2$ SCQCD: there are a few parity pairs, but their presence is not as striking and systematic as in $\mathcal{N} = 4$ SYM.

L	r	Δ_0	$\delta\Delta^P\ [g_{YM}^2 N/\pi^2]$
3	1	4.5	$\frac{3}{4}^-$, $\frac{3}{8}^+$
		5.5	$\frac{1}{96}(67 \pm \sqrt{457})^-$, $P_3(x)^+$
		6.5	$\frac{25}{24}^-$, $\frac{1}{96}(81 \pm \sqrt{473})^-$, $P_3(x)^+$
			$\frac{25}{48}^\pm$
	2	4	$\frac{3}{16}^-$, $\frac{9}{16}^+$
		5	$\frac{7}{16}^-$, $\frac{7}{24}^-$, $\frac{13}{16}^+$, $\frac{1}{24}(13 \pm \sqrt{39})^+$
4	1	6.5	$\frac{3}{4}^+$, $\frac{9}{8}^-$
		7.5	$P_3(x)^-$, $P_4(x)^+$
	2	6	$\frac{1}{16}(9 \pm \sqrt{37})^+$, $\frac{1}{96}(67 \pm \sqrt{457})^+$, $P_3(x)^-$
			$\frac{9}{16}^\pm$, $0^\pm$
		7	$\frac{1}{96}(81 \pm \sqrt{473})^+$, $P_3(x)^-$, $P_7(x)^+$, $P_8(x)^-$
			$\frac{25}{48}^\pm$
	3	5.5	$\frac{7}{16}^-$, $\frac{13}{16}^+$
		6.5	$\frac{11}{16}^+$, $\frac{17}{16}^-$, $\frac{1}{96}(71 \pm \sqrt{553})^+$, $P_3(x)^-$
5	3	6.5	$\frac{1}{16}(9 \pm \sqrt{37})^-$, $\frac{9}{16}^\pm$
Paired eigenvalues: ~ 13 %			

Table 4.5: $SU(1,1)$ primaries with $0 < r < L$ in the $SU(1,1) \times U(1|1)$ sector of $\mathcal{N} = 1$ SQCD.

4.2 The $SU(2|1)$ sector of $\mathcal{N} = 2$ SCQCD

The results of the spectral analysis seem to indicate that the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ theories are not fully integrable. They do have some integrable subsectors but these are isomorphic to the respective sectors in $\mathcal{N} = 4$ SYM, and therefore trivially integrable. In this section we will study an $SU(2|1)$ subsector of $\mathcal{N} = 2$ SCQCD that has no analog in $\mathcal{N} = 4$ SYM. By studying the scattering of magnons and the Yang-Baxter equation we will prove that the $SU(2|1)$ sector is *not* integrable.

Let us start with a symmetry analysis of the $\mathcal{N} = 2$ superconformal group.

It has the following Lorentz and R-symmetry subgroups: $SU(2_\alpha) \times SU(2_{\dot\alpha}) \times SU(2_{\mathcal{I}}) \times U(1)_R \subset SU(2_\alpha, 2_{\dot\alpha}|2_{\mathcal{I}})$. The spin chain vacuum is the chiral state $\mathrm{Tr}\,\phi^k$. It breaks the superconformal group to the subgroup $PSU(2_{\dot\alpha}|2_{\mathcal{I}}) \times SU(2_\alpha) \ltimes \mathbb{R}$, where $\mathbb{R}$ is a central generator that gets identified with the spin chain Hamiltonian. In accordance with Goldstone's theorem, broken symmetry generators are manifested as gapless excitations of the spin chain called magnons. Table 4.6 shows the symmetry generators of the $\mathcal{N} = 2$ superconformal algebra. The diagonal boxed generators correspond to the symmetry preserved by the vacuum while the off-diagonal ones are broken and correspond to Goldstone magnons, which transform in the bifundamental representation of $PSU(2_{\dot\alpha}|2_{\mathcal{I}}) \times SU(2_\alpha)$.

	$SU(2_{\dot\beta})$	$SU(2_{\mathcal{J}})$	$SU(2_\beta)$
$SU(2_{\dot\alpha})$	$\dot{\mathcal{L}}_{\dot\alpha}{}^{\dot\beta}$	$\bar{\mathcal{Q}}_{\mathcal{J}\,\dot\alpha}$	$\mathcal{D}^\dagger_{\beta\dot\alpha}$
$SU(2_{\mathcal{I}})$	$\bar{\mathcal{S}}^{\mathcal{I}\,\dot\beta}$	$\mathcal{R}_{\mathcal{J}}{}^{\mathcal{I}}$	$\lambda^{\dagger\,\mathcal{I}}_{\beta}$
$SU(2_\alpha)$	$\mathcal{D}^{\alpha\dot\beta}$	$\lambda_{\mathcal{J}}{}^{\alpha}$	$\mathcal{L}_{\beta}{}^{\alpha}$

Table 4.6: The $\mathcal{N} = 2$ superconformal generators. The boxed generators are preserved by the choice of the spin chain vacuum while the unboxed ones are broken and correspond to Goldstone excitations. The broken generators are identified with the corresponding magnon: the upper-right column contains magnon creation operators while the lower-left row contains magnon annihilation operators.

A priori, the two-body magnon S-matrix when decomposed according to $SU(2_{\dot\alpha}|2_{\mathcal{I}}) \times SU(2_\alpha)$ quantum numbers will take the form

$$S_{SU(2_{\dot\alpha},2_\alpha|2_{\mathcal{I}})} = S_{SU(2_{\dot\alpha}|2_{\mathcal{I}})} \times S^{\mathbf{1}}_{SU(2_\alpha)} + S'_{SU(2_{\dot\alpha}|2_{\mathcal{I}})} \times S^{\mathbf{3}}_{SU(2_\alpha)}\,, \tag{4.11}$$

where the superscripts **1** and **3** denote the singlet and triplet $SU(2_\alpha)$ representations. Remarkably, the product of two fundamental $SU(2|2)$ representations consists of a single irreducible representation, which implies that the $SU(2|2)$ two-body S-matrix is completely fixed by symmetry, up to an overall phase [7]. Thus, the total two-body S-matrix of our model factorizes as

$$S_{SU(2_\alpha,2_{\dot\alpha}|2_{\mathcal{I}})} = S_{SU(2_{\dot\alpha}|2_{\mathcal{I}})} \times S_{SU(2_\alpha)}\,. \tag{4.12}$$

The $S_{SU(2_{\dot\alpha}|2_{\mathcal{I}})}$ factor is the two-body S-matrix of the magnons in the $SU(2_\alpha)$ highest weight state, namely $\{\,\lambda_+{}^{\mathcal{I}}, \mathcal{D}_{+\dot\alpha}\,\}$, while $S_{SU(2_\alpha)}$ is the two-body S-matrix of the magnons in the $SU(2_{\dot{\alpha}}|2_{\mathcal{I}})$ highest weight state, namely $\{\lambda_\alpha{}^+\}$.

We can identify two "orthogonal" all-order closed subsectors, associated with either factor of the two-body S-matrix. Exciting an arbitrary number of $SU(2_\alpha)$ highest weight magnons $\{\, \lambda_+{}^{\mathcal{I}}, \mathcal{D}_{+\dot{\alpha}}\}$ above the spin chain vacuum $\mathrm{Tr}\,\phi^k$, and demanding closure of the dilation operator, we obtain a subsector with enhanced $SU(2,1|2)$ symmetry, spanned by the following letters:

$$SU(2,1|2) \text{ sector:} \qquad (\mathcal{D}_{+\dot{\alpha}})^n\{\, \phi, \lambda_+{}^{\mathcal{I}}, \mathcal{F}_{++}\,\}\,. \tag{4.13}$$

Here the covariant derivatives are understood to be totally symmetrized at each site, so for example $(\mathcal{D}_{+\dot{\alpha}})^n\phi$ is shorthand for $\mathcal{D}_{+\{\dot{\alpha}_1}\mathcal{D}_{+\dot{\alpha}_2}\ldots\mathcal{D}_{+\dot{\alpha}_n\}}\phi$. The introduction of the self-dual field strength $\mathcal{F}_{++} = [\mathcal{D}_{+\dot{+}}, \mathcal{D}_{+\dot{-}}]$ is necessary to achieve closure of the dilation operator because of the transition $\epsilon_{\mathcal{I}\mathcal{J}}\lambda_+{}^{\mathcal{I}}\lambda_+{}^{\mathcal{J}} \leftrightarrow \phi\mathcal{F}_{++}$.

Similarly, considering the $SU(2_{\dot{\alpha}}|2_{\mathcal{I}})$ highest weight magnons $\{\lambda_\alpha{}^+\}$, and demanding closure we obtain a sector with $SU(2|1)$ symmetry:

$$SU(2|1) \text{ sector:} \qquad \{\, \phi, \lambda_\alpha{}^+, \mathcal{M}^{++}\,\}\,, \tag{4.14}$$

where we have introduced the notation $\mathcal{M}^{\mathcal{I}\mathcal{J}} \equiv Q_i^{\mathcal{I}}\bar{Q}^{i\mathcal{J}}$. Inclusion of the $\mathcal{M}^{++}$ dimer is forced at two loops by the transition $\epsilon^{\alpha\beta}\lambda_\alpha{}^+\lambda_\beta{}^+ \leftrightarrow \phi\mathcal{M}^{++}$.

In the following sections we will study scattering in the $SU(2|1)$ sector. Having no analog in $\mathcal{N}=4$ SYM, this sector has the potential to reveal a new integrability structure. The $SU(2,1|2)$ sector exists in any $\mathcal{N}=2$ gauge theory, including $\mathcal{N}=4$ SYM, and will be discussed at the end of the chapter.

4.2.1 The two-loop Hamiltonian in the $SU(2|1)$ sector

In this section we will use symmetry arguments to fix the two-loop Hamiltonian of the $SU(2|1)$ sector, up to a few arbitrary coefficients. With this result at hand, we will proceed in the following section to calculate the two-body scattering of magnons and test integrability of the sector. To avoid cluttering we will suppress the "+" $SU(2)_R$ index and write the letters as

$$\{\, \phi, \lambda_\alpha, \mathcal{M}\,\}\,. \tag{4.15}$$

At one loop the sector decomposes into $\{\phi, \lambda_\alpha\}$ and $\{\phi, \mathcal{M}\}$. Each of these subsectors is separately integrable: The first one, because it is identical to the corresponding sector in $\mathcal{N}=4$ SYM. The second one, because its Hamiltonian turns out to be trivial [13] – the dimer $\mathcal{M}$ does not move on the ϕ chain so each string of ϕ's and $\mathcal{M}$'s is already an exact eigenstate. The $SU(2|1)$ sector becomes interesting at two loops, where interaction with $\mathcal{M}$ affects the scattering of the asymptotic λ_α magnons.

To avoid an explicit Feynman diagram calculation we will use the approach of [25], where the symmetry algebra was used to restrict the form of the spin chain Hamiltonian in the $SU(2|3)$ subsector of $\mathcal{N}=4$ SYM. In that case, the two-loop Hamiltonian turned out to be completely fixed by symmetry.

Parity

The parity transformations for the fields in the $SU(2|1)$ are as follows,

$$\phi^a{}_b \leftrightarrow -\phi^b{}_a\,, \quad \lambda^a{}_b \leftrightarrow -\lambda^b{}_a\,, \quad \mathcal{M}^a{}_b \leftrightarrow -\mathcal{M}^b{}_a\,. \tag{4.16}$$

This is just transposition of adjoint indices with an extra minus sign. The action on a single trace state is then (using a ket notation for the states of the chain):

$$P|A_1 \dots A_L\rangle = (-1)^{L+f(f+1)/2}|A_L \dots A_1\rangle\,, \tag{4.17}$$

where f is the number of fermionic fields and L is the length of the state considering $\mathcal{M}$ as a *single-site* object.

4.2.2 Symmetry analysis

The states of the sector furnish a representation of the $SU(2|1)$ algebra. In the interacting theory, the symmetry generators can be written as a perturbation series in the coupling constant [17, 25],

$$\mathcal{J}(g) = \sum_{k=0}^{\infty} g^k \mathcal{J}_k\,. \tag{4.18}$$

As usual when working with spin chains we will focus in the *local* action of the generators, the *complete* action being a sum of local terms. Following [25] we will represent the action of a generator by the symbol

$$\mathcal{J}_k \sim \left\{ {a_1 \dots a_n \atop b_1 \dots b_m} \right\}. \tag{4.19}$$

This replaces the string of fields $a_1 \dots a_n$ by $b_1 \dots b_m$ and gives zero otherwise. To obtain the total action we apply this transformation at each site of the closed chain. For example,

$$\left\{ {AB \atop CD} \right\}|ABEABF\rangle = |CDEABF\rangle + 0 + 0 + |ABECDF\rangle + 0 + 0\,. \tag{4.20}$$

Of course, we will pick up an extra minus sign each time a fermionic generator ($\mathcal{Q}$ or $\mathcal{S}$) hops a fermionic field. An interaction with $n+m$ entries will be said to have $n+m$ legs. Because corrections to the generators have their origin in planar perturbation theory, the number of legs is restricted by the order of the coupling constant we are considering. The counting is easier if we forget for a moment our definition of $\mathcal{M}$ and consider Q as fundamental field of our sector. The number of legs is then restricted by,

$$n + m = k + 2\,, \tag{4.21}$$

where k is the order of the coupling.[1] Now, if a Q field sits at the far right in the upper or lower row of (4.19), we know that the next field to its right will be a $\bar{Q}$, in order to have a flavor singlet. An analogous statement holds for a $\bar{Q}$ sitting in the far left. This means that after writing the $\mathcal{J}$ generators using the Q and $\bar{Q}$ fields, we can replace all the Q's($\bar{Q}$'s) in the far right(left) with an $\mathcal{M}$ symbol, in addition to the explicit $Q\bar{Q} = \mathcal{M}$ replacement.

The $SU(2|1)$ algebra

To obtain the $SU(2|1)$ algebra we start from the full $SU(2,2|2)$ generators:[2]

$$\{\, \mathcal{L}_\alpha{}^\beta, \dot{\mathcal{L}}_{\dot\alpha}{}^{\dot\beta}, \mathcal{R}_\mathcal{I}{}^\mathcal{J}, \mathcal{P}_{\alpha\dot\beta}, \mathcal{K}^{\alpha\dot\beta}, D, r, \mathcal{Q}_\alpha{}^\mathcal{I}, \mathcal{S}_\mathcal{I}{}^\alpha, \bar{\mathcal{Q}}_{\dot\alpha\mathcal{I}}, \bar{\mathcal{S}}^{\dot\alpha\mathcal{I}} \,\}\,, \tag{4.22}$$

where $\mathcal{L}$ and $\dot{\mathcal{L}}$ are the Lorentz generators, $\mathcal{R}$ and r correspond to $SU(2)_R$ and the $U(1)$ r-charge, D is the dilation operator and $\mathcal{Q}$ and $\mathcal{S}$ are the supercharges. We now define

$$\mathcal{Q}_\alpha \equiv \mathcal{Q}_\alpha{}^+\,, \tag{4.23}$$
$$\mathcal{S}^\alpha \equiv \mathcal{S}_+{}^\alpha\,, \tag{4.24}$$
$$\mathcal{U} \equiv \mathcal{R}_+{}^+ + \tfrac{1}{2}\left(D_0 - r\right)\,, \tag{4.25}$$
$$\delta\mathcal{H} \equiv \delta D\,. \tag{4.26}$$

We have split the interacting dilation generator as

$$D = D_0 + \delta D\,, \tag{4.27}$$

[1] As in [25], we use gauge invariance of cyclic states to increase the legs of the generators to its maximum value, *i.e.* $k+2$ at order k in the coupling.

[2] See Appendix A.2 for our conventions.

where D_0 measures the classical conformal dimension and δD its quantum corrections.[3] The $SU(2|1)$ generators are then:

$$\mathcal{J} = \{\mathcal{L}_\alpha{}^\beta, \mathcal{U}, \delta\mathcal{H}, \mathcal{Q}_\alpha, \mathcal{S}^\alpha\}\,. \tag{4.28}$$

As in [25], we enhanced the algebra by the extra central $U(1)$ generator δH. The commutation relations are easy to obtain from the original $SU(2,2|2)$ commutators. Generators carrying $SU(2)$ Lorentz indices transform canonically according to:

$$[\mathcal{L}_\alpha{}^\beta, \mathcal{J}_\gamma] = \delta^\beta_\gamma \mathcal{J}_\alpha - \tfrac{1}{2}\delta^\beta_\alpha \mathcal{J}_\gamma\,, \qquad [\mathcal{L}_\alpha{}^\beta, \mathcal{J}^\gamma] = -\delta^\gamma_\alpha \mathcal{J}_\beta + \tfrac{1}{2}\delta^\beta_\alpha \mathcal{J}^\gamma\,. \tag{4.29}$$

The only non-zero anti-commutator is:

$$\{\mathcal{S}^\beta, \mathcal{Q}_\alpha\} = \mathcal{L}_\alpha{}^\beta + \delta^\beta_\alpha(\mathcal{U} + \tfrac{1}{2}\delta\mathcal{H}) \tag{4.30}$$

and the non-zero $\mathcal{U}$-charges are:

$$[\mathcal{U}, \mathcal{Q}_\alpha] = -\tfrac{1}{2}\mathcal{Q}_\alpha\,, \qquad [\mathcal{U}, \mathcal{S}^\alpha] = \tfrac{1}{2}\mathcal{S}^\alpha\,. \tag{4.31}$$

Also,

$$[\mathcal{J}, \delta\mathcal{H}] = 0\,, \tag{4.32}$$

confirming that $\delta\mathcal{H}$ is indeed a central element. Note that $\mathcal{U}$ is defined in terms of generators that do not receive quantum corrections and therefore it will not be modified in the interacting theory. The same applies to $\mathcal{L}_\alpha{}^\beta$ if we choose a regularization scheme consistent with Lorentz symmetry.

4.2.3 The interacting generators

The tree-level representation of the $SU(2|1)$ algebra reads

$$\begin{aligned}
\mathcal{U} &= \left\{ {\phi \atop \phi} \right\} + \tfrac{1}{2}\left\{ {\alpha \atop \alpha} \right\}, \\
\mathcal{L}_\alpha{}^\beta &= \left\{ {\alpha \atop \beta} \right\} - \tfrac{1}{2}\delta^\alpha_\beta \left\{ {\gamma \atop \gamma} \right\}, \\
(\mathcal{Q}_\alpha)_0 &= e^{i\beta_1}\left\{ {\phi \atop \alpha} \right\}, \\
(\mathcal{S}^\alpha)_0 &= e^{-i\beta_1}\left\{ {\alpha \atop \phi} \right\},
\end{aligned} \tag{4.33}$$

where the subscript "0" indicates that we are working at tree level. The idea is to consider perturbative deformations of these generators and restrict their

[3]To be consistent with (4.26) we also define $\mathcal{H}_0 \equiv D_0$, although $\mathcal{H}_0$ is not an $SU(2|1)$ generator.

form using the $SU(2|1)$ algebra. In principle, there should be fluctuations in the length, but because we consider the dimeric impurity $\mathcal{M}$ as a single-site object, the length always stays constant. For $\mathcal{H}_2$ we have:

$$\begin{aligned}\mathcal{H}_2 = c_0\left\{ {\phi\phi \atop \phi\phi} \right\} + c_1\left\{ {\phi\mathcal{M} \atop \phi\mathcal{M}} \right\} + c_2\left\{ {\mathcal{M}\phi \atop \mathcal{M}\phi} \right\} + c_3\left\{ {\mathcal{M} \atop \mathcal{M}} \right\} + c_4\left\{ {\phi\alpha \atop \phi\alpha} \right\} + c_5\left\{ {\alpha\phi \atop \alpha\phi} \right\} \\ + c_6\left\{ {\phi\alpha \atop \alpha\phi} \right\} + c_7\left\{ {\alpha\phi \atop \phi\alpha} \right\} + c_8\left\{ {\alpha\mathcal{M} \atop \alpha\mathcal{M}} \right\} + c_9\left\{ {\mathcal{M}\alpha \atop \mathcal{M}\alpha} \right\} + c_{10}\left\{ {\alpha\beta \atop \alpha\beta} \right\} + c_{11}\left\{ {\alpha\beta \atop \beta\alpha} \right\} .\end{aligned} \tag{4.34}$$

Imposing invariance under parity we obtain:

$$c_1 = c_2 \,, \qquad c_4 = c_5 \,, \qquad c_6 = c_7 \,, \qquad c_8 = c_9 \,. \tag{4.35}$$

In addition, protection of $\phi\phi$ implies $c_0 = 0$.[4] This still leaves seven independent coefficients. Imposing that the algebra commutation relations are satisfied perturbatively eliminates six of them, leaving us with one undetermined parameter, $c_1 \equiv \alpha_1^2$, which is associated with a rescaling of the coupling and cannot be fixed by algebraic means. The procedure is now completely algorithmic and it was described in detail in [25]. For each perturbative correction we consider the most general ansatz consistent with conservation of classical energy, r-charge and equation (4.21). Consistency of the algebra commutations relations significantly reduces the number of independent parameters. As extra input we use the fact that in the $SU(1|1)$ subsector spanned by $\{ \phi, \lambda_+ \}$ the two-loop Hamiltonian of $\mathcal{N} = 2$ SCQCD should be identical to the corresponding Hamiltonian in $\mathcal{N} = 4$ SYM [14]. We present our results in tables 4.7 and 4.8. At first sight, there seems to be a high number of independent coefficients, however most of them are unphysical. The two coefficients $\{ \alpha_1, \alpha_3 \}$ can be reabsorbed by a redefinition of the coupling,[5]

$$g \to \alpha_1 g + \alpha_3 g^3 \,. \tag{4.36}$$

The six coefficients $\{ \beta_1, \beta_2, \delta_1, \delta_2, \delta_3, \delta_4 \}$ correspond to similarity transformations and never show up in physical quantities like anomalous dimensions or S-matrix elements. We are then left with $\{ \eta, \chi \}$ which do show up in physical quantities and therefore cannot be ignored. However, the S-matrix elements that we will study in the next section happen to be independent of $\{ \eta, \chi \}$.

[4] In [25] this condition was obtained using the algebra constraints, in our case we have to give it as extra input.

[5] Note of course that $\alpha_1 \neq 0$, otherwise the whole one-loop Hamiltonian $\mathcal{H}_2$ would vanish. The actual value of α_1 can be fixed by comparison with the results of Chapter 3: $\alpha_1^2 = 2$.

$$
\begin{aligned}
\mathcal{H}_0 &= \left\{{\phi \atop \phi}\right\} + 2\left\{{\mathcal{M} \atop \mathcal{M}}\right\} + \tfrac{3}{2}\left\{{\alpha \atop \alpha}\right\}, \\
\mathcal{H}_2 &= \alpha_1^2\big(\left\{{\phi\mathcal{M} \atop \phi\mathcal{M}}\right\} + \left\{{\mathcal{M}\phi \atop \mathcal{M}\phi}\right\}\big) + 2\alpha_1^2\left\{{\mathcal{M} \atop \mathcal{M}}\right\} + \alpha_1^2\big(\left\{{\phi\alpha \atop \phi\alpha}\right\} + \left\{{\alpha\phi \atop \alpha\phi}\right\}\big) - \alpha_1^2\big(\left\{{\phi\alpha \atop \alpha\phi}\right\} + \left\{{\alpha\phi \atop \phi\alpha}\right\}\big) \\
&\quad + \alpha_1^2\big(\left\{{\alpha\mathcal{M} \atop \alpha\mathcal{M}}\right\} + \left\{{\mathcal{M}\alpha \atop \mathcal{M}\alpha}\right\}\big) + \alpha_1^2\left\{{\alpha\beta \atop \alpha\beta}\right\} + \alpha_1^2\left\{{\alpha\beta \atop \beta\alpha}\right\}, \\
\mathcal{H}_3 &= -\alpha_1^3\, e^{i\beta_2}\, \varepsilon_{\alpha\beta}\big(\left\{{\alpha\beta \atop \phi\mathcal{M}}\right\} + \left\{{\alpha\beta \atop \mathcal{M}\phi}\right\}\big) - \alpha_1^3\, e^{-i\beta_2}\, \varepsilon^{\alpha\beta}\big(\left\{{\phi\mathcal{M} \atop \alpha\beta}\right\} + \left\{{\mathcal{M}\phi \atop \alpha\beta}\right\}\big), \\
\mathcal{H}_4 &= (-\tfrac{3}{2}\alpha_1^4 + 2\alpha_1\alpha_3)\big(\left\{{\phi\phi\alpha \atop \phi\phi\alpha}\right\} + \left\{{\alpha\phi\phi \atop \alpha\phi\phi}\right\}\big) + (\alpha_1^4 - \alpha_1\alpha_3)\big(\left\{{\phi\phi\alpha \atop \phi\alpha\phi}\right\} + \left\{{\alpha\phi\phi \atop \phi\alpha\phi}\right\}\big) \\
&\quad - \tfrac{1}{2}\alpha_1^2\big(\left\{{\phi\phi\alpha \atop \alpha\phi\phi}\right\} + \left\{{\alpha\phi\phi \atop \phi\phi\alpha}\right\}\big) + (\alpha_1^4 - \alpha_1\alpha_3)\big(\left\{{\phi\alpha\phi \atop \alpha\phi\phi}\right\} + \left\{{\phi\alpha\phi \atop \phi\phi\alpha}\right\}\big) \\
&\quad + (-\tfrac{5}{4}\alpha_1^2 + \alpha_1\alpha_3 - \eta + \chi)\big(\left\{{\phi\phi\mathcal{M} \atop \phi\phi\mathcal{M}}\right\} + \left\{{\mathcal{M}\phi\phi \atop \mathcal{M}\phi\phi}\right\}\big) \\
&\quad + (-\tfrac{31}{4}\alpha_1^2 + 7\alpha_1\alpha_3 + \chi)\big(\left\{{\phi\mathcal{M} \atop \phi\mathcal{M}}\right\} + \left\{{\mathcal{M}\phi \atop \mathcal{M}\phi}\right\}\big) + (\alpha_1^4 - 2\alpha_1\alpha_3 + \eta)\big(\left\{{\phi\mathcal{M} \atop \mathcal{M}\phi}\right\} + \left\{{\mathcal{M}\phi \atop \phi\mathcal{M}}\right\}\big) \\
&\quad + (\tfrac{19}{2}\alpha_1^4 - 10\alpha_1\alpha_3 + 2\eta - 2\chi)\left\{{\mathcal{M}\phi\mathcal{M} \atop \mathcal{M}\phi\mathcal{M}}\right\} + 2\eta\left\{{\mathcal{M}\mathcal{M} \atop \mathcal{M}\mathcal{M}}\right\} \\
&\quad + (-2\alpha_1^4 + 2\alpha_1\alpha_3 - \eta + \chi + i\alpha_1^2(\delta_1 + \delta_2))\big(\left\{{\alpha\phi\mathcal{M} \atop \phi\alpha\mathcal{M}}\right\} + \left\{{\mathcal{M}\phi\alpha \atop \mathcal{M}\alpha\phi}\right\}\big) \\
&\quad + (-2\alpha_1^4 + 2\alpha_1\alpha_3 - \eta + \chi - i\alpha_1^2(\delta_1 + \delta_2))\big(\left\{{\phi\alpha\mathcal{M} \atop \alpha\phi\mathcal{M}}\right\} + \left\{{\mathcal{M}\alpha\phi \atop \mathcal{M}\phi\alpha}\right\}\big) \\
&\quad + (-\tfrac{13}{4}\alpha_1^4 + 3\alpha_1\alpha_3 - \eta + \chi)\big(\left\{{\phi\alpha\mathcal{M} \atop \phi\alpha\mathcal{M}}\right\} + \left\{{\mathcal{M}\alpha\phi \atop \mathcal{M}\alpha\phi}\right\}\big) \\
&\quad + (-2\alpha_1^4 + 2\alpha_1\alpha_3 + \eta)\big(\left\{{\alpha\mathcal{M} \atop \alpha\mathcal{M}}\right\} + \left\{{\mathcal{M}\alpha \atop \mathcal{M}\alpha}\right\}\big) + (2\alpha_1^4 - 2\alpha_1\alpha_3 + \eta)\big(\left\{{\alpha\mathcal{M} \atop \mathcal{M}\alpha}\right\} + \left\{{\mathcal{M}\alpha \atop \alpha\mathcal{M}}\right\}\big) \\
&\quad + (-\tfrac{1}{4}\alpha_1^4 + \alpha_1\alpha_3)\big(\left\{{\phi\alpha\beta \atop \phi\alpha\beta}\right\} + \left\{{\beta\alpha\phi \atop \beta\alpha\phi}\right\}\big) + (-\tfrac{7}{4}\alpha_1^4 + \alpha_1\alpha_3)\big(\left\{{\phi\alpha\beta \atop \phi\beta\alpha}\right\} + \left\{{\beta\alpha\phi \atop \alpha\beta\phi}\right\}\big) \\
&\quad + (\alpha_1^4 - \alpha_1\alpha_3 - i\alpha_1^2\delta_1)\big(\left\{{\phi\alpha\beta \atop \alpha\phi\beta}\right\} + \left\{{\beta\alpha\phi \atop \beta\phi\alpha}\right\}\big) + (\alpha_1^4 - \alpha_1\alpha_3 + i\alpha_1^2\delta_1)\big(\left\{{\alpha\phi\beta \atop \phi\alpha\beta}\right\} + \left\{{\beta\phi\alpha \atop \beta\alpha\phi}\right\}\big) \\
&\quad + (\tfrac{1}{4}\alpha_1^4 + i\alpha_1^2\delta_3)\big(\left\{{\phi\alpha\beta \atop \beta\phi\alpha}\right\} + \left\{{\beta\alpha\phi \atop \alpha\phi\beta}\right\}\big) + (\tfrac{1}{4}\alpha_1^4 - i\alpha_1^2\delta_3)\big(\left\{{\beta\phi\alpha \atop \phi\alpha\beta}\right\} + \left\{{\alpha\phi\beta \atop \beta\alpha\phi}\right\}\big) \\
&\quad + (-\tfrac{7}{2}\alpha_1^4 + 4\alpha_1\alpha_3)\left\{{\alpha\phi\beta \atop \alpha\phi\beta}\right\} + \tfrac{1}{2}\alpha_1^2\left\{{\alpha\phi\beta \atop \beta\phi\alpha}\right\} \\
&\quad + (-\tfrac{7}{2}\alpha_1^4 + 4\alpha_1\alpha_3 - \eta + \chi)\big(\left\{{\mathcal{M}\alpha\beta \atop \mathcal{M}\alpha\beta}\right\} + \left\{{\beta\alpha\mathcal{M} \atop \beta\alpha\mathcal{M}}\right\}\big) \\
&\quad + (\tfrac{3}{2}\alpha_1^4 - 2\alpha_1\alpha_3 + \eta - \chi)\big(\left\{{\mathcal{M}\alpha\beta \atop \mathcal{M}\beta\alpha}\right\} + \left\{{\beta\alpha\mathcal{M} \atop \alpha\beta\mathcal{M}}\right\}\big) \\
&\quad + (-\tfrac{9}{4}\alpha_1^4 + 3\alpha_1\alpha_3)\big(\left\{{\alpha\beta\gamma \atop \alpha\gamma\beta}\right\} + \left\{{\gamma\beta\alpha \atop \beta\gamma\alpha}\right\}\big) + (\tfrac{1}{2}\alpha_1^4 - 2\alpha_1\alpha_3)\big(\left\{{\alpha\beta\gamma \atop \beta\gamma\alpha}\right\} + \left\{{\gamma\beta\alpha \atop \alpha\gamma\beta}\right\}\big) \\
&\quad + (-\tfrac{1}{2}\alpha_1^4 + 2\alpha_1\alpha_3)\left\{{\alpha\beta\gamma \atop \gamma\beta\alpha}\right\}.
\end{aligned}
$$

Table 4.7: The Hamiltonian up to order g^4.

4.2.4 The magnon S-matrix in the $SU(2|1)$ sector

We now proceed to calculate the magnon two-body S-matrix in the $SU(2|1)$ sector, and to check whether it satisfies the Yang-Baxter equation. Let us

$$
\begin{aligned}
(\mathcal{Q}_\alpha)_0 &= e^{i\beta_1}\left\{ {\phi \atop \alpha} \right\}, \\
(\mathcal{Q}_\alpha)_1 &= \alpha_1\, e^{i(\beta_1+\beta_2)}\varepsilon_{\alpha\beta}\left\{ {\beta \atop \mathcal{M}} \right\}, \\
(\mathcal{Q}_\alpha)_2 &= ie^{i\beta_1}(\delta_1+\delta_2+\delta_4)\big(\left\{ {\phi\phi \atop \phi\alpha} \right\} + \left\{ {\phi\phi \atop \alpha\phi} \right\}\big) + e^{i\beta_1}(\tfrac{1}{4}\alpha_1^2 + i\delta_4)\big(\left\{ {\phi\mathcal{M} \atop \alpha\mathcal{M}} \right\} + \left\{ {\mathcal{M}\phi \atop \mathcal{M}\alpha} \right\}\big) \\
&\quad + e^{i\beta_1}(\tfrac{1}{4}\alpha_1^2 + i\delta_3)\big(\left\{ {\phi\beta \atop \beta\alpha} \right\} - \left\{ {\beta\phi \atop \alpha\beta} \right\}\big) + ie^{i\beta_1}(\delta_2+\delta_4)\big(\left\{ {\phi\beta \atop \alpha\beta} \right\} - \left\{ {\beta\phi \atop \beta\alpha} \right\}\big), \\
(\mathcal{S}^\alpha)_0 &= e^{-i\beta_1}\left\{ {\alpha \atop \phi} \right\}, \\
(\mathcal{S}^\alpha)_1 &= \alpha_1\, e^{-i(\beta_1+\beta_2)}\varepsilon^{\alpha\beta}\left\{ {\mathcal{M} \atop \beta} \right\}, \\
(\mathcal{S}^\alpha)_2 &= -ie^{-i\beta_1}(\delta_1+\delta_2+\delta_4)\big(\left\{ {\phi\alpha \atop \phi\phi} \right\} + \left\{ {\alpha\phi \atop \phi\phi} \right\}\big) + e^{-i\beta_1}(\tfrac{1}{4}\alpha_1^2 - i\delta_4)\big(\left\{ {\alpha\mathcal{M} \atop \phi\mathcal{M}} \right\} + \left\{ {\mathcal{M}\alpha \atop \mathcal{M}\phi} \right\}\big) \\
&\quad + e^{-i\beta_1}(\tfrac{1}{4}\alpha_1^2 - i\delta_3)\big(\left\{ {\beta\alpha \atop \phi\beta} \right\} - \left\{ {\alpha\beta \atop \beta\phi} \right\}\big) - ie^{-i\beta_1}(\delta_2+\delta_4)\big(\left\{ {\alpha\beta \atop \phi\beta} \right\} - \left\{ {\beta\alpha \atop \beta\phi} \right\}\big).
\end{aligned}
$$

Table 4.8: Fermionic $SU(2|1)$ generators up to order g^2.

start by defining the momentum eigenstate of a single excitation,

$$
|\lambda_\alpha(p)\rangle = \sum_k e^{ipk}|\alpha_k\rangle \,, \tag{4.37}
$$

where k labels the position of the particle,

$$
|\alpha_k\rangle = |\ldots\phi\overset{\overset{k}{\downarrow}}{\lambda_\alpha}\phi\ldots\rangle \,. \tag{4.38}
$$

Its dispersion relation is easily obtained by acting with the Hamiltonian:

$$
\mathcal{H}|\lambda_\alpha(p)\rangle = g^2\alpha_1^2\left[(2-e^{ip}-e^{-ip}) + g^2\alpha_1^2(-3+2(e^{ip}+e^{-ip}) - \tfrac{1}{2}(e^{2ip}+e^{-2ip}))\right]|\lambda_\alpha(p)\rangle \,, \tag{4.39}
$$

hence,

$$
E^\lambda(p) = 4(g^2\alpha_1^2 - 2g^4\alpha_1^4)\sin^2\frac{p}{2} + 2g^4\alpha_1^4\sin^2 p + O(g^6)\,. \tag{4.40}
$$

To extract the S-matrix we will use the familiar perturbative asymptotic Bethe ansatz, see *e.g.* [5]. For the $SU(2_\alpha)$ singlet two-body state we define:

$$
\begin{aligned}
|\lambda_{[\alpha}\lambda_{\beta]}\rangle &= \sum_{k<l-1} \Psi_{\mathbf{1}}(k,l)|\ldots\phi\overset{\overset{k}{\downarrow}}{\lambda}_{[\alpha}\phi\ldots\phi\overset{\overset{l}{\downarrow}}{\lambda}_{\beta]}\phi\ldots\rangle \\
&\quad + \sum_k \Psi_n(k)|\ldots\phi\overset{\overset{k}{\downarrow}}{\lambda}_{[\alpha}\overset{\overset{k+1}{\downarrow}}{\lambda}_{\beta]}\phi\ldots\rangle + \sum_k \Psi_{\mathcal{M}}(k)|\ldots\phi\overset{\overset{k}{\downarrow}}{\mathcal{M}}\phi\ldots\rangle\,,
\end{aligned}
\tag{4.41}
$$

valid up to order g^2. The Ψ's correspond Schrödinger wave functions and k and l label the positions of the particles in the ϕ vacuum. At this order in perturbation theory a transition $\lambda_{[\alpha}\lambda_{\beta]} \to \mathcal{M}$ is possible and this is taken into account by the last term in (4.41). In order to solve the scattering problem we consider the following ansatz:

$$
\begin{aligned}
\Psi_{\mathbf{1}}(k,l) &= e^{i(p_1k+p_2l)} + S_{\mathbf{1}}(p_2,p_1)e^{i(p_1l+p_2k)}\,, \\
\Psi_n(k) &= S_n(p_2,p_1)e^{i(p_1+p_2)k}\,, \\
\Psi_{\mathcal{M}}(k) &= S_{\mathcal{M}}(p_2,p_1)e^{i(p_1+p_2)k}\,.
\end{aligned}
\tag{4.42}
$$

Here $S_{\mathbf{1}}(p_2,p_1)$, $S_n(p_2,p_1)$ and $S_{\mathcal{M}}(p_2,p_1)$ are functions of g and represent the different scattering amplitudes. Imposing the Schrödinger equation

$$
\mathcal{H}|\lambda_{[\alpha}\lambda_{\beta]}\rangle = E(p_1,p_2)|\lambda_{[\alpha}\lambda_{\beta]}\rangle\,,
\tag{4.43}
$$

for the separate cases $l > k+2$, $l = k+2$ and $l = k+1$ we can solve for the scattering amplitudes to order g^2. The interesting term is $S_{\mathbf{1}}(p_2,p_1)$, which governs the asymptotic magnon scattering,

$$
\begin{aligned}
S_{\mathbf{1}}(p_2,p_1) = &-\frac{1-2e^{ip_2}+e^{i(p_1+p_2)}}{1-2e^{ip_1}+e^{i(p_1+p_2)}} \\
&\times\Big(1+2ig^2\alpha_1^2\frac{(\cos p_1 - 2\cos(p_1-p_2)+\cos p_2)\sin\frac{p_1}{2}\sin\frac{p_2}{2}(\sin p_1-\sin p_2)}{\cos(\frac{p_1-p_2}{2})(3-2\cos p_1-2\cos p_2+\cos(p_1+p_2))}+O(g^4)\Big).
\end{aligned}
\tag{4.44}
$$

In the triplet sector the ansatz is simpler since $\lambda_{\{\alpha}\lambda_{\beta\}}$ does not mix with $\mathcal{M}$,

$$|\lambda_{\{\alpha}\lambda_{\beta\}}\rangle = \sum_{k<l-1} \Psi_{\mathbf{3}}(k,l)|\dots\phi\overset{\overset{k}{\downarrow}}{\lambda}_{\{\alpha}\phi\dots\phi\overset{\overset{l}{\downarrow}}{\lambda}_{\beta\}}\phi\dots\rangle + \sum_{k} \Psi_{\mathbf{3}n}(k)|\dots\phi\overset{\overset{k}{\downarrow}}{\lambda}_{\{\alpha}\overset{\overset{k+1}{\downarrow}}{\lambda}_{\beta\}}\phi\dots\rangle\,, \tag{4.45}$$

where

$$\begin{aligned}\Psi_{\mathbf{3}}(k,l) &= e^{i(p_1k+p_2l)} + S_{\mathbf{3}}(p_2,p_1)e^{i(p_1l+p_2k)}\,,\\ \Psi_{\mathbf{3}n}(k) &= S_{\mathbf{3}n}(p_2,p_1)e^{i(p_1+p_2)k}\,.\end{aligned} \tag{4.46}$$

We find

$$S_{\mathbf{3}}(p_2,p_1) = -\,1 - ig^2\alpha_1^2(\sin p_1 - \sin(p_1-p_2) - \sin p_2) + O(g^4)\,. \tag{4.47}$$

Checking the Yang-Baxter equation

We are finally ready to check the Yang-Baxter equation for the two-body magnon S-matrix. The equation reads

$$S^{\delta\epsilon}_{\alpha\beta}(p_1,p_2)S^{\tau\gamma'}_{\epsilon\gamma}(p_1,p_3)S^{\alpha'\beta'}_{\delta\tau}(p_2,p_3) = S^{\beta'\gamma'}_{\epsilon\delta}(p_1,p_2)S^{\alpha'\epsilon}_{\alpha\tau}(p_1,p_3)S^{\tau\delta}_{\beta\gamma}(p_2,p_3)\,. \tag{4.48}$$

Defining:

$$A(p_1,p_2) = S_{\mathbf{3}}(p_1,p_2)\,, \tag{4.49}$$

$$B(p_1,p_2) = \frac{1}{2}(S_{\mathbf{1}}(p_1,p_2) - S_{\mathbf{3}}(p_1,p_2))\,, \tag{4.50}$$

we can rewrite the S-matrix in terms of the identity operator $\mathbb{I}$ and the trace operator $\mathbb{K}$,

$$S(p_1,p_2) = A(p_1,p_2)\mathbb{I} + B(p_1,p_2)\mathbb{K}\,. \tag{4.51}$$

As explained *e.g.* in [13], the Yang-Baxter equation is equivalent to the single constraint

$$\begin{aligned}0 \overset{?}{=}\; & 2B(p_1,p_2)A(p_1,p_3)B(p_2,p_3) + A(p_1,p_2)A(p_1,p_3)B(p_2,p_3) + B(p_1,p_2)A(p_1,p_3)A(p_2,p_3)\\ & +B(p_1,p_2)B(p_1,p_3)B(p_2,p_3) - A(p_1,p_2)B(p_1,p_3)A(p_2,p_3)\,.\end{aligned} \tag{4.52}$$

A necessary condition for factorization of many-body scattering is the vanishing of the right-hand side. However, working at order g^2 we obtain

$$64i\alpha_1^2 e^{i(p_1+p_2+p_3)} \frac{\sin\left(\frac{p_1}{2}\right)^2 \sin\left(\frac{p_2}{2}\right)^2 \sin\left(\frac{p_3}{2}\right)^2 \tan\left(\frac{p_1-p_2}{2}\right) \tan\left(\frac{p_1-p_3}{2}\right) \tan\left(\frac{p_2-p_3}{2}\right)}{(1+e^{i(p_1+p_2)}-2e^{ip_2})(1+e^{i(p_1+p_3)}-2e^{ip_3})(1+e^{i(p_2+p_3)}-2e^{ip_3})}, \tag{4.53}$$

which is certainly non-zero. Failure of the Yang-Baxter equation conclusively shows that the $SU(2|1)$ sector is not integrable at two loops.

4.3 The universal $SU(2,1|2)$ sector

The $SU(2,1|2)$ sector (4.13) consists entirely of letters that belong to the $\mathcal{N}=2$ vector multiplet, and it is then present in any $\mathcal{N}=2$ gauge theory. Diagrammatic arguments [14] show that the planar dilation operator in this sector is the same up to *two loops* in any $\mathcal{N}=2$ superconformal theory, as it coincides to that order with a restriction of the $\mathcal{N}=4$ SYM dilation operator. The model dependence kicks in at three loops.[6]

Choosing the usual chiral vacuum $\mathrm{Tr}\,\phi^k$, the Goldstone magnons $\{\lambda_+{}^{\mathcal{I}}, \mathcal{D}_{+\dot\alpha}\}$ transform in the fundamental representation of $SU(2_{\dot\alpha}|2_{\mathcal{I}})$. Their two-body S-matrix $S_{SU(2_{\dot\alpha}|2_{\mathcal{I}})}$ is uniquely determined up to an overall phase by the $SU(2|2)$ symmetry [7], and thus, just as is the case in $\mathcal{N}=4$ SYM, it automatically satisfies the Yang-Baxter equation. This is a first hint to suspect that this sector may be generically integrable, at least in the sense of the asymptotic Bethe ansatz on the infinite chain.[7] Of course, factorization of the n-body S-matrix into two-body S-matrices is a stronger condition than Yang-Baxter, and an explicit test at three loops will be required. A three-loop diagrammatic analysis is in progress [27]. The strongest conjecture [27] suggested by this perturbative study is that the $SU(2,1|2)$ Hamiltonian of any $\mathcal{N}=2$ superconformal gauge theory can be mapped to that of $\mathcal{N}=4$ SYM by a redefinition of the 't Hooft coupling, $g^2 \to f(g^2) = g^2 + O(g^6)$. This would be a trivial operation from the viewpoint of the integrable structure. Indeed recall that it is still somewhat of a mystery why the dispersion relation of the $\mathcal{N}=4$ SYM magnons takes the exact form

$$\Delta - |r| = \sqrt{1 + 8g^2 \sin^2\frac{p}{2}}\,, \tag{4.54}$$

while integrability alone would be compatible with the replacement $g^2 \to$

[6]In the context of $\mathcal{N}=4$ SYM, the $SU(2,1|2)$ sector can be regarded as a non-compact cousin of the $SU(2|3)$ sector, whose Hamiltonian was determined up to three loops by Beisert [25] using symmetry arguments. The Hamiltonian of non-compact sectors is much harder to fix. Zwiebel's paper [26] represents the state of the art.

[7]We are postponing at this stage the harder questions about finite-size effects.

$f(g^2)$ (which is indeed what happens in the ABJM model [28]). However a redefinition of g can have drastic dynamical consequences, for example it may radically change the strong coupling behavior of anomalous dimensions (ABJM is again a case in point.)

A second indication in favor of integrability of the $SU(2,1|2)$ sector comes from the AdS/CFT correspondence – at least, that is, for the subset of models that admit a string dual. The simplest $\mathcal{N}=2$ theories with a known string description are the orbifolds of $\mathcal{N}=4$ SYM by a discrete subgroup $\Gamma \subset SU(2) \subset SU(4)_R$, which are dual to the IIB backgrounds $AdS_5 \times S^5/\Gamma$ [29, 30]. These are quiver gauge theories with product gauge group $SU(N)^k$, where k is the order of Γ. The k gauge couplings are exactly marginal parameters. If all gauge couplings are equal, the spin chain (and the dual sigma model) is completely integrable [15, 31], but when they are different, integrability of the full chain is broken.[8] However, the situation is much better in the $SU(2,1|2)$ sector.[9] At strong coupling one can study the S-matrix of the $SU(2|2)$ excitations using the dual sigma model. Changing the relative gauge couplings is dual to twisted-sector deformations in the sigma model: to leading order in α' (tree level in the sigma model) they do not change the scattering of the $SU(2|2)$ excitations, which live in directions of the target space unaffected by the orbifold. So the n-body S-matrix still factorizes into two-body S-matrices. To be more precise, the only effect of the twisted deformation felt by the $SU(2|2)$ excitations is a renormalization of the string tension. For example, in the $\mathbb{Z}_2$ case, the relation between α' and the AdS radius R reads

$$\frac{R^4}{\alpha'} = \frac{2\lambda\check{\lambda}}{\lambda+\check{\lambda}}\,, \tag{4.55}$$

where λ and $\check{\lambda}$ are the two 't Hooft couplings. It would be very interesting to confirm this picture to next order in α', where the effect of the twisted deformation is non-trivial, by an explicit one-loop calculation of the sigma-model S-matrix. Recall that the two-body $SU(2|2)$ S-matrix is completely fixed by symmetry, so to really probe integrability one would have to study factorization of the n-body S-matrix or devise some other test.

In summary, the $SU(2,1|2)$ sector(s) of $\mathcal{N}=2$ superconformal gauge the-

---

[8] For the simplest example of the $\mathbb{Z}_2$ orbifold, this phenomenon was studied in detail in [13, 14, 32], which focussed on the magnons transforming in the bifundamental representation of the $SU(N_c)\times SU(N_{\check{c}})$ gauge group, with $N_c \equiv N_{\check{c}}$. For $\lambda \neq \check{\lambda}$ their dispersion relation develops a gap. The form of their two-body S-matrix is fixed by symmetry, and fails to satisfy the Yang-Baxter equation except when $\lambda = \check{\lambda}$.

[9] There are actually k separate $SU(2,1|2)$ sectors, one for each of the $SU(N)$ vector multiplets.

ories have the same Hamiltonian as in $\mathcal{N}=4$ SYM for small λ (to two-loop order, $O(\lambda^2)$); and in theories with AdS duals, the large λ limit of the Hamiltonian is also the same as in $\mathcal{N}=4$ SYM, modulo a renormalization of the coupling. For example, in the $\mathbb{Z}_2$ quiver theory, it follows from (4.55) that for large λ and large $\check{\lambda}$ (with $\lambda/\check{\lambda}$ fixed) the dilation operator in the $SU(2,1|2)$ sector coincides with the one in $\mathcal{N}=4$ SYM if one replaces $\lambda \to 2\lambda\check{\lambda}/(\lambda+\check{\lambda})$.[10] We are led to conjecture that this remains true for all intermediate values of the coupling, with the appropriate redefinition $\lambda \to f(\lambda)$ that matches the weak and strong coupling behaviors.

$SU(2,1|1)$ sector

In closing, it is tempting to entertain the natural extrapolations of this conjecture to $\mathcal{N}=1$ conformal gauge theories. Every $\mathcal{N}=1$ superconformal gauge theory contains a closed $SU(2,1|1)$ sector, with letters belonging entirely to the $\mathcal{N}=1$ vector multiplet,

$$SU(2,1|1) \text{ sector:} \qquad (\mathcal{D}_{+\dot{\alpha}})^n\{\,\lambda_+, \mathcal{F}_{++}\,\}\,. \tag{4.56}$$

The diagrammatic arguments of [14] show again that in any $\mathcal{N}=1$ superconformal theory the dilation operator in this sector coincides up to two loops with the restriction of the $\mathcal{N}=4$ SYM dilation operator. (Of course this is a meaningful statement only for $\mathcal{N}=1$ SCFTs that have a weak coupling limit). Choosing the chiral vacuum $\mathrm{Tr}\,\lambda_+^k$, the asymptotic excitations on the chain are the massless magnons $\{\mathcal{D}_{+\dot{\alpha}}\}$, transforming as a doublet of $SU(2_{\dot{\alpha}})$. This is not enough symmetry to completely fix the form of the two-body magnon S-matrix, which makes integrability of the $SU(2,1|1)$ sector somewhat less compelling as a general conjecture. For models that admit string duals, some evidence for integrability comes again from the AdS/CFT correspondence. For example, while the generic Leigh-Strassler deformation of $\mathcal{N}=4$ SYM is not fully integrable (see [37] for a review), there is still hope for integrability in the $SU(2,1|1)$ sector. Indeed, one can argue for integrability at strong coupling (to leading order): the deformation of the $AdS_5\times S^5$ background that corresponds to the Leigh-Strassler deformation (whatever its explicit form may be) is not expected to affect the tree-level scattering of excitations in the $SU(2,1|1)$ subsector, since those excitations live entirely in AdS_5.

Seiberg duality implies that the resummation of the ϵ expansion in the electric theory must coincide with the resummation of the $\tilde{\epsilon}$ expansion in the

[10] This correspondence is also precisely confirmed [33] by considering the strong coupling limit of the matrix model [34] that calculates the expectation value of the 1/2 BPS circular Wilson loop in the $\mathbb{Z}_2$ quiver theory, following [35, 36].

magnetic theory. In the $SU(2,1|1)$ sector, the dilation operator is the same as in $\mathcal{N} = 4$ SYM, and thus obviously integrable, up to two loops in both expansions. The optimistic scenario is for the sector to remain integrable throughout the conformal window. It will be interesting to perform higher order checks in both ϵ and $\tilde{\epsilon}$. Integrability would offer the exciting prospect of much more quantitative tests of Seiberg duality than presently possible.

Intermission

Up to this point, we have concentrated our efforts in two specific theories, $\mathcal{N} = 2$ SCQCD and $\mathcal{N} = 1$ SQCD in the upper edge of the conformal window. Both these theories are supersymmetric and therefore, their conformal symmetry is enhanced to superconformal symmetry. The superconformal group is a very powerful and superconformal theories are amenable to analytical analysis. In the absence of supersymmetry and in dimensions higher than two, analytic results based on conformal symmetry alone are harder to obtain. Therefore, starting with the following chapter, our philosophy will be different to what we have been doing so far. Instead of studying a specific set of theories, we will look for structural constraints in the *space* of conformal theories. Our analysis will be embedded in the so-called "bootstrap program" and it will be mostly numeric.

Chapter 5

The Boundary Bootstrap Program

The "bootstrap" has been a recurring dream in theoretical physics. It is the ambitious aspiration that, starting from a few basic spectral assumptions, symmetries and general consistency requirements (such as unitarity and crossing) will be powerful enough to fix the form of the theory, with no reference to a Lagrangian. The dual models of the strong interactions emerged as an incarnation of the S-matrix bootstrap attempts of the 1960s and eventually led to the discovery of string theory. The bootstrap program for conformal field theories (CFTs) in d dimensions was formulated in the early 1970s [38]. Despite important formal developments such as the operator product expansion and the conformal block decomposition (see *e.g.* the early books [39, 40]), attempts to solve CFTs in arbitrary dimensions were not successful. For two-dimensional CFTs, the revolution came in the 1980s with the discovery of many exactly-solvable "rational" models. While this is a beautiful incarnation of the bootstrap idea, the methods that work in $2d$ rational CFTs[1] are too specialized to be imitated in higher dimensions, or even in two dimensions for the generic non-rational model.

The interest in CFT in various dimensions is nowadays stronger than ever, sustained by phenomenological questions in condensed matter physics ($d = 3$) and particle physics ($d = 4$), as well as by more formal motivations such as the AdS/CFT correspondence and the rich integrability structures of superconformal field theories ($d \leq 6$). A pioneering work [3] has rekindled the conformal bootstrap, turning it into a concrete computational tool. This approach has been refined and extended in a series of papers [1, 41–48].

The modern bootstrap starts with the simple question: in a generic theory,

[1] or in closely-related models such as Liouville theory

which values of operator dimensions and OPE coefficients are compatible with the constraints of crossing symmetry and unitarity for the four-point functions? There is a shift of viewpoint, from trying to find analytic answers in a specific model to deriving (by numerical methods if necessary) universal bounds valid for any model. As it turns out, one can derive strong constraints already from the analysis of a single four-point function of identical scalar operators [3]. This should be regarded as the first step in a systematic exploration of the space of CFTs. More surprisingly, important theories such as the $3d$ Ising model appear to live at interesting corners of the parameter space, sitting at "kinks" of the exclusion curves [1, 41, 48]. So even the solution of some special models in $d > 2$ may not be too far-fetched, after all.

In its simplest version, the revived conformal bootstrap works as follows. The four-point correlation function $\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle$ of a scalar operator can be written as a sum over conformal blocks in two different channels, by taking OPEs in two different limits. The conformal block decompositions in either channel must sum up to the same four-point function, giving crossing-symmetry relations for the couplings and scaling dimensions. While this was understood long ago, the main idea of [3] is that these constraints can be put to good use by taking derivatives of the four-point function at symmetric points and applying linear programming techniques to obtain contradictions if certain conditions for *e.g.* the operator spectrum are not met. The prototypical example of a constraint that arises in this way is an upper bound for the dimension of the first scalar primary φ^2 appearing in the OPE of two φ's. Crossing symmetry and unitarity imply that $\Delta_{\varphi^2} \leq f(\Delta_\varphi)$ for some numerically determined function $f(\Delta_\varphi)$. The method admits straightforward extensions to bounds on scaling dimensions of tensorial operators, central charges and OPE coefficients.[2]

In this thesis we extend this program to conformal field theories with a boundary. An Euclidean CFT in d dimensions can be defined in the half-space $x_d \geq 0$, with boundary conditions at $x_d = 0$ that preserve an $SO(d,1)$ subgroup of the original $SO(d+1,1)$ conformal symmetry [54, 55]. For a given bulk CFT, different consistent boundary conditions are usually possible. Boundary CFTs (BCFTs) are very interesting in their own right and find diverse physical applications. They describe surface phenomena in systems near criticality, with surface critical exponents related to the conformal dimensions of the boundary operators. In string theory, two-dimensional worldsheet BCFTs are interpreted as D-branes. These would be sufficient reasons to consider the boundary bootstrap, but one of the main questions we would like to address is

[2]Analogous "sum rule" techniques can also be used to obtain non-trivial bounds from modular invariant partition functions, see [49–53].

whether by probing the theory with a boundary one can constrain the original bulk theory itself.[3] One could in fact also go ahead and consider a more general setup where conformal defects of all possible codimensions (boundaries being the special case of codimension one) appear on a democratic footing.

Besides the spectrum of bulk operators and their three-point functions, which are unaffected by the boundary conditions, a BCFT is characterized by additional boundary data: the spectrum of boundary operators, their three-point functions, and the bulk-boundary two-point functions. A correlator containing both bulk and boundary operators can be decomposed in different channels, giving crossing-symmetry constraints that in general involve both bulk and boundary data. We will focus on the simplest non-trivial type of correlator, the two-point function of two bulk operators, which in the presence of a boundary is a non-trivial function of a *single* conformal cross-ratio. It can be decomposed in the bulk channel, by first fusing the two bulk operators together, or in the boundary channel, by taking the boundary OPE of each bulk operator. See figure 5.1 on page 76.

The main advantage of using the boundary bootstrap to constrain bulk dynamics is the simplicity of the setup just described. This follows from the results of section 5.1, where we discuss the two-point function of bulk scalar operators: its functional form and its conformal block decomposition in the bulk and boundary channels. The conformal blocks turn out to be simple (hypergeometric) functions of the single cross-ratio and furthermore depend analytically on the spacetime dimension d. This is to be contrasted with the standard conformal blocks for four-point functions (in a theory with no boundary), which depend on two cross-ratios and admit closed-form expressions only when d is an even integer.

5.1 Boundary crossing symmetry for scalars

In this section we introduce the general setup of boundary CFT and derive the crossing symmetry equations for the two-point function of bulk scalar operators. For background material on BCFTs see [54, 57–60], and especially the paper by McAvity and Osborn [61], whose results we borrow at several points in this and subsequent sections.

[3] A prototype is the the beautiful theory developed by Cardy [56, 57] in $2d$ rational CFTs, which relates the set of consistent boundary conditions with the bulk spectrum and its modular transformation properties.

5.1.1 Scalar two-point function

Let us start by deriving the form of the scalar two-point function in the presence of a boundary, a classic result dating back to [54]. We will use standard Euclidean coordinates $x^\mu = (x^1, \ldots, x^d)$ and consider the half-space defined by $x^d > 0$, the coordinates tangential to the boundary are denoted $\boldsymbol{x}$. It will be useful to embed this physical space in a higher dimensional space as the so-called null projective cone [62, 63]. Consider Minkowski space in d+2 dimensions in lightcone coordinates denoted by $P^A = (P^+, P^-, P^1, \ldots P^d)$. The null projective cone is defined as,

$$P^A P_A = 0 \qquad \text{with} \qquad P^A \sim \lambda P^A \,. \tag{5.1}$$

The map from the null projective cone to our physical space is given by

$$x^\mu = \frac{P^\mu}{P^+} \,. \tag{5.2}$$

One easily finds that the usual $SO(d{+}1,1)$ Lorentz group of the d+2-dimensional Minkowski space becomes the conformal group of the d-dimensional Euclidean space. The null projective cone provides a linearization of the action of the conformal group.

As we mentioned above, the presence of a boundary at $x^d = 0$ breaks the symmetry group to $SO(d,1)$. In the null projective cone this breaking can be implemented by introducting a fixed vector V with components

$$V^A = (0, \ldots, 0, 1) \,, \tag{5.3}$$

and restricting ourselves to those Lorentz transformations that leave V^A invariant. The residual conformal transformations for the coordinates x^μ are easily obtained from the linear transformations of the P^A coordinates.

Let us now consider scalar fields that are homogeneous functions of the coordinates,

$$\mathcal{O}(\lambda P) = \lambda^{-\Delta} \mathcal{O}(P) \,, \tag{5.4}$$

where Δ is the conformal dimension of the field $\mathcal{O}$. The physical CFT scalar operator is defined as

$$O(x) = (P^+)^\Delta \mathcal{O}(P) \,. \tag{5.5}$$

The two-point function of $\mathcal{O}$ should be invariant under $SO(d,1)$ and consistent with (5.4). The only $SO(d,1)$ invariants that can be formed with two coordinates and the fixed vector V^A are

$$P_1 \cdot P_2, \quad V \cdot P_1, \quad \text{and} \quad V \cdot P_2. \tag{5.6}$$

The two-point function must then be of the form

$$\langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)\rangle = \frac{1}{(2V\cdot P_1)^{\Delta_1}(2V\cdot P_2)^{\Delta_2}} f(\xi), \tag{5.7}$$

where $f(\xi)$ is an arbitrary function of the conformal invariant,

$$\xi = \frac{-P_1\cdot P_2}{2(V\cdot P_1)(V\cdot P_2)}. \tag{5.8}$$

In physical coordinates,

$$\xi = \frac{(x_1-x_2)^2}{4x_1^d x_2^d}\,. \tag{5.9}$$

We see that the limit $\xi \to 0$ corresponds to bringing the operators close together while the limit $\xi \to \infty$ amounts to bringing the operators close to the boundary. It will be useful to introduce a function $G(\xi) = \xi^{(\Delta_1+\Delta_2)/2} f(\xi)$, the two-point function then becomes

$$\langle O_1(x_1)O_2(x_2)\rangle = \frac{1}{(2x_1^d)^{\Delta_1}(2x_2^d)^{\Delta_2}} \xi^{-(\Delta_1+\Delta_2)/2} G(\xi). \tag{5.10}$$

For two identical (canonically normalized) operators $\lim_{\xi\to 0} G(\xi) = 1$, since we need to recover the usual two-point function far away from the boundary. Although using the null projective cone is somewhat of an overkill for the scalar two-point function, this formalism will become essential for the tensor calculations of section 5.3.

5.1.2 The boundary bootstrap

Much like a four-point function for a CFT without a boundary, one can decompose the correlation function (5.10) into conformal blocks. In this case there exist two different decompositions (or channels) and we review both of them below.

In the *bulk channel* we simply substitute the bulk OPE in the two-point function (5.10). For two identical scalar operators the bulk OPE takes the form (omitting tensor indices for simplicity):

$$O(x)O(y) = \frac{1}{(x-y)^{2\Delta}} + \sum_k \lambda_k C[x-y, \partial_y] O_k(y)\,, \tag{5.11}$$

where k labels conformal primary fields. The differential operators $C[x-y, \partial_y]$ are determined by the (bulk) conformal symmetry and the couplings λ_k can be taken to be real [3]. We emphasize that this OPE is a local property of the bulk CFT and therefore unaffected by the presence of a boundary. On the other hand, whereas in the absence of any boundaries only the identity operator gets a non-zero one-point function (and all other terms in the OPE therefore drop out of the two-point function of O), this is no longer the case once a boundary is present. Using the null projective cone it is easily demonstrated that boundary conformal invariance allows for one-point functions of scalar operators of the form:

$$\langle O(x) \rangle = \frac{a_O}{(2x^d)^\Delta} \,, \tag{5.12}$$

with a coefficient a_O whose magnitude is unambiguous as we have normalized the operator using the first term in (5.11). One-point functions for operators with spin are not allowed by conformal invariance, see section 5.3.2 below. Substituting now (5.11) in (5.10) and using (5.12) one arrives at the bulk channel conformal block decomposition:

$$G(\xi) = 1 + \sum_k \lambda_k a_k \, f_{\text{bulk}}(\Delta_k; \xi) \,, \tag{5.13}$$

where the *bulk conformal blocks* $f_{\text{bulk}}(\Delta_k; \xi)$ can be determined by working out the expression:

$$C[x-y, \partial_y] \frac{1}{(y^d)^{\Delta_k}} \,. \tag{5.14}$$

This computation was performed in [61], with the result that (see appendix C for a derivation)

$$f_{\text{bulk}}(\Delta_k; \xi) = \xi^{\Delta_k/2} {}_2F_1\left(\frac{\Delta_k}{2}, \frac{\Delta_k}{2}; \Delta_k + 1 - \frac{d}{2}; -\xi\right) . \tag{5.15}$$

Equations (5.13) with the explicit expression (5.15) summarize the bulk block decomposition of the two-point function. Notice that the blocks are naturally defined as a series expansion around $\xi = 0$, which is when the two operators approach each other. Convergence of the OPE away from the boundary however implies that the conformal block decomposition should converge for all physical values of ξ, that is for all $0 < \xi < \infty$.

In the *boundary channel* we use the bulk-to-boundary OPE where a bulk operator is written as an infinite sum over boundary operators. For a scalar operator this OPE takes the form:

$$O(x) = \frac{a_O}{(2x^d)^\Delta} + \sum_l \mu_l D[x^d, \partial_{\boldsymbol{x}}] \hat{O}_l(\boldsymbol{x}) \,, \tag{5.16}$$

where the index l runs over boundary primary fields, the differential operators $D[x^d, \partial_{\boldsymbol{x}}]$ are again completely determined by (boundary) conformal symmetry and the couplings μ_l are again assumed to be real. The first term in (5.16) corresponds to the one-point function of $O(x)$ and represents the contribution of the boundary identity operator. Subsequent operators all have to be scalars by boundary Lorentz invariance. Notice also that in equation (5.16) we used a hat to denote operators living on the boundary (and such operators obviously can depend only on $\boldsymbol{x}$).

The constraints of boundary conformal invariance for the correlation functions of boundary operators $\hat{O}(\boldsymbol{x})$ are exactly the same as those of ordinary conformal invariance in $d-1$ dimensions. This implies in particular that boundary operators cannot get one-point functions and their two-point functions take the canonical form,

$$\langle \hat{O}(\boldsymbol{x}) \hat{O}(\boldsymbol{y}) \rangle = \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|^{2\Delta}} \,, \tag{5.17}$$

which also provides a normalization for boundary operators. Combining now (5.16) and (5.10) and using (5.17) one arrives at the boundary channel conformal block decomposition:

$$G(\xi) = \xi^{\Delta} \left(a_O^2 + \sum_l \mu_l^2 \, f_{\text{bdy}}(\Delta_l; \xi) \right) \,, \tag{5.18}$$

where the *boundary conformal blocks* $f_{\text{bdy}}(\Delta_l; \xi)$ can now be determined from:

$$D[x^d, \partial_{\boldsymbol{x}}] D[y^d, \partial_{\boldsymbol{y}}] \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|^{2\Delta}} \,. \tag{5.19}$$

Just as for the bulk blocks, this computation was done in [61] (and rederived in appendix C),

$$f_{\text{bdy}}(\Delta; \xi) = \xi^{-\Delta} {}_2F_1 \left(\Delta, \Delta + 1 - \frac{d}{2}; 2\Delta + 2 - d; -\frac{1}{\xi} \right) \,. \tag{5.20}$$

The boundary blocks have a good series expansion when both operators approach the boundary, that is around $\xi = \infty$.

The boundary block decomposition is summarized by equations (5.18) and (5.20). The convergence of the bulk-boundary OPE away from other operator insertions implies that this conformal block decomposition should converge for all $0 < \xi < \infty$ as well.

The statement of *crossing symmetry* is nothing more than the fact that the two decompositions (5.13) and (5.18) should agree,

$$\boxed{G(\xi) = 1 + \sum_k \lambda_k a_k \, f_{\text{bulk}}(\Delta_k;\xi) = \xi^{\Delta}\left(a_O^2 + \sum_l \mu_l^2 \, f_{\text{bdy}}(\Delta_l;\xi)\right)} \; . \tag{5.21}$$

A pictorial representation of this equation is shown in figure 5.1. The aim of this thesis is to explore how equation (5.21) can be used to constrain the space of boundary conformal field theories.

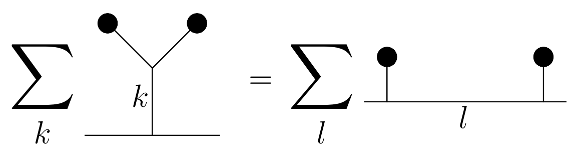

Figure 5.1: Two-point function crossing symmetry in boundary CFT.

5.2 The boundary bootstrap in the epsilon expansion

In this section we demonstrate that in a few special cases it is possible to obtain an *analytic* solution of the crossing symmetry equation (5.21). As we will see below, in this way we can in fact *bootstrap* the outcome of a one-loop computation and recover the order ϵ critical exponents of the Wilson-Fisher fixed point! This is possible because our solutions turn out to have only one or two blocks in either channel and equation (5.21) reduces to a finite-dimensional linear system. This should be constrasted with the conformal block decomposition for the bulk four-point function, whose asymptotic properties dictate that it always decomposes into an infinite number of conformal blocks [3], which makes the problem much harder. The results in this section therefore highlight the relative simplicity of the boundary bootstrap program. At higher orders in the epsilon expansion, the problem becomes infinite-dimensional even in the boundary case, and more powerful methods will have to be developed.

5.2.1 The simplest bootstrap

Let us begin our exploration of the constraining power of the crossing symmetry equation (5.21) by considering the following question: is it possible to satisfy crossing symmetry with just a *single* block in either channel? It turns out that this question can be answered affirmatively and leads to a rederivation of the free-field theory two-point functions. In formulas, our question becomes whether there exists a solution to the equation

$$1 + \lambda a_\eta \, f_{\rm bulk}(\eta;\xi) = \xi^\Delta \left(a_O^2 + \mu^2 \, f_{\rm bdy}(\eta';\xi)\right) , \tag{5.22}$$

for all ξ and with unknowns $\lambda a_\eta, \eta, \Delta, a_O^2$ and η'. We use η and η' to denote the dimensions of the single bulk and boundary operator, respectively.

In order to find a solution we will expand both sides in ξ. The bulk conformal blocks (5.15) have a natural series expansion in powers of ξ around $\xi = 0$, which is when we bring the two points close together. On the other hand, the boundary conformal blocks of equation (5.20) are naturally defined via a series expansion around $\xi = \infty$ where both points approach the boundary.

Now, using standard hypergeometric transformation formulas (see for example [64]), we can expand a boundary block around $\xi = 0$,

$$f_{\rm bdy}(\eta';\xi) = c_1(1 + \ldots) + c_2 \xi^{1-d/2}(1 + \ldots) , \tag{5.23}$$

with the dots representing subleading integer powers of ξ and c_1 and c_2 certain constants. Substituting this expansion into (5.22) and simply matching the powers of ξ to those possibly appearing on the left hand side of (5.22), we directly find that:

$$\Delta = \Delta_\phi \equiv \frac{d}{2} - 1 , \qquad\qquad \eta = 2\Delta_\phi = d - 2 . \tag{5.24}$$

This is our first non-trivial result: the scaling dimension Δ has to be that of a free field ϕ and the value of η reflects the simple free-field bulk OPE, $\phi \times \phi = \mathbf{1} + \phi^2$.

Our next step is to notice that the bulk block with $\eta = 2\Delta_\phi$ becomes particularly simple,

$$f_{\rm bulk}(2\Delta_\phi;\xi) = \left(\frac{\xi}{\xi+1}\right)^{\Delta_\phi} , \tag{5.25}$$

and expanding now both sides of (5.22) around $\xi = \infty$ we find that

$$1 + \lambda a_\eta \left(1 + \frac{1 - d/2}{\xi} + \ldots\right) = \xi^{\Delta_\phi} \left(a_O^2 + \mu^2 \xi^{-\eta'} \left(1 - \frac{\eta'}{2\xi} + \ldots\right)\right) , \tag{5.26}$$

which allows us to solve for all the other coefficients. We find two possible solutions:

$$\begin{aligned} +: \qquad & \lambda a_\eta = +1\,, \qquad a_O^2 = 0\,, \qquad \eta' = \Delta_\phi\,, \qquad \mu^2 = 2\,, \\ -: \qquad & \lambda a_\eta = -1\,, \qquad a_O^2 = 0\,, \qquad \eta' = \Delta_\phi + 1\,, \qquad \mu^2 = \frac{d-2}{2}\,. \end{aligned} \tag{5.27}$$

Although we have only used the series expansions of the conformal blocks around the endpoints $\xi = 0$ and $\xi = \infty$, it turns out that for the above values of the coefficients the crossing symmetry equation is miraculously satisfied at every order in ξ. Therefore, the two functions

$$\begin{aligned} G^+(\xi) &= 1 + f_{\text{bulk}}(2\Delta_\phi;\xi) = \xi^{\Delta_\phi}\Big(2 f_{\text{bdy}}(\Delta_\phi;\xi)\Big) = 1 + \left(\frac{\xi}{\xi+1}\right)^{\Delta_\phi}\,, \\ G^-(\xi) &= 1 - f_{\text{bulk}}(2\Delta_\phi;\xi) = \xi^{\Delta_\phi}\left(\frac{d-2}{2} f_{\text{bdy}}(\Delta_\phi+1;\xi)\right) = 1 - \left(\frac{\xi}{\xi+1}\right)^{\Delta_\phi}\,, \end{aligned} \tag{5.28}$$

are valid solutions to the crossing symmetry equation (5.21) with just a single block in each channel. Using (5.10) we find that they correspond to two-point functions of the form:

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{(x-y)^{2\Delta_\phi}} \pm \frac{1}{(x-y^r)^{2\Delta_\phi}}\,, \tag{5.29}$$

where y^r is the coordinate vector y reflected in the boundary, so if $y = (\boldsymbol{y}, y^d)$ then $y^r = (\boldsymbol{y}, -y^d)$. This equation informs us that we have derived the two possible two-point functions of a free field on a half-space, with the $+$ sign corresponding to Neumann boundary conditions and the $-$ sign corresponding to Dirichlet boundary conditions.

Let us offer a few more comments on the above solutions. First of all, the bulk-to-boundary OPE is consistent with the boundary conditions. Indeed, the bulk-to-boundary OPE of a free field ϕ contains a priori a boundary field $\hat{\phi}$ and its normal derivative $\partial_d\hat{\phi}$ of dimensions Δ_ϕ and $\Delta_\phi + 1$, respectively. (Notice that these are both $SO(d,1)$ primaries.) As expected, in the Dirichlet case the operator $\hat{\phi}$ vanishes by the boundary conditions and only the block corresponding to $\partial_d\hat{\phi}$ is present. In the Neumann case the situation is reversed. Finally, the operator ϕ^2 is the only operator appearing in the bulk channel and the sign of its one-point function is reversed between the two boundary conditions.

5.2.2 Order ϵ bootstrap

Having obtained the scalar two-point function for the free theory, let us apply the bootstrap technique to the interacting theory in the epsilon expansion. In this section we will allow for N massless scalars with strength $\frac{\lambda}{4!}(\phi^2)^2$. The N-dependence of the free two-point function comes from the overall normalization, so the results of the previous section remain unchanged. Defining $d = 4 - \epsilon$, the Wilson-Fisher fixed point is given by

$$\frac{\lambda_*}{16\pi^2} = \frac{3\epsilon}{N+8} + O(\epsilon^2)\,. \tag{5.30}$$

We can now write the bootstrap equations as a perturbation series in ϵ. Following the strategy used in the free case we will assume a finite number of blocks in each channel. In particular, we will consider two non-trivial blocks in the bulk channel and a single block in the boundary channel. This ansatz has some partial justification in Feynman diagrams. In order for an operator O to appear in the bulk OPE of ϕ with itself, the three-point function $\langle\phi\phi O\rangle$ should be non-zero. For operators of the form ϕ^{2n} (ignoring $O(N)$ indices) the only allowed possibilities at order ϵ are ϕ^2 and ϕ^4. For $n > 2$ the correlator is higher order in ϵ, two or more vertices are needed to contract all the legs. In the boundary channel[4] we are only considering the operator $\hat{\phi}$, similarly to the bulk case, the bulk-to-boundary OPE between ϕ and $\hat{\phi}^{2n+1}$ for $n > 0$ is higher order in ϵ. Let us then proceed to bootstrap the order ϵ correlator and comment on the validity of our ansatz at the end of this section.

We want to solve the following equation,

$$1 + \lambda a_{\phi^2} f_{\text{bulk}}(\Delta_{\phi^2};\xi) + \lambda a_{\phi^4} f_{\text{bulk}}(\Delta_{\phi^4};\xi) = \mu^2 \xi^{\Delta_\phi} f_{\text{bdy}}(\Delta_{\hat{\phi}},\xi)\,. \tag{5.31}$$

Because we are working perturbatively we will write all coefficients as a power series in ϵ. For the spacetime dimension d and the external dimension conformal dimension Δ_ϕ we have

$$\begin{aligned} d &= 4 - \epsilon\,, \\ \Delta_\phi &= \frac{d}{2} - 1 + \delta\Delta_\phi \epsilon + O(\epsilon^2)\,. \end{aligned} \tag{5.32}$$

[4] For concreteness we will consider the Neumann case but a parallel analysis can be done for Dirichlet boundary conditions.

For the internal conformal dimensions we write,

$$\begin{aligned}
\Delta_{\phi^2} &= d - 2 + \delta\Delta_{\phi^2}\epsilon + O(\epsilon^2)\,,\\
\Delta_{\phi^4} &= 2d - 4 + \delta\Delta_{\phi^4}\epsilon + O(\epsilon^2)\,,\\
\Delta_{\hat{\phi}} &= \frac{d}{2} - 1 + \delta\Delta_{\hat{\phi}}\epsilon + O(\epsilon^2)\,.
\end{aligned} \tag{5.33}$$

Finally, for the coefficients multiplying the blocks,

$$\begin{aligned}
\lambda a_{\phi^2} &= 1 + \delta\lambda a_{\phi^2}\epsilon + O(\epsilon^2)\,,\\
\lambda a_{\phi^4} &= \delta\lambda a_{\phi^4}\epsilon + O(\epsilon^2)\,,\\
\mu^2 &= 2 + \delta\mu^2\epsilon + O(\epsilon^2)\,,
\end{aligned} \tag{5.34}$$

where the quantities denoted by "δ" correspond to deviations from the free-field solution. For example, λa_{ϕ^4} has only a correction term since it is not present in the free theory. We will again use the transformation formulas that led to (5.23) in order to expand the boundary blocks around $\xi = 0$. The procedure now is the same as before, we Taylor expand both sides of the equation and match equal powers of the parameter ξ. As in the free case, after matching the first few coefficients, equation (5.31) is solved to all orders in ξ. The order ϵ solution is,

$$\begin{aligned}
\delta\Delta_{\phi} &= 0\,, & \delta\Delta_{\phi^2} &= 2\alpha\,, & \delta\Delta_{\hat{\phi}} &= -\alpha\,,\\
\delta\lambda a_{\phi^2} &= \alpha\,, & \delta\lambda a_{\phi^4} &= \frac{\alpha}{2}\,, & \delta\mu^2 &= 0\,,
\end{aligned} \tag{5.35}$$

where α is an arbitrary coefficient. The zero one-loop anomalous dimension for ϕ is not a surprise, the anomalous dimension of ϕ^2 is also well known and can be used to fix the value of α,

$$\alpha = \frac{1}{2}\left(\frac{N+2}{N+8}\right)\,. \tag{5.36}$$

The first order corrections to the OPE coefficients of the ϕ^2 and ϕ^4 blocks are positive, while the order ϵ correction to μ^2 is zero, as expected from Feynman diagrams. We find a negative anomalous dimension for the boundary operator corresponding to $\hat{\phi}$. The anomalous dimension for ϕ^4 does not enter the equations at this order in the expansion. The complete corrected two-point function is then

$$
\begin{aligned}
G^{+}_{\phi\phi} &= 1 + \left(\frac{\xi}{\xi+1}\right)^{1-\frac{\epsilon}{2}} + \frac{\epsilon}{2}\left(\frac{N+2}{N+8}\right)\Big(\frac{\xi}{\xi+1}\log(\xi) + \log(\xi+1)\Big) + O(\epsilon^2) \\
&= 1 + \Big(1 + \frac{\epsilon}{2}\left(\frac{N+2}{N+8}\right)\Big) f_{\text{bulk}}(2-\epsilon+\epsilon\left(\frac{N+2}{N+8}\right);\xi) + \frac{\epsilon}{4}\left(\frac{N+2}{N+8}\right) f_{\text{bulk}}(4;\xi) + O(\epsilon^2) \\
&= \xi^{1-\frac{\epsilon}{2}}\Big(2 f_{\text{bdy}}(1 - \frac{\epsilon}{2} - \frac{\epsilon}{2}\left(\frac{N+2}{N+8}\right);\xi)\Big) + O(\epsilon^2)\,, \qquad (5.37)
\end{aligned}
$$

where the $+$ sign indicates Neumann boundary conditions. An analogous calculation can be done for the Dirichlet case. We simply quote the result:

$$
\begin{aligned}
G^{-}_{\phi\phi} &= 1 - \left(\frac{\xi}{\xi+1}\right)^{1-\frac{\epsilon}{2}} + \frac{1}{2}\epsilon\left(\frac{N+2}{N+8}\right)\Big(-\frac{\xi}{\xi+1}\log(\xi) + \log(\xi+1)\Big) + O(\epsilon^2) \\
&= 1 - \Big(1 - \frac{1}{2}\epsilon\left(\frac{N+2}{N+8}\right)\Big) f_{\text{bulk}}(2-\epsilon+\epsilon\left(\frac{N+2}{N+8}\right);\xi) + \frac{\epsilon}{4}\left(\frac{N+2}{N+8}\right) f_{\text{bulk}}(4;\xi) + O(\epsilon^2) \\
&= \xi^{1-\frac{\epsilon}{2}}\Big(\Big(1 - \frac{\epsilon}{2} + \frac{\epsilon}{2}\left(\frac{N+2}{N+8}\right)\Big) f_{\text{bdy}}(2 - \frac{\epsilon}{2} - \frac{\epsilon}{2}\left(\frac{N+2}{N+8}\right);\xi)\Big) + O(\epsilon^2)\,,
\end{aligned}
\qquad (5.38)
$$

which features only minor changes with respect to the previous case. Comparison of these expressions with the explicit calculation of [61] shows perfect agreement. We have used the bootstrap equations to obtain a one-loop result!

Let us now return to our original ansatz. We did not consider primary operators with derivatives acting on ϕ, which we denote schematically by $\Box^k\phi^2$ and $\Box^k\phi^4$. For the first family, we can never have $\partial_\mu\partial_\mu$ acting on the same field, because the equations of motion imply $\partial_\mu\partial_\mu\phi \sim \epsilon\phi^3$ and the operator is not really of the form $\Box^k\phi^2$. The only possibility is to have $\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_k}\phi\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_k}\phi$, but these operators are conformal descendants, and their contribution is already taken into account by the ϕ^2 block. For the second family, the equations of motion argument still holds, but not all operators are conformal descendants. In fact, there is an infinite number of primaries of the schematic form $\Box^k\phi^4$.[5] Our original ansatz was thus incomplete, we should have added an infinite number of blocks to the left-hand side of equation (5.31) with tree level dimension $\Delta_k = 2(d-2)+2k$. As we obtained the correct answer, it is clear that these operators do not appear at one loop. We believe that this is due to the vanishing of the three-point functions $\langle\phi\phi\Box^k\phi^4\rangle$ for $k>0$, a fact which should follow from the higher-spin Ward identities of the free theory.

Starting at order ϵ^2, crossing symmetry can no longer be solved with a finite

[5]This statement can be checked using conformal characters.

number of blocks. It would be nice to find more powerful analytic techniques to deal with the infinite-dimensional linear system, and develop a bootstrap apprach to the all-order epsilon expansion. At each order a new infinite family of bulk primary operators appears. Perhaps the constraints of sligthly broken higher-spin symmetry [65, 66] could help in organizing the information contained in (5.21).

5.3 Boundary crossing symmetry for stress tensors

In section 5.1 we derived the crossing symmetry equation (5.21) for the two-point function of scalar operators using the bulk and boundary conformal block decompositions. In this section we will derive a similar equation for the two-point function of the stress tensor.

5.3.1 Summary of results

As we show in equation (5.55) below, the two-point function of a spin two operator in the presence of a boundary features three independent tensor structures. Each tensor structure comes multiplied with its own scalar function of ξ and we find it convenient to collect these three functions in a three-component vector of the form $(f(\xi),\, g(\xi),\, h(\xi))$. Furthermore, for the stress tensor the Ward identities relate the three components in the following way:

$$\begin{aligned}(d-2)\xi^2\frac{d}{d\xi}g &= (d^2+3d-2)h - 2(d-1)\xi(1+\xi)\frac{d}{d\xi}h\\ 4d\xi^3\frac{d}{d\xi}f &= -4(1+\xi)h + \Big(\xi(d^2+2d-4) - 2d\xi^2(1+\xi)\frac{d}{d\xi}\Big)g\,,\end{aligned} \tag{5.39}$$

so up to a few integration constants there is effectively only one independent function of ξ.

In the following subsections we derive the conformal block decompositions of the functions (f, g, h) in the bulk and the boundary channel. The main result of these subsections will be the following crossing symmetry equation:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \sum_k \lambda_k a_{\mathcal{O}_k} \begin{pmatrix} f_{\text{bulk}}(\Delta_k;\xi) \\ g_{\text{bulk}}(\Delta_k;\xi) \\ h_{\text{bulk}}(\Delta_k;\xi) \end{pmatrix}$$
$$= \mu^2_{(0)} \begin{pmatrix} f^{(0)}_{\text{bdy}}(d;\xi) \\ g^{(0)}_{\text{bdy}}(d;\xi) \\ h^{(0)}_{\text{bdy}}(d;\xi) \end{pmatrix} + \mu^2_{(1)} \begin{pmatrix} f^{(1)}_{\text{bdy}}(d;\xi) \\ g^{(1)}_{\text{bdy}}(d;\xi) \\ h^{(1)}_{\text{bdy}}(d;\xi) \end{pmatrix} + \sum_n \mu^2_{(2),n} \begin{pmatrix} f^{(2)}_{\text{bdy}}(\Delta_n;\xi) \\ g^{(2)}_{\text{bdy}}(\Delta_n;\xi) \\ h^{(2)}_{\text{bdy}}(\Delta_n;\xi) \end{pmatrix}, \tag{5.40}$$

where all the functions (f, g, h) are explicitly known functions of ξ. Equation (5.40) is the analogue of (5.21) for scalars and we will use it in section 6.4 to obtain bounds on operator dimensions and OPE coefficients. Let us now discuss it in a bit more detail.

First of all, because of the three independent tensor structures we get a three-dimensional vector of equations (and the conformal blocks themselves also become three-dimensional vectors). It is then important to realize that the Ward identities are operator equations and therefore they must be true for the individual conformal blocks as well. Each vector appearing in (5.40) thus *individually* satisfies the Ward identities (5.39).

The left-hand side of (5.39) is the bulk channel conformal block decomposition. As in (5.21), we separated out the conformal block corresponding to the identity operator. For the other operators we should recall that $SO(d,1)$ conformal symmetry dictates that only scalars can get non-zero one-point functions and therefore only scalar blocks can contribute to the bulk channel expansion.

The right-hand side of (5.39) represents the boundary channel conformal block decomposition. A priori, a spin 2 operator has a boundary OPE decomposition involving operators with spins ranging from 0 to 2 and indeed we find all these possibilities in (5.39), where the spins of the exchanged operator is written as the superscript in parentheses. However in this case the Ward identities turn out to further constrain the conformal block decomposition. More specifically, the boundary scalar and vector appearing in the boundary OPE decomposition of $T_{\mu\nu}$ *must* have scaling dimensions equal to the spacetime dimensions, so $\Delta^{(0)} = \Delta^{(1)} = d$. There is thus a unique block for the exchange of a scalar of dimension d and also for a vector of dimension d. These two blocks are the first two terms on the right-hand side of (5.40). On the other hand, the dimensions of the spin 2 fields are not constrained in this way and there can therefore in principle be infinitely many spin 2 blocks, represented by the final sum in (5.40).

Let us offer a few more comments on the spin 0 and 1 boundary operators. As one may have anticipated, in physical theories they correspond to the $\hat{T}_{d\,d}$ and $\hat{T}_{i\,d}$ (i being a tangential index) components of the bulk stress tensor, restricted to the boundary. These operators are intimately related to infinitesimal variations in the location of the boundary surface which explains the "non-renormalization" of their scaling dimensions, see [59] for details. For physical BCFTs the displacement operator $\hat{T}_{d\,d}$ is generically present on the boundary and we encountered it already in the discussion of the extraordinary transition in section 5.2. On the other hand, the vector operator is only present if there is a non-zero energy flow across the boundary. For BCFTs this is an unphysical boundary condition and we can then set $\mu^2_{(1)} = 0$. (Notice that an energy flow would be allowed if the surface $x^d = 0$ was actually an $SO(d,1)$ preserving *interface* between two different theories, one defined for $x^d > 0$ and the other for $x^d < 0$, and in such cases the vector block will generically be present.)

In appendix E we present a few explicit solutions to the crossing symmetry equation (5.40). We discuss the universal solution in two dimensions (which is fully determined by the Virasoro algebra), the free-field theory solutions in d dimensions and the extraordinary transition to leading order for the Wilson-Fisher fixed point.

5.3.2 Correlation functions of tensor operators

In this section we discuss correlation functions of operators with spin in conformal field theories. We will use the results of [67], see also [68], and adapt them to conformal field theories with a boundary. Many of the results in this and the next two subsections were also obtained in [59, 61] but we present here an independent derivation which is straightforwardly implemented on a computer.

The index structures appearing in correlation functions of tensor operators are easily found in the null projective cone formalism discussed in section 5.1.1. According to [67], a generic tensor field $f_{\mu_1\ldots\nu_n}(x)$ lifts to a tensor field $F_{A_1\ldots A_n}(P)$ in the null projective cone with the following properties:

- equal symmetries in the indices of $F_{A_1\ldots A_n}(P)$ and of $f_{\mu_1\ldots\nu_n}(x)$;
- transversality, so $P^{A_i}F_{A_1\ldots A_i\ldots A_n}(P) = 0$ for $1 \leq i \leq n$;

- a gauge equivalence defined as $F_{A_1\dots A_n}(P) \sim P_{A_i}\Lambda_{A_1\dots\hat{A}_i\dots A_n}$ for any Λ and $1 \le i \le n$.

For symmetric traceless tensors it is convenient to contract the indices on F with auxiliary variables Z^A and write $F(P,Z) \equiv F_{A_1\dots A_n}(P)Z^{A_1}\dots Z^{A_n}$. Tracelessness implies that we may restrict ourselves to the subspace defined by $Z^2=0$ and the gauge equivalence implies that we may take $Z\cdot P=0$ as well. The transversality condition becomes:

$$P\cdot\frac{\partial}{\partial Z}F(P,Z)=0\,. \tag{5.41}$$

Correlation functions of n symmetric traceless tensor primary operators can now be written as scalar functions $G(P_i,Z_i)$ with $1\le i\le n$ with the following properties:

- the dependence on Z_i should be a homogeneous polynomial of degree l_i;
- the dependence on P_i should be homogeneous of degree $-\Delta_i$;
- transversality dictates that $P_i\cdot\partial_{Z_i}G=0$ for $1\le i\le n$;
- for any conserved tensor there is a Ward identity of the form [67]

  $$(\partial_P\cdot D^{(d)})\,G=0\,, \tag{5.42}$$

  with

  $$D^{(d)}_A=\Big(\frac{d}{2}-1+Z\cdot\frac{\partial}{\partial Z}\Big)\frac{\partial}{\partial Z^A}-\frac{1}{2}Z_A\frac{\partial^2}{\partial Z\cdot\partial Z}\,, \tag{5.43}$$

  where P and Z are the variables corresponding to the conserved tensor, for example P_1 and Z_1 if the conserved tensor is the first operator.

As an example, let us review the well-known result for the three-point function of two stress tensors and one scalar operator $G_{TT\mathcal{O}}(P_1,P_2,P_3,Z_1,Z_2)$. The first three constraints together dictate that there are three different invariant tensor structures,

$$G_{TT\mathcal{O}}=\frac{1}{(-2P_1\cdot P_2)^{d-\Delta/2}(-2P_2\cdot P_3)^{\Delta/2}(-2P_3\cdot P_1)^{\Delta/2}}\Big(a(W_{12})^2+bH_{12}^2+cH_{12}W_{12}\Big)\,, \tag{5.44}$$

with for now arbitrary constants a, b, c and with building blocks

$$
\begin{aligned}
W_{12} &= \frac{\Big((Z_1 \cdot P_2)(P_1 \cdot P_3) - (Z_1 \cdot P_3)(P_1 \cdot P_2)\Big)\Big((Z_2 \cdot P_1)(P_2 \cdot P_3) - (Z_2 \cdot P_3)(P_1 \cdot P_2)\Big)}{(P_1 \cdot P_2)(P_2 \cdot P_3)(P_3 \cdot P_1)}\,, \\
H_{12} &= \frac{(Z_1 \cdot Z_2)(P_1 \cdot P_2) - (Z_1 \cdot P_2)(Z_2 \cdot P_1)}{P_1 \cdot P_2}\,.
\end{aligned}
\tag{5.45}
$$

The Ward identities for the stress tensor furthermore dictate that:

$$
\begin{aligned}
a &= \frac{\Delta(\Delta+2)}{4d(d+1)}\lambda_{TT\mathcal{O}}\,, \\
b &= \frac{(\Delta-d)^2(d-1)-2d}{d(d+1)(d-2)}\lambda_{TT\mathcal{O}}\,, \\
c &= \frac{\Delta((\Delta-d)(d-1)-2)}{d(d+1)(d-2)}\lambda_{TT\mathcal{O}}\,,
\end{aligned}
\tag{5.46}
$$

where $\lambda_{TT\mathcal{O}}$ is an undetermined overall coefficient. Upon sending $\Delta \to 0$ we find that $a, c \to 0$ but $b \to \lambda_{TT\mathbf{1}}$ and we recover the unit normalized stress tensor two-point function,

$$
\langle T(P_1, Z_1) T(P_2, Z_2) \rangle = \frac{H_{12}^2}{(-2P_1 \cdot P_2)^d}\,, \tag{5.47}
$$

provided we set $\lambda_{TT\mathbf{1}} = 1$. The normalization in (5.46) is therefore such that $\lambda_{TT\mathcal{O}}$ is a natural three-point coupling coefficient.

Let us finally take the OPE limit by sending $P_1 \to P_2$. In that case H_{12} remains finite whilst

$$
W_{12} \to W_{12}^{\mathrm{OPE}} \equiv \frac{(Z_1 \cdot P_2)(Z_2 \cdot P_1)}{(P_1 \cdot P_2)} \tag{5.48}
$$

and therefore

$$
G_{TT\mathcal{O}} \to \frac{a(W_{12}^{\mathrm{OPE}})^2 + bH_{12}^2 + cH_{12}W_{12}^{\mathrm{OPE}}}{(-2P_1 \cdot P_2)^{d-\Delta/2}(-2P_1 \cdot P_3)^{\Delta}}\,, \tag{5.49}
$$

and we infer that the $T \times T \to \mathcal{O}$ operator product expansion becomes to leading order

$$
T(P_1, Z_1) T(P_2, Z_2) \sim \ldots + \frac{a(W_{12}^{\mathrm{OPE}})^2 + bH_{12}^2 + cH_{12}W_{12}^{\mathrm{OPE}}}{(-2P_1 \cdot P_2)^{d-\Delta/2}}\mathcal{O}(P_1) + \ldots \tag{5.50}
$$

where we assumed that $\mathcal{O}$ is normalized such that $\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\rangle = (-2P_1 \cdot P_2)^{-\Delta}$.

As we mentioned in section 5.1.1, the breaking of $SO(d+1,1)$ to $SO(d,1)$ due to the presence of a boundary is implemented by introducing an additional fixed vector

$$V^A = (0, 0, \ldots, 0, 1)\,, \tag{5.51}$$

representing the unit normal to the boundary. Correlation functions are still required to be $SO(d+1,1)$ scalars with the same four properties as above but they can now depend on V^A as well. For example, we have already mentioned that the one-point function of a scalar operator can take the form:

$$\langle \mathcal{O}(P)\rangle = \frac{a_{\mathcal{O}}}{(V\cdot P)^{\Delta}}\,, \tag{5.52}$$

with arbitrary coefficient $a_{\mathcal{O}}$. For one-point functions of tensor operators one directly sees that the numerator would have to involve a factor $(V \cdot Z)^l$ but this is not transverse and so higher-spin one-point functions must vanish.

With two points we can build the invariant object ξ of section 5.1.1,

$$\xi = \frac{-P_1\cdot P_2}{2(V\cdot P_1)(V\cdot P_2)} = \frac{(x_1-x_2)^2}{4x_1^d x_2^d}\,, \tag{5.53}$$

and conformal symmetry thus determines two-point functions only up to arbitrary functions of ξ. For the scalar two-point function this leads to equation (5.10),

$$\langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)\rangle = \frac{1}{(2V\cdot P_1)^{\Delta_1}(2V\cdot P_2)^{\Delta_2}} f_{\mathcal{O}_1\mathcal{O}_2}(\xi)\,, \tag{5.54}$$

where $f_{\mathcal{O}_1\mathcal{O}_2}(\xi)$ is not fixed by conformal symmetry. Two-point functions involving tensors are easily found, e.g.

$$\begin{aligned}
Z_2^A\langle \mathcal{O}(P_1)\mathcal{J}_A(P_2)\rangle &= \frac{(Z_2\cdot V)(P_2\cdot P_1)-(P_2\cdot V)(Z_2\cdot P_1)}{(V\cdot P_1)^{\Delta_{\mathcal{O}}+1}(V\cdot P_2)^{\Delta_{\mathcal{J}}+1}} f_{\mathcal{O}\mathcal{J}}(\xi)\,,\\
Z_2^A Z_2^B\langle \mathcal{O}(P_1)\mathcal{T}_{AB}(P_2)\rangle &= \frac{\Big((Z_2\cdot V)(P_2\cdot P_1)-(P_2\cdot V)(Z_2\cdot P_1)\Big)^2}{(V\cdot P_1)^{\Delta_{\mathcal{O}}+2}(V\cdot P_2)^{\Delta_{\mathcal{T}}+2}} f_{\mathcal{O}\mathcal{T}}(\xi)\,,\\
Z_1^A Z_2^B\langle \mathcal{J}_A(P_1)\mathcal{J}_B(P_2)\rangle &= \frac{f_{\mathcal{J}\mathcal{J}}(\xi)H_{12}+g_{\mathcal{J}\mathcal{J}}(\xi)Q_{12}}{\xi^{\Delta_1}(V\cdot P_1)^{\Delta_1}(V\cdot P_2)^{\Delta_2}}\,,\\
Z_1^A Z_1^B Z_2^C Z_2^D\langle \mathcal{T}_{AB}(P_1)\mathcal{T}_{CD}(P_2)\rangle &= \frac{f_{\mathcal{T}\mathcal{T}}(\xi)H_{12}^2+g_{\mathcal{T}\mathcal{T}}(\xi)H_{12}Q_{12}+h_{\mathcal{T}\mathcal{T}}(\xi)Q_{12}^2}{(4\xi)^{\Delta_1}(V\cdot P_1)^{\Delta_1}(V\cdot P_2)^{\Delta_2}}\,,
\end{aligned} \tag{5.55}$$

with H_{12} already defined above and with

$$Q_{12} = \left(\frac{(V\cdot P_1)(Z_1\cdot P_2)}{(P_1\cdot P_2)} - (V\cdot Z_1)\right)\left(\frac{(V\cdot P_2)(Z_2\cdot P_1)}{(P_1\cdot P_2)} - (V\cdot Z_2)\right). \tag{5.56}$$

If the above tensors are conserved then we write J and T instead of $\mathcal{J}$ and $\mathcal{T}$. In that case $\Delta_J = d-1$ and $\Delta_T = d$ and from the Ward identities we also find that:

$$\begin{aligned}
f_{\mathcal{O}J}(\xi) &= c_{\mathcal{O}J}(\xi(1+\xi))^{-d/2}\,, \\
f_{\mathcal{O}T}(\xi) &= c_{\mathcal{O}T}(\xi(1+\xi))^{-1-d/2}\,, \\
0 &= \left((d+1) - 2\xi\frac{d}{d\xi}\right) g_{JJ} - 2\xi^2\frac{d}{d\xi}\Big(f_{JJ}+g_{JJ}\Big)\,, \\
(d-2)\xi^2 g'_{TT} &= (d^2+3d-2)h_{TT} - 2(d-1)\xi(1+\xi)h'_{TT}\,, \\
4d\xi^3 f'_{TT} &= -4(1+\xi)h_{TT} + \left(\xi(d^2+2d-4) - 2d\xi^2(1+\xi)\frac{d}{d\xi}\right) g_{TT}\,,
\end{aligned} \tag{5.57}$$

with $c_{\dots}$ denoting an integration constant. We see that the two-point function of two stress tensors and the two-point function of two currents are both fixed up to a single function of ξ. The last two equations in (5.57) were already presented in equation (5.39). They agree with equation (2.27) and (2.31) of [61] with the replacements $f(\xi) = C(v)$, $g(\xi) = 4v^2 B(v)$ and $h(\xi) = v^4 A(v)$ and with $v^2 = \xi/(\xi+1)$.

We can also insert operators at boundary points labelled X satisfying $X\cdot V = 0$. As before, we will denote such operators with a hat. We project the indices of such operators to lie along the boundary, which in the null projective cone is implemented by the constraint $V\cdot D^{(d)} = 0$ with the operator $D^{(d)}_A$ already given by (5.43). The correlation functions of interest are those with a single stress tensor in the bulk. We find:

$$\begin{aligned}
Z_2^A Z_2^B \langle \hat{\mathcal{O}}(X_1) T_{AB}(P_2)\rangle &= \delta_{d,\Delta_{\hat{\mathcal{O}}}}\; c_{\hat{\mathcal{O}}T}\frac{\Big((Z_2\cdot V)(P_2\cdot X_1) - (P_2\cdot V)(Z_2\cdot X_1)\Big)^2}{(-2X_1\cdot P_2)^{d+2}}\,, \\
Z_1^A Z_2^B Z_2^C \langle \hat{\mathcal{J}}_A(X_1) T_{BC}(P_2)\rangle &= \delta_{d,\Delta_{\hat{\mathcal{J}}}}\; c_{\hat{\mathcal{J}}T}\frac{\Big((Z_2\cdot V)(P_2\cdot X_1) - (P_2\cdot V)(Z_2\cdot X_1)\Big)\hat{H}_{12}}{(-2X_1\cdot P_2)^{d+1}}\,, \\
Z_1^A Z_1^B Z_2^C Z_2^D \langle \hat{\mathcal{T}}_{AB}(X_1) T_{CD}(P_2)\rangle &= c_{\hat{\mathcal{T}}T}\frac{\hat{H}_{12}^2 - \frac{1}{d-1}Q_{12}^2}{(-2X_1\cdot P_2)^{\Delta_{\hat{\mathcal{T}}}}(V\cdot P_2)^{d-\Delta_{\hat{\mathcal{T}}}}}\,,
\end{aligned} \tag{5.58}$$

with

$$\hat{H}_{12} = \frac{(\hat{Z}_1 \cdot Z_2)(P_1 \cdot P_2) - (\hat{Z}_1 \cdot P_2)(Z_2 \cdot P_1)}{P_1 \cdot P_2}\,, \qquad \hat{Z}_1^A \equiv Z_1^A - (Z_1 \cdot V)V^A\,. \tag{5.59}$$

Notice that for scalars and vectors the scaling dimension is required to be d whereas the dimension of $\hat{\mathcal{T}}$ is unconstrained by the Ward identity.

Up to terms that ensure that $V \cdot D^{(d)}$ annihilates the correlator, two-point functions of boundary operators are of the same form as two-point functions of bulk operators in the absence of a boundary. In particular we find that:

$$\begin{aligned}
\langle \hat{\mathcal{O}}(X_1, Z_1)\hat{\mathcal{O}}(X_2, Z_2)\rangle &= \frac{1}{(-2X_1 \cdot X_2)^{\Delta}}\,,\\
\langle \hat{\mathcal{J}}(X_1, Z_1)\hat{\mathcal{J}}(X_2, Z_2)\rangle &= \frac{H_{12} - (V \cdot Z_1)(V \cdot Z_2)}{(-2X_1 \cdot X_2)^{\Delta}}\,,\\
\langle \hat{\mathcal{T}}(X_1, Z_1)\hat{\mathcal{T}}(X_2, Z_2)\rangle &= \frac{\Big(H_{12} - (V \cdot Z_1)(V \cdot Z_2)\Big)^2 - \frac{1}{d-1}(V \cdot Z_1)^2(V \cdot Z_2)^2}{(-2X_1 \cdot X_2)^{\Delta}}\,.
\end{aligned} \tag{5.60}$$

Equation (5.60) defines our normalization of the boundary operators. Notice that H_{12} descends from the projective cone to $z_1^\mu z_2^\nu(\delta_{\mu\nu} - 2x_{12,\mu}x_{12,\nu}/x_{12}^2)$ so it is easily verified that our normalization is consistent with reflection positivity. Using (5.58) and (5.60) we find the bulk-to-boundary OPE of the stress tensor,

$$T(P, Z) \to c_{\hat{\mathcal{O}}T}(Z \cdot V)^2\hat{\mathcal{O}}(X) - c_{\hat{\mathcal{J}}T}(Z \cdot V)\hat{\mathcal{J}}(X, Z) + \frac{c_{\hat{\mathcal{T}}T}}{(V \cdot P)^{d-\Delta_{\hat{\mathcal{T}}}}}\hat{\mathcal{T}}(X, Z) + \dots \tag{5.61}$$

5.3.3 Bulk channel blocks for the stress tensor

In this subsection we compute the conformal blocks for the two-point function of the stress tensor using the conformal Casimir differential equation method of [69]. These are the conformal blocks appearing on the left-hand side of (5.40).

On a symmetric traceless tensor $F(P, Z)$ the action of an element L_{AB} of $SO(d + 1, 1)$ takes the form:

$$L_{AB}F(P, Z) = \Big(P_A\frac{\partial}{\partial P^B} - P_B\frac{\partial}{\partial P^A} + \frac{1}{\frac{d}{2} + l - 1}(Z_A D_B^{(d+2)} - Z_B D_A^{(d+2)})\Big)F(P, Z)\,, \tag{5.62}$$

with the operator $D_A^{(d+2)}$ given by (5.43) but with $d \to d+2$ since we are rotating in $d+2$ dimensions. The conformal Casimir equation is then:

$$\frac{1}{2} L_{AB} L^{AB} F(P,Z) = -C_{\Delta,l} F(P,Z) \,, \tag{5.63}$$

with $C_{\Delta,l} = \Delta(\Delta-d) + l(l+d-2)$. We used this equation in appendix C to find the result (C.6) for the conformal block in the bulk channel for a scalar two-point function. For two stress tensors the conformal block can be written as:

$$G_b^\Delta(P_1,P_2,Z_1,Z_2) = \frac{f_b(\xi) H_{12}^2 + g_b(\xi) H_{12} Q_{12} + h_b(\xi) Q_{12}^2}{(4\xi)^d (V\cdot P_1)^d (V \cdot P_2)^d} \,, \tag{5.64}$$

and the constraint $\frac{1}{2}(L^{(1)}_{AB} + L^{(2)}_{AB})(L^{(1)AB} + L^{(2)AB}) G^\Delta = -C_{\Delta,0} G^\Delta$ together with the Ward identities leads to the unique solution for the coefficients:

$$h_b = \frac{\Delta(\Delta+2)}{16d(d+1)} (4\xi)^{\Delta/2+2} {}_2F_1\left(2+\frac{\Delta}{2}, 2+\frac{\Delta}{2}; 1-\frac{d}{2}+\Delta; -\xi\right) \,, \tag{5.65}$$

with f_b and g_b determined by the Ward identities (5.57). Let us verify the normalization by taking the OPE limit $\xi \to 0$. We already mentioned that H_{12} then remains finite and it is not hard to find that

$$Q_{12} \to -\frac{1}{2\xi} W_{12}^{\text{OPE}} \,, \tag{5.66}$$

with W_{12}^{OPE} defined in (5.48). From the expansion of (5.65) and the Ward identities we find

$$\begin{aligned}
h_b &= (4\xi)^{\Delta/2}(4\xi^2 \hat{a} + O(\xi)) \,, & \hat{a} &= \frac{\Delta(\Delta+2)}{4d(d+1)} \,, \\
f_b &= (4\xi)^{\Delta/2}\Big(\hat{b} + O(\xi)\Big) \,, & \hat{b} &= \frac{(\Delta-d)^2(d-1)-2d}{d(d+1)(d-2)} \,, \\
g_b &= (4\xi)^{\Delta/2}\Big(-2\xi\hat{c} + O(\xi) \,,\Big) & \hat{c} &= \frac{\Delta((\Delta-d)(d-1)-2)}{d(d+1)(d-2)} \,,
\end{aligned} \tag{5.67}$$

and the entire block behaves as:

$$G_b^\Delta(P_1,P_2,Z_1,Z_2) = \frac{\hat{a}(W_{12}^{\text{OPE}})^2 + \hat{b} H_{12}^2 + \hat{c} H_{12} W_{12}^{\text{OPE}}}{(-2P_1\cdot P_2)^{d-\Delta/2} (V\cdot P_1)^\Delta} \,, \tag{5.68}$$

which is compatible with (5.46), (5.50) and (5.52).

Explicit expressions for f_b and g_b are also available in terms of linear combinations of ${}_2F_1$ hypergeometric functions.

The identity block can be found by sending $\Delta \to 0$. We then find that $f_b = 1$ and $g_b = h_b = 0$.

5.3.4 Boundary channel blocks for the stress tensor

We label the boundary block associated to a primary operator of dimension Δ and spin l as $G_s^{(\Delta,l)}$ (with a subscript "s" for surface). Each block has again the same form as the TT two-point function given in (5.55) with three associated functions $f_s^{(\Delta,l)}$, $g_s^{(\Delta,l)}$ and $h_s^{(\Delta,l)}$. In the two-point function of the stress tensor there are three types of boundary blocks, $G_s^{(d,0)}$, $G_s^{(d,1)}$ and $G_s^{(\Delta,2)}$. To find these blocks we act with the $SO(d,1)$ Casimir operator on one of the two points and solve the resulting differential equation. In the equations below we use $h \equiv d/2$.

For a block corresponding to the exchange of a boundary scalar of dimension d we find:

$$\begin{aligned}
h_s^{(d,0)} &= \frac{1}{2h(2h+1)} \xi^{h+1} (1+\xi)^{-h-3} \Big(2h(2h+1)\xi^2 + 2(2h+1)(h-1)\xi + h(h-1) \Big) \, , \\
g_s^{(d,0)} &= \frac{1}{h(2h+1)} \xi^{h} (1+\xi)^{-h-2} (h + \xi + 2h\xi) \, , \\
f_s^{(d,0)} &= \frac{1}{4h(2h+1)} \xi^{h-1} (1+\xi)^{-h-1} \, ,
\end{aligned} \tag{5.69}$$

where we already fixed the normalization. In the limit where $\xi \to \infty$ we find that only the third tensor structure contributes and

$$G_s^{(d,0)}(P_1, P_2, Z_1, Z_2) \sim \frac{(V \cdot Z_1)^2 (V \cdot Z_2)^2}{(-2P_1 \cdot P_2)^{2h}} \, , \tag{5.70}$$

which agrees with (5.61) and the first equation in (5.60).

For the block corresponding to the exchange of a boundary vector of dimension d we find:

$$
\begin{aligned}
h_s^{(d,1)} &= \frac{1}{2(2h+1)}\xi^{h+1}(1+\xi)^{-h-3}\Big(-2(2h+1)\xi^2+2h(h-1)\xi+h(h-1)\Big)\,,\\
g_s^{(d,1)} &= \frac{1}{(2h+1)}\xi^{h}(1+\xi)^{-h-2}\Big(\xi^2+h(1+2\xi+2\xi^2)\Big)\,,\\
f_s^{(d,1)} &= \frac{1}{4(2h+1)}\xi^{h-1}(1+\xi)^{-h-1}(1+2\xi)\,,
\end{aligned}
\tag{5.71}
$$

and the block behaves for $\xi\to\infty$ as

$$
G_s^{(d,1)}(P_1,P_2,Z_1,Z_2)\sim\frac{(V\cdot Z_1)(V\cdot Z_2)\Big(H_{12}-(V\cdot Z_1)(V\cdot Z_2)\Big)}{(-2P_1\cdot P_2)^{2h}}\,, \tag{5.72}
$$

which is again consistent with the formulas given above.

Finally, for the spin two blocks:

$$
\begin{aligned}
h_s^{(\Delta,2)} &= \frac{2(h-1)}{2h-1}(4\xi)^{2h-\Delta}{}_3F_2\Big(2+\Delta,3-2h+\Delta,1-h+\Delta;1-2h+\Delta,2-2h+2\Delta;-\frac{1}{\xi}\Big)\,,\\
g_s^{(\Delta,2)} &= -2(4\xi)^{2h-\Delta}+O(\xi^{-1})\,,\\
f_s^{(\Delta,2)} &= (4\xi)^{2h-\Delta}+O(\xi^{-1})\,,
\end{aligned}
\tag{5.73}
$$

where $g_s^{(\Delta,2)}$ and $f_s^{(\Delta,2)}$ can also be explicitly written as a sum over two hypergeometric functions. As we send $\xi\to\infty$ we recover that

$$
G_s^{(\Delta,2)}(P_1,P_2,Z_1,Z_2)\sim\frac{\Big(H_{12}-(V\cdot Z_1)(V\cdot Z_2)\Big)^2-\frac{1}{d-1}(V\cdot Z_1)^2(V\cdot Z_2)^2}{(V\cdot P_1)^{2h-\Delta}(V\cdot P_2)^{2h-\Delta}(-2P_1\cdot P_2)^{\Delta}}\,, \tag{5.74}
$$

which is again consistent with the formulas given above.

Chapter 6

Numerical Analysis and Results

Review of Statistical Mechanics

Before proceeding with the numerical analysis it will be useful to review some statistical mechanics concepts. In the study of critical systems with a boundary it is known that Neumann boundary conditions for the Landau-Ginzburg field ϕ (which corresponds to the bulk spin operator σ) describe the so-called *special transition*, while Dirichlet boundary conditions describe the *ordinary transition*. The phase diagram of the Ising model in the presence of a boundary is shown in figure 6.1.

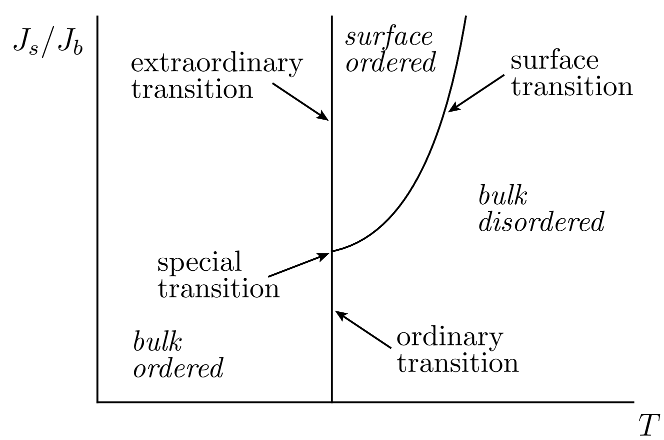

Figure 6.1: Phase diagram for the surface critical behavior of the Ising model in dimension $2 < d < 4$. Temperature is plotted on the horizontal axis and the (relative) surface interaction strength on the vertical axis. The extraordinary transition disappears for $d = 4$, while the special transition is absent in $d = 2$.

In our investigations the bulk is always critical so we are always on the vertical line in figure 6.1. For weak boundary interactions one finds the *ordinary* transition where the boundary follows the bulk and also orders at the bulk critical temperature. There is a critical boundary interaction strength where the boundary and bulk critical temperature just coincide which is the *special* transition. Finally, in the presence of strong boundary interactions the boundary can order at a higher temperature than the bulk. The bulk transition where the boundary is already ordered is then called an *extraordinary* transition. This transition breaks the $\mathbb{Z}_2$ symmetry of the Ising model, as ϕ should acquires a one-point function of the form (5.12). The extraordinary transition cannot be described in free-field theory (such a one-point function does not satisfy the free equations of motion), but it appears at first order in the Wilson-Fisher fixed point in $4-\epsilon$ dimensions, see appendix D.4. We refer the reader to [70, 71] for introductions to boundary critical phenomena.

The BCFT associated to the extraordinary transition is the most "stable" as there are no relevant boundary scalar operators. In fact it is believed that its lowest-dimensional boundary scalar is the "displacement operator" $\hat{T}_{dd}$, which is the boundary limit of the bulk stress tensor with both indices pointing in the direction normal to the boundary. The displacement operator has protected conformal dimension exactly equal to d, and it is thus irrelevant on the $(d-1)$-dimensional boundary. The BCFTs associated to the ordinary and special transitions preserve the $\mathbb{Z}_2$ symmetry, which thus remains a good quantum number for boundary operators. The boundary spectrum of the BCFT associated to the ordinary transition contains a single relevant scalar operator which is $\mathbb{Z}_2$ odd, and corresponds to $\partial_d\hat{\phi}$ in the Landau-Ginzburg description. Finally there are *two* relevant scalars in the BCFT for the special transition, one $\mathbb{Z}_2$ odd and the other $\mathbb{Z}_2$ even, corresponding respectively to $\hat{\phi}$ and $\hat{\phi}^2$.

In $d=2$, the extraordinary transition is associated to the Cardy boundary states $|\mathbf{1}\rangle\!\rangle$ and $|\varepsilon\rangle\!\rangle$ labelled by the identity and the energy, respectively. We have

$$
\begin{aligned}
|\mathbf{1}\rangle\!\rangle &= \frac{1}{\sqrt{2}}|\mathbf{1}\rangle + \frac{1}{\sqrt{2}}|\varepsilon\rangle + \frac{1}{\sqrt[4]{2}}|\sigma\rangle\,, \\
|\varepsilon\rangle\!\rangle &= \frac{1}{\sqrt{2}}|\mathbf{1}\rangle + \frac{1}{\sqrt{2}}|\varepsilon\rangle - \frac{1}{\sqrt[4]{2}}|\sigma\rangle\,,
\end{aligned}
\tag{6.1}
$$

where the kets on the right-hand side denote Ishibashi states. We see that the two states are physically equivalent since they are related by $\mathbb{Z}_2$ conjugation. The ordinary transition is associated instead to the Cardy boundary state $|\sigma\rangle\!\rangle$

labelled by the spin, which is given by

$$|\sigma\rangle\rangle = |\mathbf{1}\rangle - |\varepsilon\rangle\,. \tag{6.2}$$

There is no $2d$ BCFT associated to the special transition, since the one-dimensional boundary cannot order dynamically at non-zero temperature and so the surface transition is absent.

6.1 Implementation for Scalars

In this section we adapt the numerical methods of [3] to the boundary bootstrap equations and derive exclusion curves for operator dimensions and OPE coefficients. The results we obtain below will depend sensitively on some assumptions about the boundary operator spectrum and thereby fall naturally into different categories related to the different possible boundary conditions. Following [1] we will focus mainly on correlation functions of the σ operator in the three-dimensional Ising model, whose possible boundary conditions were presented in figure 6.1. For reasons to be discussed in subsection 6.1, our focus will be on the *special* and *extraordinary* transitions, which will respectively be discussed in subsections 6.2 and 6.3 below. The relevant bulk and boundary operator product expansions and scaling dimensions are summarized in table 6.1. For $d = 4$ there are several operators that do not appear in OPE and we indicated this with a dash. The quoted values for the Ising model in $d = 3$ are of course approximate, but good enough for the numerical precision of this paper. We were unable to find a reliable estimate of the dimension of the $\hat{\sigma}'$ operator for the special transition.

Let us review how to implement the optimization problem numerically. The following techniques were explained in great detail in [3, 44] so we shall be brief. We start by isolating the contribution of the identity operator in equation (5.21),

$$1 = -\sum_k \lambda_k a_k\, f_{\mathrm{bulk}}(\Delta_k;\xi) + \xi^{\Delta_{\mathrm{ext}}}\left(a_O^2 + \sum_l \mu_l^2\, f_{\mathrm{bdy}}(\Delta_l;\xi)\right)\,, \tag{6.3}$$

and introduce the compact notation,

$$1 = \sum_\Delta p_\Delta F_\Delta(\xi)\,, \tag{6.4}$$

bulk			
$\sigma \times \sigma = 1 + \varepsilon + \varepsilon' + \varepsilon'' + \cdots$			
d	2	3	4
Δ_σ	$\frac{1}{8}$	0.5182(3)	1
Δ_ϵ	1	1.413(1)	2
$\Delta_{\epsilon'}$	4	3.84(4)	-
$\Delta_{\epsilon''}$	8	4.67(11)	-

special		
$\sigma = \hat{\sigma} + \hat{\sigma}' + \cdots$		
d	3	4
$\Delta_{\hat{\sigma}}$	0.42	1
$\Delta_{\hat{\sigma}'}$	?	-

extraordinary
$\sigma = 1 + \hat{T}_{dd} + \cdots$

Table 6.1: Bulk and boundary operator product expansions and operator dimensions in the Ising model in various dimensions. There is no special transition in two dimensions. For the extraordinary transition the first boundary operator is $\hat{T}_{dd}$ whose dimension is always equal to the spacetime dimension d. The results for $d = 3$ are approximate and were obtained from [1, 2] whereas the results for $d = 2$ and $d = 4$ can be found in the appendices.

where

$$
\begin{aligned}
p_\Delta &= \left(\lambda_k a_k \,, a_O^2 \,, \mu_l^2\right) , && (6.5)\\
F_\Delta(\xi) &= \left(-f_{\text{bulk}}(\Delta_k;\xi) \,, \xi^{\Delta_{\text{ext}}} \,, \xi^{\Delta_{\text{ext}}} f_{\text{bdy}}(\Delta_l;\xi)\right) . && (6.6)
\end{aligned}
$$

With these definitions equation (6.4) is analogous to the sum rule of [3]. There is however a crucial difference between the boundary problem that we are studying compared to the four-point function crossing symmetry of [3]: even assuming unitarity (as we shall always do) the coefficients p_Δ are not all guaranteed to be positive. They are certainly positive in the boundary channel, since they are squares of real numbers, but in the bulk channel the combination $\lambda_k a_k$ is not manifestly positive. Indeed it is not difficult to find counterexamples (such as a free scalar with Dirichlet boundary conditions). In the following, we will *assume* positivity for the bulk expansion such that $p_\Delta \geq 0$ as in the four-point function case. The conjecture is that for a given bulk CFT, there exists a choice of boundary conditions that exhibits positivity. In the Ising model, the ordinary transition is excluded from our analysis, since both signs occur in the bulk expansion (as can be demonstrated in $d = 2$ and in $d = 4 - \epsilon$ dimensions). We will however assume positivity for the special and the extraordinary transitions. This assumption is supported by the results in the previous section as well as in appendix D. We have found positivity of the bulk block coefficients around $d = 4$, both for the free field and the Wilson-Fisher fixed point at order ϵ, as well as in $d = 2$ where it is a consequence of the positivity of the first two coefficients in the first line of (6.1). In appendix D.7 we also found that the coefficients for the special transition are positive in the $O(N)$ model at large N for any dimension.

We are now ready to start extracting information from the sum rule (6.4). The simplest possible bound can be obtained as follows: We allow for the bulk spectrum to span all possible values consistent with unitarity,

$$\Delta_{\text{bulk}} \geq \frac{d-1}{2}\,, \tag{6.7}$$

while restricting the boundary spectrum to be greater than a given value,

$$\Delta_{\text{bdy}} \geq \Delta_{\text{min}}\,. \tag{6.8}$$

Then, we consider a functional Λ with the following properties,

$$\begin{aligned} \Lambda(1) \ &< \ 0\,, & (6.9)\\ \Lambda(F_\Delta) \ &\geq \ 0\,, & (6.10) \end{aligned}$$

where, according to our definitions, F_Δ stands for any of the blocks appearing in (6.6) with scaling dimensions obeying (6.7) and (6.8). If such a functional is found, equation (6.4) becomes inconsistent and we can rule out that particular CFT. The idea then is to see how high we can push Δ_{min}.

Before implementing the machinery of linear functionals we need to choose a set of "coordinates" in our function space. We will parametrize the blocks by an infinite vector of derivatives $\{F^k_\Delta\}$ evaluated at $\xi = 1$,

$$F^k_\Delta = \left.\frac{\partial^k F_\Delta(\xi)}{\partial \xi^k}\right|_{\xi=1}\,, \tag{6.11}$$

and crossing symmetry becomes now an infinite set of algebraic equations. In order to make the problem numerically tractable we will discretize the spectrum of bulk and boundary dimensions and consider a maximum number of derivatives. With this truncation we have an optimization problem with a finite dimensional set of inequalities, this is an example of a *linear program*. In order to solve the linear programs we used the *Mathematica* routine `LinearProgamming` and the IBM ILOG CPLEX Optimizer. In all our plots below we used a grid of 0.01 and a total of 15 derivatives.

6.2 Special transition

In the following we present our numerical results for the special transition. The one-point function of the bulk spin operator σ vanishes since the $\mathbb{Z}_2$ symmetry is unbroken by the (Neumann) boundary conditions. As we have emphasized in the previous subsection, positivity of the bulk channel coefficients will be a working assumption.

6.2.1 Simplest bound for the boundary channel

Let us start by plotting the simplest possible bound of the form described above. Our only assumption for the bulk spectrum will be the three-dimensional unitarity bound, $\Delta_{\text{bulk}} \geq 0.5$, but otherwise bulk operators of any dimension are allowed to appear in the OPE. Crossing symmetry and positivity however imply that the conformal dimension of the lowest dimension boundary operator cannot be arbitrary. Instead, we found that depending on the external dimension the first boundary operator has to lie below the curve of figure 6.2.

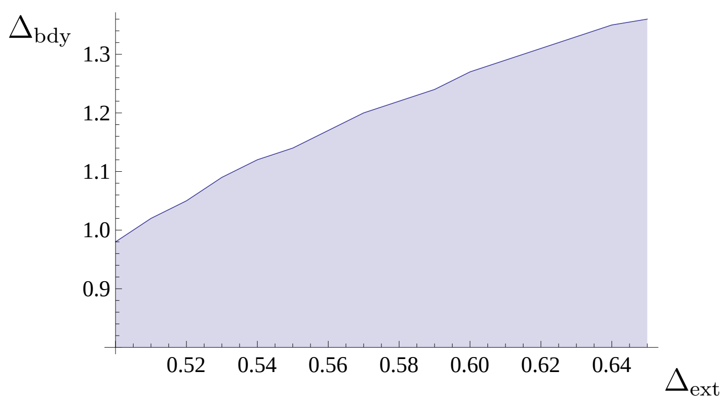

Figure 6.2: Upper bound for the first boundary operator in the special transition.

Although this is a correct bound, we should mention the following caveat: The bulk block blows up at the unitarity bound and our more precise assumption for the bulk spectrum was actually $\Delta_{\text{bulk}} \geq 0.5 + 10^{-6}$. Unfortunately, it turns out that the numerics are quite sensitive around this point. For example, the bound becomes much stronger if we change our assumptions on the bulk spectrum to $\Delta_{\text{bulk}} \geq 0.51$. Because of this, we do not consider this plot to be particularly relevant but it serves as a good warm-up example before tackling the most interesting cases below.

6.2.2 Improved bound for the boundary channel

The boundary bound obtained above can be improved by making further assumptions. In the bulk channel decomposition of a scalar two-point function we expect, on physical grounds, a “gap” between the unitarity bound and the conformal dimension of the first operator appearing in the bulk OPE. For example, according to table 6.1, in the three-dimensional Ising model the first bulk operator appearing in the OPE of the spin operator σ is the energy operator ε with $\Delta_\varepsilon = 1.41$, far above the unitarity bound. Clearly, allowing for the bulk spectrum to go all the way down to the unitarity bound is very unphysical. In figure 6.3 we present an improved bound in which we assumed that the bulk spectrum satisfies $\Delta_{\text{bulk}} \geq 2\Delta_{\text{ext}}$.

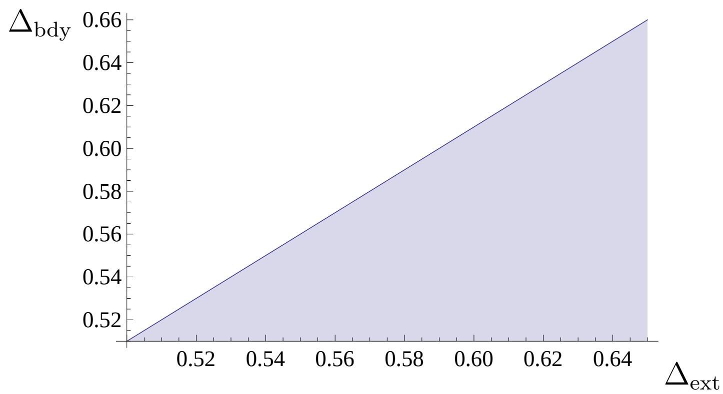

Figure 6.3: Improved bound for the first boundary operator in the special transition. The bulk spectrum is assumed to satisfy $\Delta_{\text{bulk}} \geq 2\Delta_{\text{ext}}$.

Our solution seems to indicate that the bound cannot go below the straight line where $\Delta_{\text{bdy}} = \Delta_{\text{ext}}$. The reason for this is the trivial solution $(x_1 - x_2)^{-2\Delta_{\text{ext}}}$ which we discuss in appendix D.5. This two-point function contains no non-trivial bulk blocks and thus effectively has an infinite gap in the bulk spectrum. On the other hand, it also has a boundary channel expansion which starts with a block of dimension Δ_{ext} and our bound of course cannot get past this particular solution. In a sense, the bound is optimal in this case, going down until it hits a known solution to crossing symmetry.

For the Ising model the dimension of the first boundary operator has a value of ~ 0.42 and is well inside the allowed region of figure 6.3. Ideally, we would have found a plot with some striking feature around this value, like the kink of [1]. However, in our case the trivial solution is standing in the way. A qualitative explanation for this difference appears in the epsilon expansion results. Namely, the anomalous dimension of the ε operator (which is ϕ^2 in

$d = 4$) is *positive* at one loop, so the Ising model lies *above* any trivial (mean field-like) solutions for the bulk four-point function. On the other hand, the one-loop anomalous dimension of the first boundary operator is *negative*, so we end up *below* the trivial solution. This was of course largely a coincidence - we are not aware of any fundamental reason requiring these anomalous dimensions to have a definite sign. Some effort was made in order to circumvent the trivial solution but we did not succeed in obtaining reliable "kinks" that highlight the presence of the Ising model.

We would like to stress however that our plot is still teaching us something very non-trivial: the lowest boundary dimension can never be greater than the external dimension. Interestingly, this result precisely implies that the bulk-to-boundary OPE is never regular, see equation (5.21). It would be very interesting to find a more direct argument for this result —perhaps even one that does not rely on our specific assumptions.

6.2.3 Bounding the second boundary operator in the Ising model

Our assumptions in the previous section were almost minimal, and the result is a general bound valid on the space of BCFTs. In this section we will take a closer look at the three-dimensional Ising model and attempt to bound the second boundary operator. We will do so for both the $\langle\sigma\sigma\rangle$ and $\langle\varepsilon\varepsilon\rangle$ correlators. Using the results from table 6.1 we can assume that[1]

$$\begin{aligned}
\Delta_{\text{ext}} &= 0.518\,, \\
\Delta_{\text{bulk}} &\geq 1.41\,, \\
\Delta^{(1)}_{\text{bdy}} &\sim 0.42\,, \\
\Delta^{(2)}_{\text{bdy}} &\geq \Delta^{(2)}_{\text{min}}\,.
\end{aligned} \tag{6.12}$$

In the boundary channel the first block corresponds to $\hat{\sigma}$. We assume that it sits isolated at $\Delta^{(1)}_{\text{bdy}} \sim 0.42$ and that all the subsequent blocks have a scaling dimension greater than $\Delta^{(2)}_{\text{min}}$. Proceeding as before we push $\Delta^{(2)}_{\text{min}}$ as high as possible until the CFT becomes inconsistent. This will give us an upper bound for the dimension of the second operator $\hat{\sigma}'$, only valid for the $\langle\sigma\sigma\rangle$ correlator of the 3d Ising model. Because ~ 0.42 is our less precise value we will explore a range around this number. Our result is shown in figure 6.4.

The same can be done for the $\langle\varepsilon\varepsilon\rangle$ correlator. The statistical mechanics

[1]The number in parenthesis label the operators on the boundary channel,"(1)" being the lowest.

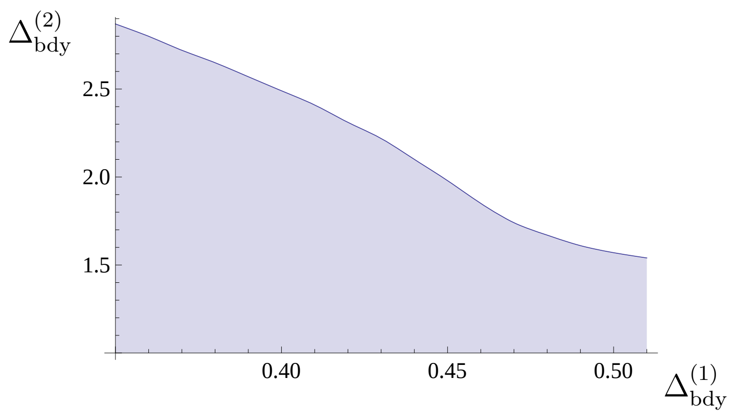

Figure 6.4: Upper bound for the dimension of the second boundary operator in $\langle\sigma\sigma\rangle$ as a function of the dimension of the first boundary operator.

data [2] in this case are

$$\begin{aligned} \Delta_{\text{ext}} &= 1.41\,, \\ \Delta_{\text{bulk}} &\geq 3.80\,, \\ \Delta^{(1)}_{\text{bdy}} &\sim 0.75\,, \\ \Delta^{(2)}_{\text{bdy}} &\geq \Delta^{(2)}_{\text{min}}\,. \end{aligned} \tag{6.13}$$

and the resulting bound is shown in figure 6.5.

Unfortunately, we were unable to find reliable estimates of the scaling dimensions of the second boundary operators in the statistical mechanics literature. It would of course be interesting to compare our values with *e.g.* a two-loop computation for the Wilson-Fisher fixed point.

6.3 Extraordinary transition

In the extraordinary transition the boundary identity operator is always present, so bounding the lowest boundary dimension is not an interesting exercise in this case. The second boundary scalar operator is expected to be $\hat{T}_{dd}$, the energy momentum tensor with indices in the normal direction, evaluated on the boundary. This operator is always present in the boundary spectrum and has conformal dimension exactly equal to d, see [59] for details. Having so much information about the boundary channel we would like to address the following question: can we bound the bulk spectrum using the boundary bootstrap? We will show below that this is indeed possible, although our bound is weaker

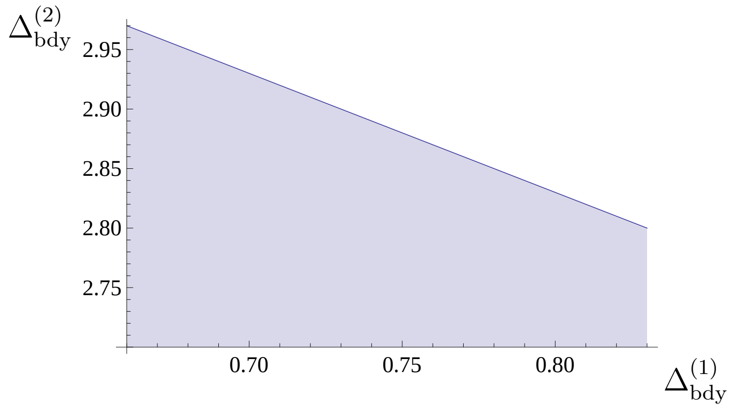

Figure 6.5: Upper bound for the second boundary operator in $\langle\varepsilon\varepsilon\rangle$ as a function of the first boundary operator.

than the one obtained in [1] who used the crossing symmetry equations for the bulk four-point function.

6.3.1 Bound for the bulk channel

The assumptions for the extraordinary transition are

$$\begin{aligned} \Delta_{\rm bulk} &\geq \Delta_{\rm min}\,. \\ \Delta^{(1)}_{\rm bdy} &= 0\,, \\ \Delta^{(2)}_{\rm bdy} &\geq d\,, \end{aligned} \tag{6.14}$$

where we used a notation familiar from the previous subsection. The fact that $\Delta^{(1)}_{\rm bdy} = 0$ corresponds to the boundary identity operator which sits isolated, and we then allow for any operator with a dimension greater than (or equal to) d to be present in the boundary channel. $\Delta_{\rm min}$ is the lowest bulk dimension and the quantity we want to bound. In figure 6.6 we plot our bound as a function of the external dimension. Because figure 6.6 can be directly compared with the bound of [1] we have superimposed their result on our plot. We can see that the bound obtained using the boundary bootstrap is qualitatively different, it is weaker and has no kink at the Ising point. Since we successfully found an "optimal" bound for the boundary spectrum in the previous subsection, it is surprising that our bulk bound does not exhibit any of the expected features.

There are two possible explanations for the discrepancy seen in figure 6.6. First, there may be a spurious solution to crossing symmetry that we have not found yet and that prevents the bound from going lower. If such a solution

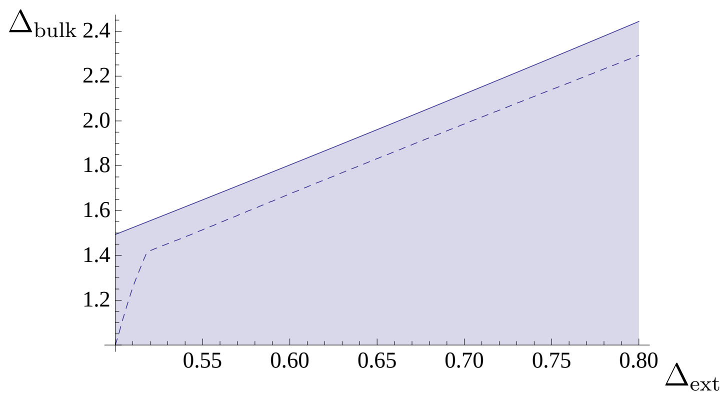

Figure 6.6: Bulk bound for the extraordinary transition as a function of the external dimension. The dashed line corresponds to the (stronger) bound obtained in [1] using the bulk crossing symmetry equations.

exists then it would be interesting to understand whether it corresponds to a full-fledged BCFT or not. Notice that this solution would appear to violate the bound of [1] but this may be due to the fact that certain operators do not get one-point functions and therefore do not appear in our bulk block expansion. The second explanation is that our numerics are not precise enough and that we would be able to lower the bound by increasing our numerical precision. We offer some comments on this second possibility below.

Bulk bound for arbitrary d

One of the advantages of studying the boundary problem is that the blocks are an analytic function of d. In figure 6.7 we plot the bulk bound obtained above for different dimensions including non-integer values.

The bound we find is always significantly different from any known solutions to crossing symmetry. In particular, in the figure we have shown the line interpolating through the minimal models in $d = 2$ and the Ising model for the integral dimensions. Again, it would be interesting to understand if this is due to our finite numerical precision or whether there exist 'spurious' solutions to the crossing symmetry equations at the current bounds.

6.3.2 Upper bound for $\hat{T}_{dd}$ OPE coefficient

The method of linear functionals can also be used to bound OPE coefficients. In [42] a universal upper bound for the OPE coefficient of three scalars was found using the four-point function bootstrap. The same technique was used

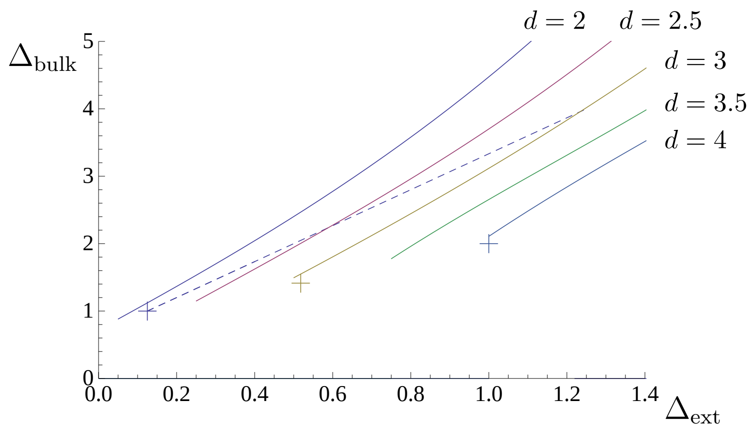

Figure 6.7: Bulk bound for different spacetime dimensions in the extraordinary transition. We highlighted the Ising model in various dimensions with the crosses. The dashed line is a specific solution for $d = 2$ which interpolates through the minimal models, see appendix D.2.

in [43, 44] to obtain an upper bound for the OPE coefficient of the stress tensor. This coefficient is inversely proportional to the central charge c of the theory so the result translates into a lower bound for c.

In this section we will use the boundary bootstrap to bound the coefficient μ_d^2 of the $\hat{T}_{dd}$ boundary block $f_{\text{bdy}}(d,\xi)$. We recall that this block is always present in the extraordinary transition, see the OPE in table 6.1. We start by imposing,

$$\begin{aligned} \Lambda(\xi^{\Delta_{\text{ext}}} f_{\text{bdy}}(d,\xi)) &= 1\,, \qquad (6.15)\\ \Lambda(F_\Delta) &\geq 0\,. \qquad (6.16)\end{aligned}$$

Applying this functional to the crossing symmetry relation (6.4) we obtain,

$$\mu_d^2 \leq \Lambda(1)\,, \qquad (6.17)$$

where μ_d^2 is the OPE coefficient of $f_{\text{bdy}}(d,\xi)$. The best bound is obtained by minimizing the action of Λ on the identity. For the spectrum we require,

$$\begin{aligned} &\Delta_{\text{bulk}} \geq 2\Delta_{\text{ext}}\,,\\ &\Delta_{\text{bdy}}^{(1)} = 0\,, \qquad (6.18)\\ &\Delta_{\text{bdy}}^{(2)} \geq d\,.\end{aligned}$$

Notice that we have again assumed a gap of $2\Delta_{\text{ext}}$ in the bulk. We plot our result as a function of the external dimension in figure 6.8. Let us try to justify

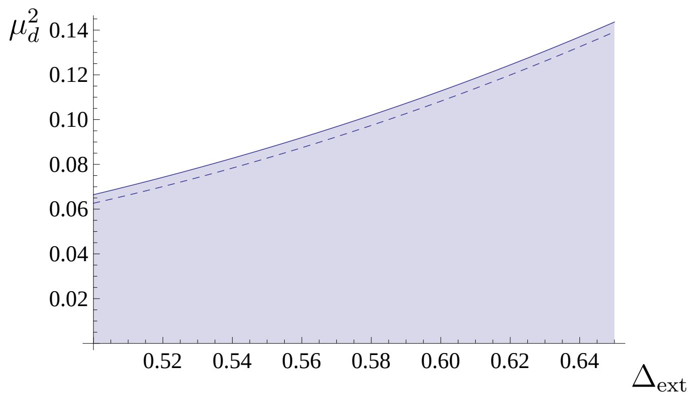

Figure 6.8: Upper bound for the coefficient of the $\hat{T}_{dd}$ block as a function of the external dimension. The dashed line represents and improved bound with a stronger assumption for the gap, following the dashed line of figure 6.6 (see text).

our choice of $\Delta_{\text{bulk}} \geq 2\Delta_{\text{ext}}$. A way to make the bound stronger would be to increase the bulk gap above this value, the maximum value we can assume for the gap is dictated by the bulk bound of [1], obtained using the four-point bootstrap equations. In figure 6.8 we have thus plotted an improved upper bound (dashed line) assuming $\Delta_{\text{bulk}} \geq f(\Delta_{\text{ext}})$, where $f(\Delta)$ is the function represented by the dashed line of figure 6.6. It is clear that the upper bound is not too sensitive to the assumed gap. For example, for the Ising model $\Delta_{\text{ext}} = 0.518$, and the upper bounds are $\mu_d^2 \lesssim 0.0734$ and $\mu_d^2 \lesssim 0.0693$ for $\Delta_{\text{bulk}} \geq 2(0.518) \sim 1.04$ and $\Delta_{\text{bulk}} \geq f(0.518) = 1.41$ respectively. A change of ~ 0.37 in the bulk gap translates into a change of ~ 0.0041 in the bound, so at least for this example $2\Delta_{\text{ext}}$ does a good job as a representative gap for the space of CFTs.

The procedure used above generalizes with no major changes to arbitrary dimensions, let us then make a quick comparison with some known values. For the $2d$ Ising model the coefficient μ_d^2 can be read from the conformal block expansion in (D.5), it has the value $\mu_d^2 = \frac{1}{32\sqrt{2}} \sim 0.0221$ whereas the Linear Programming methods result in an upper bound $\mu_d^2 \lesssim 0.0309$. For the extraordinary transition in the ϵ-expansion equation (D.29) tells us $\mu_d^2 = \frac{1}{10} = 0.10$, whereas we obtained the upper bound $\mu_d^2 \lesssim 0.119$ in four dimensions. We see that the numbers agree reasonably well.

6.3.3 Towards the Ising model

In analogy with [1] we may try to isolate the Ising model in various dimensions. To this end we will improve the results of the previous subsection by using as additional knowledge the dimension of the next scalar operator ε' which appears in the $\sigma \times \sigma$ OPE. According to table 6.1, in three dimensions this operator has a scaling dimension $\Delta_{\epsilon'}$ of approximately 3.84 whereas in two dimensions it has dimension 4 (it corresponds to $L_{-2}\bar{L}_{-2}\mathbf{1}$). We again assumed a boundary channel spectrum consistent with the extraordinary transition, *i.e.* a possible one-point function and a gap equal to the spacetime dimensions d. Summarizing,

$$\begin{aligned} \Delta^{(2)}_{\text{bulk}} &\geq \Delta_{\epsilon'}\,, \\ \Delta^{(1)}_{\text{bdy}} &= 0\,, \\ \Delta^{(2)}_{\text{bdy}} &\geq d\,. \end{aligned} \tag{6.19}$$

with $\Delta_{\epsilon'}$ fixed to the values of table 6.1. Our aim is now to find the possible range of values that $\Delta^{(1)}_{\text{bulk}}$ can take. The resulting plots are shown in figure 6.9.

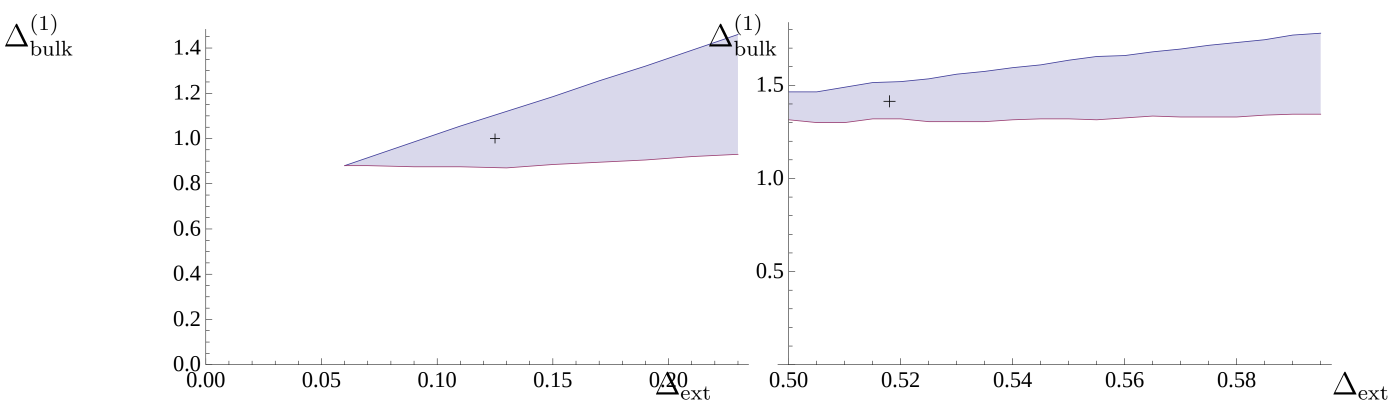

Figure 6.9: Locating the Ising model in $d = 2$ (left) and $d = 3$ (right). The plots show the dimension of a bulk operator versus the external dimension. With the assumptions explained in the main text, we need at least one bulk operator in the shaded regions. The Ising model is indicated with the cross in both plots.

Notice that the plots give results that are qualitatively similar to those of [1], in a considerably simpler setup. This is of course an encouraging result. Furthermore, we also did not rule out the Ising model and this provides some *a posteriori* justification for our assumption of positivity in three dimensions.

It is however rather unfortunate that the bounds we obtain are relatively weak. For this specific example we have tried different numerical implementations as well, for example we included more derivatives or evaluated the blocks at different points like $\xi = 1/2$ or $\xi = 2$. In each case we were unable to significantly lower the bounds. We have also attempted to improve the results by imposing an additional gap between the second and the *third* operator in the bulk channel. The third bulk operator has scaling dimensions 8 in $d = 2$ and approximately 4.6 in $d = 3$. Imposing this additional gap significantly improved the bounds for $d = 2$ but unfortunately this was not the case for $d = 3$.

6.4 Numerical results for stress tensors

The numerical analysis of equation (5.40) proceeds largely as for the scalar two-point function. In particular, we again translate the constraints of crossing symmetry to an infinite vector of derivatives at $\xi = 1$ and apply a linear functional in order to exclude certain spectra, using the same numerical methods as described above. Notice that the Ward identities (5.39) can be used to express derivatives of f and g in terms of derivatives of h. We therefore do not need to include more than the zeroth derivative for the f and g components if we include many derivatives of the h component. There is again no guarantee that the coefficients of the conformal blocks are positive in the bulk channel. Just as before we will therefore have to assume this condition of positivity in order to obtain bounds.

6.4.1 Bound on the bulk gap

In order to turn equation (5.40) into a useful equation to constrain conformal field theories we have to decide which parameters we are going to vary. In previous computations of this sort the canonical parameter was always the dimension of the external field but for the stress tensor this dimension is fixed to be d. In our first analysis we instead chose to vary the dimension of the lowest spin 2 boundary block which we denote as $\Delta_{(2)}$. We then obtain an upper bound for the lowest bulk operator dimension as a function of $\Delta_{(2)}$ which we plot as the upper curve in figure 6.10.

We may rephrase this result by saying that the upper curve in figure 6.10 informs us that the crossing symmetry equation (5.40) can only be satisfied if there is at least one "critical" bulk operator with a scaling dimension somewhere below the curve. We can however subsequently ask whether this "critical" operator really could be sitting anywhere below the curve (and above the unitarity bound $\Delta_{\text{bulk}} > 1/2$). In fact it turns out that the region where such an operator has to appear can be constrained even further: we can limit it to the shaded region in figure 6.10. We conclude that for every $\Delta_{(2)}$ there has to be at least one bulk operator somewhere within this region. (There could in addition be other operators, for example somewhere in the white "band" or multiple operators in the shaded region, but none of this modifies the validity of our claim.)

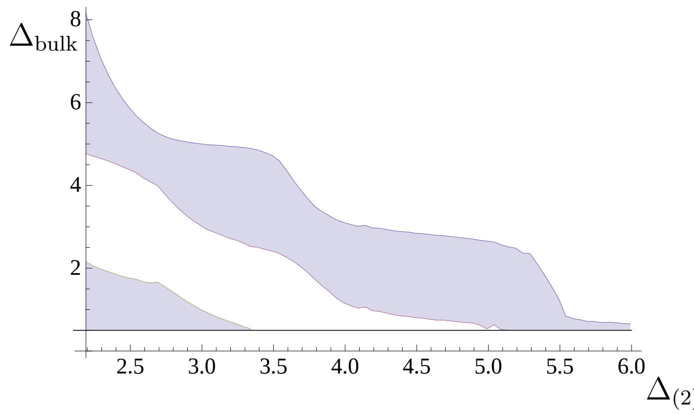

Figure 6.10: Bounds for the energy momentum tensor two-point function in three spacetime dimensions. The upper curve is the upper bound Δ_{bulk} for the first bulk operator as a function of the gap $\Delta_{(2)}$ for the first spin 2 boundary operator. The other lines denote further constraints for such a bulk operator, to the extent that for every $\Delta_{(2)}$ there has to be at least one bulk scalar somewhere in the shaded region.

In figure 6.10 we assumed that the vector block was *not* present in the boundary OPE of $T_{\mu\nu}$. Upon repeating the analysis with a vector block we obtained exactly the same curves for $\Delta_{(2)} > 3$ (up to small deviations due to the finite numerical precision), whereas for $\Delta_{(2)} \leq 3$ we would not be able to bound the bulk gap at all. The latter phenomenon has an easy explanation: the bulk identity operator can be decomposed in the boundary channel into the scalar block, the vector block and an infinite series of spin 2 blocks starting with $\Delta_{(2)} = 3$. For $\Delta_{(2)} \leq 3$ and with the vector block present it is therefore possible to have an infinite gap in the bulk (*i.e.* no bulk operators apart from the identity) and so Δ_{bulk} cannot be bounded. This is reminiscent of the "trivial" solution for the scalar two-point function discussed in appendix D.5 which we found numerically in section 6.2.

The curves shown in figure 6.10 have several "bumps" and other features whose origins are unfortunately unclear to us. For example, we were unable to find specific solutions of crossing symmetry that reflect the existence of these bumps. It would be interesting to see if such solutions exist and whether a conformal field theory is associated to them.

6.4.2 Bound on OPE coefficients in the three-dimensional Ising model

In subsection 6.3.2 we discussed how to bound OPE coefficients in the conformal block decomposition. Here we repeat the same procedure for the two-point function of the stress tensor. We will again bound the coefficient of the boundary operator $\hat{T}_{dd}$ which in equation (5.40) corresponds to the coefficient $\mu^2_{(0)}$ of the scalar block in the boundary channel. In addition, we decided to focus our attention on the three-dimensional Ising model. In particular, we have assumed that the bulk spectrum consists of operators with dimensions equal to 1.41, 3.84, and any operator with a scaling dimension greater than 4.6. We then obtain an upper bound on $\mu^2_{(0)}$ as a function of the unknown scaling dimension $\Delta_{(2)}$ of the lowest spin two operator in the boundary channel. We assumed that no vector operator was present in the boundary channel. We plot our results in figure 6.11.

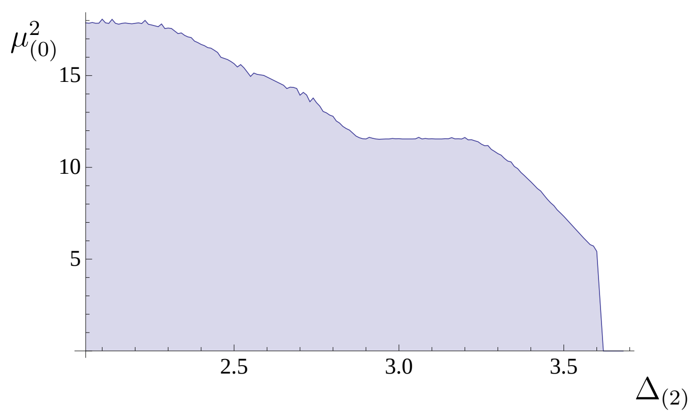

Figure 6.11: Bounds for the coefficient of the scalar boundary block in the two-point function of the stress tensor as a function of the gap $\Delta_{(2)}$ in the spin 2 boundary dimensions.

We find a rather surprising plateau for $\Delta_{(2)}$ between approximately 2.9 and 3.2 where $\mu^2_{(0)} \sim 11.5$. From the results in appendix E we find that $\mu^2_{(0)} = 4$ in two dimensions and that $\mu^2_{(0)} = 640/\epsilon + O(\epsilon^0)$ in $4-\epsilon$ dimensions so at the very least our estimate appears to have the right order of magnitude. It would be interesting to compute the dimension of the first spin 2 operator appearing in the boundary channel in the epsilon expansion, since it is natural to expect that the Ising model lies at one of the corners of this plateau.

Chapter 7

Discussion and Future Work

The work presented in this thesis is naturally divided into two parts. Let us summarize our results for each of them.

In the first part consisting of chapters 2, 3, and 4 we found $\mathcal{N} = 2$ superconformal symmetry is more constraining than naively expected: it fixes the one-loop Hamiltonian of $\mathcal{N} = 2$ SCQCD completely, and that of the interpolating quiver theory up to a single parameter. The same is true for the $\mathcal{N} = 1$ superconformal algebra. Using the same techniques as in the $\mathcal{N} = 2$ theory, we constrained the one-loop dilation operator of $\mathcal{N} = 1$ SQCD. Both results generalize the scalar sector calculations of [13] and [19]. For the $\mathcal{N} = 1$ theory we worked in the "electric" description of the theory, at the Banks-Zaks fixed point near the upper edge of the conformal window. It would be interesting to apply the same strategy to the dual magnetic theory, at the Banks-Zaks fixed point in the lower edge of the conformal window.

One-loop integrable subsectors are easy to identify in both theories, but they are trivially isomorphic to analogous sectors in $\mathcal{N} = 4$ SYM. The question we would like to answer is whether these theories are *completely* integrable. Armed with the full Hamiltonian we performed a systematic search of parity pairs, a hallmark of integrability, in chapter 4. Our analysis indicates that the presence of degenerate pairs is not as systematic in in the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ theories as it is in $\mathcal{N} = 4$ SYM. These preliminary results are not particularly encouraging, but also not conclusive.

In order to give a definite answer to the question of integrability, at least for the $\mathcal{N} = 2$ theory, in chapter 4 we studied the $SU(2|1)$ sector spanned by,

$$\{\phi, \lambda_\alpha, \mathcal{M}\}\,. \tag{7.1}$$

We fixed the two-loop Hamiltonian using algebraic techniques up to two undetermined coefficients. It turned out that the Yang-Baxter equation for magnon scattering is not satisfied in this sector. This proves that $\mathcal{N}=2$ SCQCD cannot be completely integrable. For $\mathcal{N}=1$ SQCD we don't have a proof of non-integrability, but the results for $\mathcal{N}=2$ together with the parity pairs analysis seem to indicate that this theory is not fully integrable as well.

Even if the complete theories turn out to be non-integrable, we can still ask whether there is scope for integrability in some closed subsectors. In chapter 4 we argued in favor of the the $SU(2,1|2)$ subsector spanned by

$$\{\mathcal{D}^k_{+\dot\alpha}\,\phi\,,\mathcal{D}^k_{+\dot\alpha}\,\lambda_{\mathcal{I}+}\,,\;\mathcal{D}^k_{+\dot\alpha}\,\mathcal{F}_{++}\}\,, \tag{7.2}$$

and the $SU(2,1|1)$ subsector spanned by

$$\{\mathcal{D}^k_{+\dot\alpha}\,\lambda_{+}\,,\;\mathcal{D}^k_{+\dot\alpha}\,\mathcal{F}_{++}\}\,. \tag{7.3}$$

In the $\mathcal{N}=2$ case, while at one loop its Hamiltonian coincides with that of $\mathcal{N}=4$ SYM, it will start differing from it at sufficiently high order. It will be very interesting to investigate whether integrability is preserved. The two-body magnon S-matrix is completely fixed by symmetry (up to the overall dynamical phase) to be the $SU(2|2)$ S-matrix of [7], which automatically satisfies the Yang-Baxter equation. Still, integrability is by no means obvious, since factorization of the n-body S-matrix is a stronger condition than Yang-Baxter. If this sector turns out to be integrable to all loops, its difference from the analogous sector of $\mathcal{N}=4$ SYM would be fully encoded in the expression of the dynamical phase and of the magnon dispersion relation. The difference with $\mathcal{N}=4$ SYM should start appearing at three loops [14]. For $\mathcal{N}=1$ SQCD, the $SU(2,1|1)$ sector exists of course also in the dual magnetic theory, so its integrability may allow to interpolate across the whole conformal window. We look forward to future investigations of this scenario.

In the second part of this thesis, consisting of chapter 5 and 6, we explored the constraining power of crossing symmetry for BCFTs in general spacetime dimension. After discussing the basic setup in section 5.1, we illustrated the relative simplicity of the "boundary bootstrap" in section 5.2 where we found exact solutions with at most two blocks in each channel. We have then applied the linear programming methods of [3] to the boundary crossing symmetry equations for both scalar operators and stress tensors. With our assumption of positivity for the bulk expansion coefficients, we have demonstrated that these methods can be useful in the BCFT setup as well and that they lead to interesting universal bounds on scaling dimensions and OPE coefficients. Several of our results warrant a more detailed theoretical investigation. For example, the bound on the second boundary operator in the special transition and the T_{dd} OPE coefficient in the extraordinary transition should be compared with computations in the epsilon expansion. Similarly, our numerical results of section 6.2 indicate that the bulk-to-boundary OPE always has to be singular, a result that should be put on a more solid theoretical footing. Finally, our results for the stress tensor are rather mysterious and certainly call for further investigations, beginning with the one-loop anomalous dimension of the spin two boundary operator in the extraordinary transition.

It is unfortunate that the distinct "kinks" of [1] appear not to be generically present in the BCFT bounds. We emphasize that this (negative) result is completely independent from our positivity assumption, indeed in $d = 2$ we see no kink but we know that the exact result does exhibit positivity. It would be interesting to see if there is another solution to crossing symmetry "standing in the way" and thereby preventing us from obtaining such a kink. More generally our results are a reflection of the fact that there is currently no deep understanding of why and when such kinks will appear. It would of course be very interesting to understand this phenomenon better. We hope that our numerical results will be helpful in further investigations.

The weakest point of our analysis is admittedly the assumption of positivity for the bulk expansion coefficients. While we have presented strong evidence that it is satisfied for the special and extraordinary Ising BCFTs, it would be desirable to find a proof. A possible approach would be to derive rigorous inequalties for boundary correlators on the lattice.

It is clear that the possibilities for further numerical exploration are practically unlimited. This paper is a first attempt to investigate the boundary bootstrap with a focus on the three-dimensional Ising model, but we feel we have just scratched the surface and that there are many interesting open questions. To mention a few, there are many other operator dimensions and OPE coefficients to be bounded, one may extend our results to supersymmetric theories, and to spacetime dimensions greater than four. Furthermore, the relatively simple form of the conformal blocks makes the boundary bootstrap especially suitable for investigations involving tensor operators, a research direction that is much more involved for the bulk four-point function in a theory with no boundary. Finally there is the prospect to broaden the setup and include conformal defects of all possible codimensions.

The Supersymmetric Bootstrap

The two lines of research presented in this thesis can be somehow combined by adding supersymmetry to the bootstrap equations. In this last section we will discuss its implementation and some preliminary results. Let us start by considering the more familiar case of crossing symmetry for a scalar four-point function in a “bulk” CFT (*i.e.* without the presence of a boundary). This correlator is fixed by conformal symmetry up to two conformal invariants u and v:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{x_{12}^{2\Delta_\phi}x_{24}^{2\Delta_\phi}}\,, \tag{7.4}$$

where Δ_ϕ is the conformal dimension of ϕ. The function $g(u,v)$ is arbitrary and using the OPE decomposition

$$\phi(x)\phi(0) = \sum_{\mathcal{O}\in\phi\times\phi}\lambda_{\mathcal{O}}C[x,\partial]\mathcal{O}(0) \tag{7.5}$$

we can write it as a sum of conformal blocks,

$$g(u,v) = \sum_{\mathcal{O}\in\phi\times\phi}\lambda^2 g_{\Delta,\ell}(u,v)\,. \tag{7.6}$$

The sum goes over the conformal primaries $\mathcal{O}$ of dimension Δ and spin ℓ and the block $g_{\Delta,\ell}(u,v)$ encodes the contribution of the whole conformal family generated by $\mathcal{O}$. For four dimensions, explicit expressions for these blocks in terms of hypergeometric functions were found in [72]. Invariance under $x_1 \leftrightarrow x_2$ and $x_1 \leftrightarrow x_3$ implies:

$$\sum_{\mathcal{O}\in\phi\times\phi}\lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi}\sum_{\mathcal{O}\in\phi\times\phi}\lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(v,u)\,. \tag{7.7}$$

Isolating the contribution of the identity operator we can write a “sum rule” suitable for linear programming techniques [3]:

$$1 = \sum_{\mathcal{O}}\lambda_{\mathcal{O}}^2 F_{\Delta\,,\ell}(u,v) \tag{7.8}$$

where

$$F_{\Delta\,,\ell}(u,v) = \frac{v^{\Delta_\phi}g_{\Delta,\ell}(u,v) - u^{\Delta_\phi}g_{\Delta,\ell}(v,u)}{u^{\Delta_\phi} - v^{\Delta_\phi}}\,. \tag{7.9}$$

Now, the sum goes over all the primaries $\mathcal{O}$ appearing in the $\phi\times\phi$ OPE, in supersymmetric theories, different conformal primaries can be related by supersymmetry transformations. This usually implies that different three-point couplings are related to each other and we can further constraint the conformal block expansion.

7.1 An $\mathcal{N} = 2$ superconformal fixed point with E_6 symmetry

Our discussion will be more transparent if we consider a particular example. Let us study then how crossing symmetry works for the $\mathcal{N} = 2$ theory of Minahan and Nemeschansky found in [73].[1] This is a strongly correlated theory with no Lagrangian description and therefore hard to study. It has an E_6 flavor group with an associated current $J_{\alpha\dot{\alpha}}$, this current sits in a superconformal multiplet whose lowest weight state is a scalar. The structure of the multiplet is shown in table 7.1.

$$\begin{tikzcd}
 & & \phi^{\mathcal{I}\mathcal{J}} \arrow[dl] \arrow[dr] & & \\
 & \psi^{\mathcal{I}}_{\alpha} \arrow[dl] \arrow[dr] & & \bar{\psi}_{\mathcal{I}\,\dot{\alpha}} \arrow[dl] \arrow[dr] & \\
\rho & & J_{\alpha\dot{\alpha}} & & \bar{\rho}
\end{tikzcd}$$

Table 7.1: Field content of the $\hat{\mathcal{B}}_1$ multiplet. The arrows $\swarrow$ and $\searrow$ correspond to the action of the $\mathcal{Q}^{\mathcal{I}}_{\alpha}$ and $\bar{\mathcal{Q}}_{\dot{\alpha}\mathcal{I}}$ supercharges respectively.

Being a scalar, the lowest weight state can be studied using the crossing symmetry techniques described above. There is however a technicality that we did not discuss in this work, the fact that the scalar $\phi^{\mathcal{I}\mathcal{J}}$ is in the triplet of $SU(2)_R$ and in the adjoint ($\mathbf{78}$) of the flavor group (we are suppressing flavor indices). This means that there are several channels in the OPE decomposition of $\phi \times \phi$. Apart from the $SU(2)_R$ channels

$$\mathbf{3} \times \mathbf{3} = \mathbf{1} + \mathbf{3} + \mathbf{5}\,, \tag{7.10}$$

we also have the E_6 Clebsch-Gordan decomposition,

$$\mathbf{78} \times \mathbf{78} = \mathbf{1} + \mathbf{650} + \mathbf{2430} + \mathbf{78} + \mathbf{2925}\,. \tag{7.11}$$

In principle, we have 15 different channels all with their associated crossing symmetry equation. It turns out that the three $SU(2)_R$ channels are related

[1] Bootstrap equations with $\mathcal{N} = 1$ supersymmetry were studied in [44].

to each other [74] and so we are left with five different crossing symmetry equations. Crossing symmetry for scalars with flavor quantum numbers were studied in [45–47].

Having understood the flavor structure what remains is to impose the constraints coming from $\mathcal{N} = 2$ superconformal symmetry. This analysis was done in [74] and we will quote their results here. As usual, we will follow the conventions of [20] for superconformal multiplets. The flavor current multiplet shown in figure 7.1, denoted by $\hat{\mathcal{B}}_1$, has the following OPE decomposition,

$$\hat{\mathcal{B}}_1 \times \hat{\mathcal{B}}_1 \sim \mathbf{1} + \hat{\mathcal{B}}_1 + \hat{\mathcal{B}}_2 + \hat{\mathcal{C}}_{R=0(j,j)} + \hat{\mathcal{C}}_{R=1(j,j)} + \mathcal{A}^{\Delta}_{R=0,r=0,(j,j)} \,. \tag{7.12}$$

Here $\mathbf{1}$ correspond the identity operator, $\hat{\mathcal{B}}_2$ is a short multiplet in the same family as $\hat{\mathcal{B}}_1$, $\hat{\mathcal{C}}_{R=0(j,j)}$ and $\hat{\mathcal{C}}_{R=1(j,j)}$ are semi-short multiplets and $\mathcal{A}^{\Delta}_{R=0,r=0,(j,j)}$ denotes the long-multiplets. The $\hat{\mathcal{C}}_{R=0(0,0)}$ and $\hat{\mathcal{B}}_1$ multiplets contain the stress tensor $T_{\alpha\dot{\alpha},\beta\dot{\beta}}$ and the flavor current $J_{\alpha\dot{\alpha}}$ respectively. Its three-point couplings are therefore proportional to the central charge c and the flavor central charge κ. From the point of view of the bootstrap equations, these central charges are arbitrary parameters. For the E_6 theory we are studying we will fix them to their known values: $c = \frac{13}{6}$ and $\kappa = 6$.

Each of the superconformal multiplets appearing in the OPE (7.12) is composed of a finite number of conformal multiplets. This means that we can assemble different conformal blocks into one "superconformal block". For example, the superconformal block that encodes the contribution of the long multiplet $\mathcal{A}^{\Delta}_{R=0,r=0,(j,j)}$ is given by [74],

$$\begin{aligned}
\mathcal{G}^{\mathcal{N}=2}_{\Delta,\ell} \;=\;& g_{\Delta,\ell} - g_{\Delta+1,\ell+1} - \frac{1}{4} g_{\Delta+1,\ell-1} + \frac{1}{4} g_{\Delta+2,\ell} \\
&+ \frac{(\Delta+\ell+2)^2}{4(\Delta+\ell+1)(\Delta+\ell+3)} g_{\Delta+2,\ell+2} - \frac{(\Delta+\ell+2)^2}{16(\Delta+\ell+1)(\Delta+\ell+3)} g_{\Delta+3,\ell+1} \\
&+ \frac{(\Delta-\ell)^2}{64(\Delta-\ell-1)(\Delta-\ell+1)} g_{\Delta+2,\ell-2} - \frac{(\Delta-\ell)^2}{64(\Delta-\ell-1)(\Delta-\ell+1)} g_{\Delta+3,\ell-1} \\
&+ \frac{(\Delta+\ell+2)^2(\Delta-\ell)^2}{256(\Delta+\ell+1)(\Delta+\ell+3)(\Delta-\ell-1)(\Delta-\ell+1)} g_{\Delta+4,\ell} \,.
\end{aligned} \tag{7.13}$$

As stated above, we have several conformal multiplets contributing with their respective three-point couplings related by supersymmetry. The conformal blocks for the short and semi-short multiplets in (7.12) were also calculated in [74]. We can now set up the crossing symmetry equations with the extra constraints coming from supersymmetry.

Let us skip the details and present some preliminary results. In table 7.2

we show an upper bound for the first non-protected scalar in each of the E_6 channels. The columns correspond to the number of derivatives considered (12,18,22,26). As the derivatives increase, the bound becomes stronger.

	12	18	22	26
1	6.55	5.10	4.80	4.72
650	3.54	3.42	3.40	3.36
2430	4.84	4.66	4.62	4.60
78	5.05	4.75	4.65	4.55
2925	5.30	5.20	5.15	5.15

Table 7.2: Upper bound for the first non-protected scalar in each of the E_6 channels. The columns correspond to the number of derivatives considered in the linear programming.

From the table is clear that the bound has not reached a stable value and that we need to increase our vector space by adding more derivatives and maybe using a finer grid. The hope is that once the bound reaches a stopping point it would correspond to the actual value of the dimension of the first operator in the OPE, a piece of information that is inaccessible by other means.

The supersymmetric bootstrap also allows us to study the *space* of superconformal theories. We could plot the bounds for the first non-protected scalars as a function of c and κ (generalizing table 7.2) and look for kinks or features (in the spirit of [1]), with the hope of discovering new superconformal theories. We can also generalize our analysis to flavor groups other than E_6.

Moreover, the $\hat{\mathcal{B}}_1$ four-point function is not the only one that can be bootstrapped. For example, the stress tensor multiplet $\mathcal{C}_{R=0(0,0)}$ also has a scalar as a lowest weight state and can be analysed using bootstrap techniques. The superconformal blocks for this multiplet however, have not been calculated yet, and that is a research project on its own.

As we can see, the supersymmetric bootstrap is a beautiful merging of the two lines of research presented in this thesis. It is a promising research direction with a lot of new results waiting to be uncovered and we hope to come back to it elsewhere.

Appendix A

$\mathcal{N} = 2$ Superconformal Algebra

In this appendix we present a collection of results for the $\mathcal{N} = 2$ superconformal algebra, necessary for the calculation of the complete one-loop Hamiltonian presented in the main text.

A.1 $\mathcal{N} = 2$ Superconformal Multiplets

Detailed studies of the possible shortening conditions for the $\mathcal{N} = 2$ superconformal algebra were performed in [20, 75, 76]. In this appendix we summarize their findings in table A.1, following the conventions of [20].

A generic long multiplet of the $\mathcal{N} = 2$ superconformal algebra is denoted by $\mathcal{A}^{\Delta}_{R,r(j,\bar{j})}$. It is generated by the action of the 8 Poincaré supercharges $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ on a superconformal primary, which by definition is annihilated by all the conformal supercharges $\mathcal{S}$. When some combination of the $\mathcal{Q}$'s also annihilates the primary, the corresponding multiplet is shorter. $|R, r\rangle^{h.w.}_{(j,\bar{j})}$ is the highest weight state with eigenvalues $(R, r, j\bar{j})$ under the Cartan generators of the $SU(2)_R \times U(1)_r$ R-symmetry and of the Lorentz group. The multiplet built on this state is denoted as $\mathcal{X}_{R,r(j,\bar{j})}$, where the letter $\mathcal{X}$ characterizes the shortening condition. The left column of table A.1 labels the condition. A superscript on the label corresponds to the index $\mathcal{I} = 1, 2$ of the supercharge that kills the primary: for example $\mathcal{B}_1$ refers to $\mathcal{Q}_{\alpha}{}^{1}$. Similarly a "bar" on the label refers to the conjugate condition: for example $\bar{\mathcal{B}}_2$ corresponds to $\bar{Q}_{2\,\dot{\alpha}}$ annihilating the state; this would result in the short anti-chiral multiplet $\bar{\mathcal{B}}_{R,r(j,0)}$, obeying $\Delta = 2R - r$. Note that conjugation reverses the sign of r and exchanges j and $\bar{j}$ in the expression of the conformal dimension.

Shortening Conditions				Multiplet
$\mathcal{B}_1$	$\mathcal{Q}^1_\alpha \vert R,r\rangle^{h.w.} = 0$	$j=0$	$\Delta = 2R+r$	$\mathcal{B}_{R,r(0,\bar{j})}$
$\bar{\mathcal{B}}_2$	$\bar{\mathcal{Q}}_{2\dot{\alpha}} \vert R,r\rangle^{h.w.} = 0$	$\bar{j}=0$	$\Delta = 2R-r$	$\bar{\mathcal{B}}_{R,r(j,0)}$
$\mathcal{E}$	$\mathcal{B}_1 \cap \mathcal{B}_2$	$R=0$	$\Delta = r$	$\mathcal{E}_{r(0,\bar{j})}$
$\bar{\mathcal{E}}$	$\bar{\mathcal{B}}_1 \cap \bar{\mathcal{B}}_2$	$R=0$	$\Delta = -r$	$\bar{\mathcal{E}}_{r(j,0)}$
$\hat{\mathcal{B}}$	$\mathcal{B}_1 \cap \bar{B}_2$	$r=0,\ j,\bar{j}=0$	$\Delta = 2R$	$\hat{\mathcal{B}}_R$
$\mathcal{C}_1$	$\epsilon^{\alpha\beta}\mathcal{Q}^1_\beta \vert R,r\rangle^{h.w.}_\alpha = 0$		$\Delta = 2+2j+2R+r$	$\mathcal{C}_{R,r(j,\bar{j})}$
	$(\mathcal{Q}^1)^2 \vert R,r\rangle^{h.w.} = 0$ for $j=0$		$\Delta = 2+2R+r$	$\mathcal{C}_{R,r(0,\bar{j})}$
$\bar{\mathcal{C}}_2$	$\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\mathcal{Q}}_{2\dot{\beta}} \vert R,r\rangle^{h.w.}_{\dot{\alpha}} = 0$		$\Delta = 2+2\bar{j}+2R-r$	$\bar{\mathcal{C}}_{R,r(j,\bar{j})}$
	$(\bar{\mathcal{Q}}_2)^2 \vert R,r\rangle^{h.w.} = 0$ for $\bar{j}=0$		$\Delta = 2+2R-r$	$\bar{\mathcal{C}}_{R,r(j,0)}$
$\mathcal{F}$	$\mathcal{C}_1 \cap \mathcal{C}_2$	$R=0$	$\Delta = 2+2j+r$	$\mathcal{C}_{0,r(j,\bar{j})}$
$\bar{\mathcal{F}}$	$\bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$	$R=0$	$\Delta = 2+2\bar{j}-r$	$\bar{\mathcal{C}}_{0,r(j,\bar{j})}$
$\hat{\mathcal{C}}$	$\mathcal{C}_1 \cap \bar{\mathcal{C}}_2$	$r=\bar{j}-j$	$\Delta = 2+2R+j+\bar{j}$	$\hat{\mathcal{C}}_{R(j,\bar{j})}$
$\hat{\mathcal{F}}$	$\mathcal{C}_1 \cap \mathcal{C}_2 \cap \bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$	$R=0, r=\bar{j}-j$	$\Delta = 2+j+\bar{j}$	$\hat{\mathcal{C}}_{0(j,\bar{j})}$
$\mathcal{D}$	$\mathcal{B}_1 \cap \bar{\mathcal{C}}_2$	$r=\bar{j}+1$	$\Delta = 1+2R+\bar{j}$	$\mathcal{D}_{R(0,\bar{j})}$
$\bar{\mathcal{D}}$	$\bar{\mathcal{B}}_2 \cap \mathcal{C}_1$	$-r=j+1$	$\Delta = 1+2R+j$	$\bar{\mathcal{D}}_{R(j,0)}$
$\mathcal{G}$	$\mathcal{E} \cap \bar{\mathcal{C}}_2$	$r=\bar{j}+1, R=0$	$\Delta = r = 1+\bar{j}$	$\mathcal{D}_{0(0,\bar{j})}$
$\bar{\mathcal{G}}$	$\bar{\mathcal{E}} \cap \mathcal{C}_1$	$-r=j+1, R=0$	$\Delta = -r = 1+j$	$\bar{\mathcal{D}}_{0(j,0)}$

Table A.1: Shortening conditions and short multiplets for the $\mathcal{N}=2$ superconformal algebra.

A.2 Oscillator Representation

In this appendix we descibe the oscillator representation of the $\mathcal{N}=2$ superconformal algebra $SU(2,2|2)$. We introduce two sets of bosonic oscillators $(\mathbf{a}^\alpha, \mathbf{a}^\dagger_\alpha)$, $(\mathbf{b}^{\dot{\alpha}}, \mathbf{b}^\dagger_{\dot{\alpha}})$ and one set of fermionic oscillators $(\mathbf{c}^{\mathcal{I}}, \mathbf{c}^\dagger_{\mathcal{I}})$, where $(\alpha, \dot{\alpha})$

are Lorentz indices and $\mathcal{I}$ is an $SU(2)_R$ index. In addition we will need two more "auxiliary" fermionic operators $(\mathbf{d}, \mathbf{d}^\dagger)$ and $(\tilde{\mathbf{d}}, \tilde{\mathbf{d}}^\dagger)$. The non-zero (anti)commutation relations are

$$\begin{aligned}
[\mathbf{a}^\alpha, \mathbf{a}^\dagger_\beta] &= \delta^\alpha_\beta\,, && \text{(A.1)}\\
[\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\beta}] &= \delta^{\dot\alpha}_{\dot\beta}\,, && \text{(A.2)}\\
\{\mathbf{c}^{\mathcal{I}}, \mathbf{c}^\dagger_{\mathcal{J}}\} &= \delta^{\mathcal{I}}_{\mathcal{J}}\,, && \text{(A.3)}\\
\{\mathbf{d}, \mathbf{d}^\dagger\} &= \{\tilde{\mathbf{d}}, \tilde{\mathbf{d}}^\dagger\} = 1\,. && \text{(A.4)}
\end{aligned}$$

In this oscillator representation the generators of $SU(2,2|2)$ read

$$\begin{aligned}
&\mathcal{Q}_\alpha{}^{\mathcal{I}} = \mathbf{a}^\dagger_\alpha \mathbf{c}^{\mathcal{I}}\,, && \bar{\mathcal{Q}}_{\dot\alpha\mathcal{I}} = \mathbf{b}^\dagger_{\dot\alpha}\mathbf{c}^\dagger_{\mathcal{I}}\,, && \text{(A.5)}\\
&\mathcal{S}_{\mathcal{I}}{}^\alpha = \mathbf{c}^\dagger_{\mathcal{I}}\mathbf{a}^\alpha\,, && \bar{\mathcal{S}}^{\dot\alpha\mathcal{I}} = \mathbf{b}^{\dot\alpha}\mathbf{c}^{\mathcal{I}}\,, && \text{(A.6)}\\
&\mathcal{P}_{\alpha\dot\beta} = \mathbf{a}^\dagger_\alpha \mathbf{b}^\dagger_{\dot\beta}\,, && \mathcal{K}^{\alpha\dot\beta} = \mathbf{a}^\alpha\mathbf{b}^{\dot\beta}\,, && \text{(A.7)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_\beta{}^\alpha &= \mathbf{a}^\dagger_\beta\mathbf{a}^\alpha - \frac{1}{2}\delta^\alpha_\beta\mathbf{a}^\dagger_\gamma\mathbf{a}^\gamma\,, && \text{(A.8)}\\
\dot{\mathcal{L}}_{\dot\beta}{}^{\dot\alpha} &= \mathbf{b}^\dagger_{\dot\beta}\mathbf{b}^{\dot\alpha} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\mathbf{b}^\dagger_{\dot\gamma}\mathbf{b}^{\dot\gamma}\,, && \text{(A.9)}\\
\mathcal{R}_{\mathcal{J}}{}^{\mathcal{I}} &= \mathbf{c}^\dagger_{\mathcal{J}}\mathbf{c}^{\mathcal{I}} - \frac{1}{2}\delta^{\mathcal{I}}_{\mathcal{J}}\mathbf{c}^\dagger_{\mathcal{K}}\mathbf{c}^{\mathcal{K}}\,, && \text{(A.10)}\\
r &= -\frac{1}{2}\mathbf{c}^\dagger_{\mathcal{K}}\mathbf{c}^{\mathcal{K}} + \frac{1}{2}\mathbf{d}^\dagger\mathbf{d} + \frac{1}{2}\tilde{\mathbf{d}}^\dagger\tilde{\mathbf{d}}\,, && \text{(A.11)}\\
D &= 1 + \frac{1}{2}\mathbf{a}^\dagger_\gamma\mathbf{a}^\gamma + \frac{1}{2}\mathbf{b}^\dagger_{\dot\gamma}\mathbf{b}^{\dot\gamma}\,, && \text{(A.12)}\\
C &= 1 - \frac{1}{2}\mathbf{a}^\dagger_\gamma\mathbf{a}^\gamma + \frac{1}{2}\mathbf{b}^\dagger_{\dot\gamma}\mathbf{b}^{\dot\gamma} - \frac{1}{2}\mathbf{c}^\dagger_{\mathcal{K}}\mathbf{c}^{\mathcal{K}} - \frac{1}{2}\mathbf{d}^\dagger\mathbf{d} - \frac{1}{2}\tilde{\mathbf{d}}^\dagger\tilde{\mathbf{d}}\,. && \text{(A.13)}
\end{aligned}$$

Here C is a central charge that must kill any physical state. It could be eliminated from the algebra by redefining $r + C \to r$, but it is useful for implementing the harmonic action so we will keep it. The quadratic Casimir operator is

$$\begin{aligned}
J^2 &= \frac{1}{2}D^2 + \frac{1}{2}\mathcal{L}_\alpha{}^\beta\mathcal{L}_\beta{}^\alpha + \frac{1}{2}\dot{\mathcal{L}}_{\dot\alpha}{}^{\dot\beta}\dot{\mathcal{L}}_{\dot\beta}{}^{\dot\alpha} - \frac{1}{2}\mathcal{R}_{\mathcal{I}}{}^{\mathcal{J}}\mathcal{R}_{\mathcal{J}}{}^{\mathcal{I}}\\
&\quad -\frac{1}{2}[\mathcal{Q}_\alpha{}^{\mathcal{I}}, \mathcal{S}_{\mathcal{I}}{}^\alpha] - \frac{1}{2}[\bar{\mathcal{Q}}_{\dot\alpha\mathcal{I}}, \bar{\mathcal{S}}^{\dot\alpha\mathcal{I}}] - \frac{1}{2}\{\mathcal{P}_{\alpha\dot\beta}, \mathcal{K}^{\alpha\dot\beta}\} \qquad \text{(A.14)}\\
&\quad -\frac{1}{2}(r+C)(r+C)\,.
\end{aligned}$$

A.2.1 Vector multiplets $\mathcal{V}$ and $\bar{\mathcal{V}}$

We define a vacuum state $|0\rangle$ annihilated by all the lowering operators. Then we identify

$$\begin{aligned} \mathcal{D}^k\mathcal{F} &= (\mathbf{a}^\dagger)^{k+2}(\mathbf{b}^\dagger)^k(\mathbf{c}^\dagger)^0|0\rangle\,, && \text{(A.15)}\\ \mathcal{D}^k\lambda &= (\mathbf{a}^\dagger)^{k+1}(\mathbf{b}^\dagger)^k(\mathbf{c}^\dagger)^1|0\rangle\,, && \text{(A.16)}\\ \mathcal{D}^k\phi &= (\mathbf{a}^\dagger)^{k}\quad(\mathbf{b}^\dagger)^k(\mathbf{c}^\dagger)^2|0\rangle\,, && \text{(A.17)} \end{aligned}$$

and

$$\begin{aligned} \mathcal{D}^k\bar{\mathcal{F}} &= (\mathbf{a}^\dagger)^{k}(\mathbf{b}^\dagger)^{k+2}(\mathbf{c}^\dagger)^2\mathbf{d}^\dagger\tilde{\mathbf{d}}^\dagger|0\rangle\,, && \text{(A.18)}\\ \mathcal{D}^k\bar{\lambda} &= (\mathbf{a}^\dagger)^{k}(\mathbf{b}^\dagger)^{k+1}(\mathbf{c}^\dagger)^1\mathbf{d}^\dagger\tilde{\mathbf{d}}^\dagger|0\rangle\,, && \text{(A.19)}\\ \mathcal{D}^k\bar{\phi} &= (\mathbf{a}^\dagger)^{k}(\mathbf{b}^\dagger)^{k}\quad(\mathbf{c}^\dagger)^0\mathbf{d}^\dagger\tilde{\mathbf{d}}^\dagger|0\rangle\,. && \text{(A.20)} \end{aligned}$$

For example,

$$\lambda_{\mathcal{I}\alpha} = \mathbf{a}^\dagger_\alpha\mathbf{c}^\dagger_{\mathcal{I}}|0\rangle\,,\quad \bar{\lambda}_{\mathcal{I}\dot{\alpha}} = \mathbf{b}^\dagger_{\dot{\alpha}}\mathbf{c}^\dagger_{\mathcal{I}}\mathbf{d}^\dagger\tilde{\mathbf{d}}^\dagger|0\rangle\,. \qquad \text{(A.21)}$$

It's easy to see that all the quantum numbers match, including the zero central charge constraint.

A.2.2 Hypermultiplet $\mathcal{H}$

Similarly, for the hypermultiplet we identify

$$\begin{aligned} \mathcal{D}^kQ &= (\mathbf{a}^\dagger)^{k}(\mathbf{b}^\dagger)^k(\mathbf{c}^\dagger)^1\mathbf{d}^\dagger|0\rangle\,, && \text{(A.22)}\\ \mathcal{D}^k\bar{Q} &= (\mathbf{a}^\dagger)^{k}(\mathbf{b}^\dagger)^k(\mathbf{c}^\dagger)^1\tilde{\mathbf{d}}^\dagger|0\rangle\,, && \text{(A.23)}\\ \mathcal{D}^k\psi &= (\mathbf{a}^\dagger)^{k+1}(\mathbf{b}^\dagger)^k\mathbf{d}^\dagger|0\rangle\,, && \text{(A.24)}\\ \mathcal{D}^k\tilde{\psi} &= (\mathbf{a}^\dagger)^{k+1}(\mathbf{b}^\dagger)^k\tilde{\mathbf{d}}^\dagger|0\rangle\,, && \text{(A.25)}\\ \mathcal{D}^k\bar{\psi} &= (\mathbf{a}^\dagger)^{k}(\mathbf{b}^\dagger)^{k+1}(\mathbf{c}^\dagger)^2\tilde{\mathbf{d}}^\dagger|0\rangle\,, && \text{(A.26)}\\ \mathcal{D}^k\bar{\tilde{\psi}} &= (\mathbf{a}^\dagger)^{k}(\mathbf{b}^\dagger)^{k+1}(\mathbf{c}^\dagger)^2\mathbf{d}^\dagger|0\rangle\,. && \text{(A.27)} \end{aligned}$$

A.3 Two-letter Superconformal Primaries

By demanding that they are annihilated by all the conformal supercharges and by the appropriate combinations of Poincaré supercharges, we have worked out the expressions for the superconformal primaries of the irreducible modules that appear on the right hand side of the tensor products (2.9)–(2.14).

The grassmannOps.m oscillator package by Jeremy Michelson and Matthew Headrick was extremely useful for this task. We simply quote the results:

$\underline{\mathcal{V}\times\mathcal{V}}$:

$$\begin{aligned}
\bar{\mathcal{E}}_{2(0,0)} &= \phi\phi\,, &\text{(A.28)}\\
\bar{\mathcal{D}}_{\frac{1}{2}(\frac{1}{2},0)} &= \lambda_{1+}\phi-\phi\lambda_{1+}\,, &\text{(A.29)}\\
\hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q-1}{2})} &= \sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q-1}{k}\binom{q}{k}\left(\mathcal{D}^{q-k-1}\lambda_{1+}\mathcal{D}^k\lambda_{2+}-\mathcal{D}^{q-k-1}\lambda_{2+}\mathcal{D}^k\lambda_{1+}\right)\\
&\quad+\frac{1}{q+1}\left(\sum_{k=0}^{q-1}(-1)^k\binom{q-1}{k}\binom{q+1}{k}\mathcal{D}^{q-k-1}\mathcal{F}_{\dot{+}\dot{+}}\mathcal{D}^k\phi\right.\\
&\quad\left.+\sum_{k=0}^{q-1}(-1)^k\binom{q-1}{k}\binom{q+1}{k+2}\mathcal{D}^{q-k-1}\phi\mathcal{D}^k\mathcal{F}_{\dot{+}\dot{+}}\right)\,. &\text{(A.30)}
\end{aligned}$$

For $\bar{\mathcal{V}}\times\bar{\mathcal{V}}$ the expressions are identical with $(\phi,\lambda,\mathcal{F})$ replaced by $(\bar{\phi},\bar{\lambda},\bar{\mathcal{F}})$. The Casimir operator acting on these modules gives

$$\begin{aligned}
J_{12}^2\bar{\mathcal{E}}_{2(0,0)} &= 0\,, &\text{(A.31)}\\
J_{12}^2\hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q-1}{2})} &= (q+1)(q+2)\hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q-1}{2})},\quad q\geq -1\,. &\text{(A.32)}
\end{aligned}$$

$\underline{\mathcal{V}\times\mathcal{H}}$:

$$\begin{aligned}
\bar{\mathcal{D}}_{\frac{1}{2}(0,0)} &= \phi Q_1\,, &\text{(A.33)}\\
\hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q}{2})} &= \sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q+1}{k}\left(\mathcal{D}^{q-k}\lambda_{2+}\mathcal{D}^kQ_1-\mathcal{D}^{q-k}\lambda_{1+}\mathcal{D}^kQ_2\right)\\
&\quad-\sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q+1}{k+1}\mathcal{D}^{q-k}\phi\mathcal{D}^k\psi_+\\
&\quad+q\sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q-1}{k}\binom{q+1}{k}\mathcal{D}^{q-k-1}\mathcal{F}_{++}\mathcal{D}^k\bar{\bar{\psi}}_{\dot{+}}\,. &\text{(A.34)}
\end{aligned}$$

As before, for $\bar{\mathcal{V}}\times H$ we replace $(\phi,\lambda,\mathcal{F})$ and $(\psi,\bar{\bar{\psi}})$ by its conjugates. The action of the Casimir is

$$J_{12}^2\hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q}{2})}=(q+\frac{3}{2})(q+\frac{5}{2})\hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q}{2})},\quad q\geq -1\,. \qquad \text{(A.35)}$$

$\underline{\mathcal{H}\times\mathcal{V}}$**:**

$$\bar{\mathcal{D}}_{\frac{1}{2}(0,0)} = Q_1\check{\phi}\,, \tag{A.36}$$

$$\begin{aligned}\hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q}{2})} &= \sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q+1}{k+1}\left(\mathcal{D}^{q-k}Q_2\mathcal{D}^k\check{\lambda}_{1+}-\mathcal{D}^{q-k}Q_1\mathcal{D}^k\check{\lambda}_{2+}\right)\\ &\quad+\sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q+1}{k}\mathcal{D}^{q-k}\psi_+\mathcal{D}^k\check{\phi}\\ &\quad+q\sum_{k=0}^{q-1}\frac{(-1)^k}{k+2}\binom{q-1}{k}\binom{q+1}{k+1}\mathcal{D}^{q-k-1}\check{\bar{\bar{\psi}}}_{\dot{+}}\mathcal{D}^k\check{\mathcal{F}}_{++}\,.\end{aligned} \tag{A.37}$$

$\underline{\mathcal{H}\times\mathcal{H}}$**:**

$$\hat{\mathcal{B}}_1 = Q_1\bar{Q}_1\,, \tag{A.38}$$

$$\begin{aligned}\hat{\mathcal{C}}_{0(\frac{q}{2},\frac{q}{2})} &= \sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q}{k}\left(\mathcal{D}^{q-k}Q_1\mathcal{D}^k\bar{Q}_2-\mathcal{D}^{q-k}Q_2\mathcal{D}^k\bar{Q}_1\right)\\ &\quad+q\sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q}{k}\binom{q+1}{k}\mathcal{D}^{q-k}\psi_+\mathcal{D}^k\bar{\psi}_{\dot{+}}\\ &\quad-q\sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q}{k}\binom{q+1}{k}\mathcal{D}^{q-k}\tilde{\bar{\bar{\psi}}}_{\dot{+}}\mathcal{D}^k\tilde{\psi}_+\,,\end{aligned} \tag{A.39}$$

with

$$J_{12}^2\hat{\mathcal{C}}_{0(\frac{q}{2},\frac{q}{2})} = (q+1)(q+2)\hat{\mathcal{C}}_{0(\frac{q}{2},\frac{q}{2})},\quad q\geq -1\,. \tag{A.40}$$

$\underline{\mathcal{V}\times\bar{\mathcal{V}}}$**:**

$$\begin{aligned}\hat{\mathcal{C}}_{0(\frac{q}{2},\frac{q}{2})} &= \sqrt{\frac{2(q+2)}{q+1}}\Bigg(\sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q}{k}\mathcal{D}^{q-k}\phi\mathcal{D}^k\bar{\phi}\\ &\quad-q\sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q}{k}\binom{q+1}{k}\left(\mathcal{D}^{q-k}\lambda_{1+}\mathcal{D}^k\bar{\lambda}_{2\dot{+}}-\mathcal{D}^{q-k}\lambda_{2+}\mathcal{D}^k\bar{\lambda}_{1\dot{+}}\right)\\ &\quad+q\sum_{k=0}^{q-2}\frac{(-1)^k}{k+2}\binom{q+1}{k+1}\binom{q+2}{k}\mathcal{D}^{q-k}\mathcal{F}_{++}\mathcal{D}^k\bar{\mathcal{F}}_{\dot{+}\dot{+}}\Bigg)\,.\end{aligned} \tag{A.41}$$

For $\bar{\mathcal{V}}\times\mathcal{V}$ we conjugate all fields.

Appendix B

$\mathcal{N} = 1$ Superconformal Algebra

In this appendix we present a collection of results for the $\mathcal{N} = 1$ superconformal algebra, necessary for the calculation of the complete one-loop Hamiltonian presented in the main text.

B.1 $\mathcal{N} = 1$ superconformal multiplets

In this appendix we summarize some basic facts about $\mathcal{N} = 1$ superconformal representation theory [22]. A generic long multiplet $\mathcal{A}^{\Delta}_{r(j_1,j_2)}$ is generated by the action of 4 Poincaré supercharges $\mathcal{Q}_\alpha$ and $\bar{\mathcal{Q}}_{\dot\alpha}$ on a superconformal primary which by definition is annihilated by all the conformal supercharges $\mathcal{S}$. In table B.1 we have summarized the possible shortening and semi-shortening conditions.

B.2 Oscillator Representation

In this appendix we describe the oscillator representation of the $\mathcal{N} = 1$ superconformal algebra $SU(2,2|1)$. We introduce two sets of bosonic oscillators $(\mathbf{a}^\alpha, \mathbf{a}^\dagger_\alpha)$, $(\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\alpha})$ and one fermionic oscillator $(\mathbf{c}, \mathbf{c}^\dagger)$, where $(\alpha, \dot\alpha)$ are Lorentz indices. In addition we will need three more "auxiliary" fermionic operators

Shortening Conditions				Multiplet
$\mathcal{B}$	$\mathcal{Q}_\alpha \vert r\rangle^{h.w.} = 0$	$j_1 = 0$	$\Delta = -\frac{3}{2}r$	$\mathcal{B}_{r(0,j_2)}$
$\bar{\mathcal{B}}$	$\bar{\mathcal{Q}}_{\dot\alpha} \vert r\rangle^{h.w.} = 0$	$j_2 = 0$	$\Delta = \frac{3}{2}r$	$\bar{\mathcal{B}}_{r(j_1,0)}$
$\hat{\mathcal{B}}$	$\mathcal{B} \cap \bar{\mathcal{B}}$	$j_1, j_2, r = 0$	$\Delta = 0$	$\hat{\mathcal{B}}$
$\mathcal{C}$	$\epsilon^{\alpha\beta}\mathcal{Q}_\beta \vert r\rangle^{h.w.}_\alpha = 0$ $(\mathcal{Q})^2 \vert r\rangle^{h.w.} = 0$ for $j_1 = 0$		$\Delta = 2 + 2j_1 - \frac{3}{2}r$ $\Delta = 2 - \frac{3}{2}r$	$\mathcal{C}_{r(j_1,j_2)}$ $\mathcal{C}_{r(0,j_2)}$
$\bar{\mathcal{C}}$	$\epsilon^{\dot\alpha\dot\beta}\bar{\mathcal{Q}}_{\dot\beta} \vert r\rangle^{h.w.}_{\dot\alpha} = 0$ $(\bar{\mathcal{Q}})^2 \vert r\rangle^{h.w.} = 0$ for $j_2 = 0$		$\Delta = 2 + 2j_2 + \frac{3}{2}r$ $\Delta = 2 + \frac{3}{2}r$	$\bar{\mathcal{C}}_{r(j_1,j_2)}$ $\bar{\mathcal{C}}_{r(j_1,0)}$
$\hat{\mathcal{C}}$	$\mathcal{C} \cap \bar{\mathcal{C}}$	$\frac{3}{2}r = (j_1 - j_2)$	$\Delta = 2 + j_1 + j_2$	$\hat{\mathcal{C}}_{(j_1,j_2)}$
$\mathcal{D}$	$\mathcal{B} \cap \bar{\mathcal{C}}$	$j_1 = 0, -\frac{3}{2}r = j_2 + 1$	$\Delta = -\frac{3}{2}r = 1 + j_2$	$\mathcal{D}_{(0,j_2)}$
$\bar{\mathcal{D}}$	$\bar{\mathcal{B}} \cap \mathcal{C}$	$j_2 = 0, \frac{3}{2}r = j_1 + 1$	$\Delta = \frac{3}{2}r = 1 + j_1$	$\bar{\mathcal{D}}_{(j_1,0)}$

Table B.1: Possible shortening conditions for the $\mathcal{N} = 1$ superconformal algebra.

$(\mathbf{d}_i, \mathbf{d}_i^\dagger)$, $i = 1, 2, 3$. The non-zero (anti)commutation relations are

$$
\begin{aligned}
[\mathbf{a}^\alpha, \mathbf{a}^\dagger_\beta] &= \delta^\alpha_\beta\,, && \text{(B.1)}\\
[\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\beta}] &= \delta^{\dot\alpha}_{\dot\beta}\,, && \text{(B.2)}\\
\{\mathbf{c}, \mathbf{c}^\dagger\} &= 1\,, && \text{(B.3)}\\
\{\mathbf{d}_i, \mathbf{d}^\dagger_j\} &= \delta_{ij}\,. && \text{(B.4)}
\end{aligned}
$$

In this oscillator representation the generators of $SU(2,2|1)$ are given by

$$
\begin{aligned}
\mathcal{Q}_\alpha &= \mathbf{a}^\dagger_\alpha \mathbf{c}\,, & \bar{\mathcal{Q}}_{\dot\alpha} &= \mathbf{b}^\dagger_{\dot\alpha}\mathbf{c}^\dagger\,, && \text{(B.5)}\\
\mathcal{S}^\alpha &= \mathbf{c}^\dagger \mathbf{a}^\alpha\,, & \bar{\mathcal{S}}^{\dot\alpha} &= \mathbf{b}^{\dot\alpha}\mathbf{c}\,, && \text{(B.6)}\\
\mathcal{P}_{\alpha\dot\beta} &= \mathbf{a}^\dagger_\alpha \mathbf{b}^\dagger_{\dot\beta}\,, & \mathcal{K}^{\alpha\dot\beta} &= \mathbf{a}^\alpha \mathbf{b}^{\dot\beta}\,, && \text{(B.7)}
\end{aligned}
$$

$$\mathcal{L}_\beta{}^\alpha = \mathbf{a}^\dagger_\beta \mathbf{a}^\alpha - \frac{1}{2}\delta^\alpha_\beta \mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma \,, \tag{B.8}$$

$$\dot{\mathcal{L}}_{\dot\beta}{}^{\dot\alpha} = \mathbf{b}^\dagger_{\dot\beta} \mathbf{b}^{\dot\alpha} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta} \mathbf{b}^\dagger_{\dot\gamma} \mathbf{b}^{\dot\gamma} \,, \tag{B.9}$$

$$r = \mathbf{c}^\dagger \mathbf{c} - \frac{1}{3}\mathbf{d}^\dagger_1 \mathbf{d}_1 - \frac{1}{3}\mathbf{d}^\dagger_2 \mathbf{d}_2 - \mathbf{d}^\dagger_3 \mathbf{d}_3 \,, \tag{B.10}$$

$$D = 1 + \frac{1}{2}\mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma + \frac{1}{2}\mathbf{b}^\dagger_{\dot\gamma} \mathbf{b}^{\dot\gamma} \,, \tag{B.11}$$

$$C = 1 - \frac{1}{2}\mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma + \frac{1}{2}\mathbf{b}^\dagger_{\dot\gamma} \mathbf{b}^{\dot\gamma} - \frac{1}{2}\mathbf{c}^\dagger \mathbf{c} - \frac{1}{2}\mathbf{d}^\dagger_1 \mathbf{d}_1 - \frac{1}{2}\mathbf{d}^\dagger_2 \mathbf{d}_2 - \frac{3}{2}\mathbf{d}^\dagger_3 \mathbf{d}_3 \,, \tag{B.12}$$

Here C is a central charge that must kill any physical state. It could be eliminated from the algebra by redefining $r + C \to r$, but it is useful for implementing the harmonic action so we keep it.

B.2.1 Vector multiplets $\mathcal{V}$ and $\bar{\mathcal{V}}$

We define a vacuum state $|0\rangle$ annihilated by all the lowering operators. Then we identify

$$\mathcal{D}^k \mathcal{F} = (\mathbf{a}^\dagger)^{k+2} (\mathbf{b}^\dagger)^k |0\rangle \,, \tag{B.13}$$

$$\mathcal{D}^k \lambda = (\mathbf{a}^\dagger)^{k+1} (\mathbf{b}^\dagger)^k \mathbf{c}^\dagger |0\rangle \,, \tag{B.14}$$

and

$$\mathcal{D}^k \bar{\mathcal{F}} = (\mathbf{a}^\dagger)^{k} (\mathbf{b}^\dagger)^{k+2} \mathbf{c}^\dagger \mathbf{d}^\dagger_3 |0\rangle \,, \tag{B.15}$$

$$\mathcal{D}^k \bar\lambda = (\mathbf{a}^\dagger)^{k} (\mathbf{b}^\dagger)^{k+1} \mathbf{d}^\dagger_3 |0\rangle \,. \tag{B.16}$$

B.2.2 Chiral multiplets $\mathcal{X}$ and $\bar{\mathcal{X}}$

Similarly, for the chiral multiplets we identify

$$\mathcal{D}^k Q = (\mathbf{a}^\dagger)^{k} (\mathbf{b}^\dagger)^k \mathbf{c}^\dagger \mathbf{d}^\dagger_1 |0\rangle \,, \tag{B.17}$$

$$\mathcal{D}^k \psi = (\mathbf{a}^\dagger)^{k+1} (\mathbf{b}^\dagger)^k \mathbf{d}^\dagger_1 |0\rangle \,, \tag{B.18}$$

and

$$\mathcal{D}^k \bar{Q} = (\mathbf{a}^\dagger)^{k} (\mathbf{b}^\dagger)^k \mathbf{d}^\dagger_1 \mathbf{d}^\dagger_2 |0\rangle \,, \tag{B.19}$$

$$\mathcal{D}^k \bar\psi = (\mathbf{a}^\dagger)^{k} (\mathbf{b}^\dagger)^{k+1} \mathbf{c}^\dagger \mathbf{d}^\dagger_1 \mathbf{d}^\dagger_2 |0\rangle \,. \tag{B.20}$$

B.3 Two-letter Superconformal Primaries

By demanding that they are annihilated by all the conformal supercharges and by the appropriate combinations of Poincaré supercharges, we have worked out the expressions for the superconformal primaries of the irreducible modules that appear on the right hand side of the tensor products (2.30–2.39). The grassmannOps.m oscillator package by Jeremy Michelson and Matthew Headrick was extremely useful for this task. We simply quote the results:

$\underline{\mathcal{V}\times\mathcal{V}}$**:**

$$\bar{\mathcal{B}}_{2(0,0)} = \lambda_+\lambda_- - \lambda_-\lambda_+ \,, \tag{B.21}$$

$$\bar{\mathcal{B}}_{2(1,0)} = \lambda_+\lambda_+ \,, \tag{B.22}$$

$$\begin{aligned}\hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q-2}{2})} &= \sum_{k=0}^{q-2}\frac{(-1)^k}{q(q+1)}\binom{q+1}{k+2}\binom{q-2}{k}\mathcal{D}^{q-k-2}\lambda_+\mathcal{D}^k\mathcal{F}_{++} \\ &+\frac{1}{q}\sum_{k=0}^{q-2}\frac{(-1)^{q-k}}{k+2}\binom{q-2}{k}\binom{q}{k+1}\mathcal{D}^k\mathcal{F}_{++}\mathcal{D}^{q-k-2}\lambda_+\end{aligned} \tag{B.23}$$

For $\bar{\mathcal{V}}\times\bar{\mathcal{V}}$ the expressions are identical with $(\lambda,\mathcal{F})$ replaced by $(\bar{\lambda},\bar{\mathcal{F}})$.

$\underline{\mathcal{V}\times\mathcal{X}}$**:**

$$\bar{\mathcal{B}}_{\frac{5}{3}(\frac{1}{2},0)} = \lambda_+Q\,, \tag{B.24}$$

$$\begin{aligned}\hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q-1}{2})} &= \sum_{k=0}^{q-1}(-1)^k\binom{q-1}{k}\binom{q+1}{k}\mathcal{D}^{q-k-1}\mathcal{F}_{++}\mathcal{D}^kQ \\ &+(q+1)\sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q-1}{k}\binom{q}{k}\mathcal{D}^{q-k-1}\lambda_+\mathcal{D}^k\psi_+\,.\end{aligned} \tag{B.25}$$

For the $\tilde{\mathcal{X}}\times\mathcal{V}$ primary we replace (Q,ψ) by $(\tilde{Q},\tilde{\psi})$ and interchange the order of the fields (taking into account fermionic minus signs).

$\underline{\bar{\mathcal{V}}\times\mathcal{X}}$**:**

$$\hat{\mathcal{C}}_{(\frac{q}{2},\frac{q+1}{2})} = \sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q+1}{k}\mathcal{D}^{q-k}\bar{\lambda}_{\dot{+}}\mathcal{D}^kQ \tag{B.26}$$

$$-q\sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q-1}{k}\binom{q+1}{k}\mathcal{D}^{q-k-1}\bar{\mathcal{F}}_{\dot{+}\dot{+}}\mathcal{D}^k\psi_+\,. \tag{B.27}$$

For the $\tilde{\mathcal{X}}\times\bar{\mathcal{V}}$ primary we replace (Q,ψ) by $(\tilde{Q},\tilde{\psi})$ and interchange the order of the fields.

$\underline{\bar{\mathcal{X}} \times \mathcal{V}}$**:**

$$\hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q}{2})} = \sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q+1}{k+1}\mathcal{D}^{q-k}\bar{Q}\mathcal{D}^k\lambda_+ + q\sum_{k=0}^{q-1}\frac{(-1)^k}{k+2}\binom{q-1}{k}\binom{q+1}{k+1}\mathcal{D}^{q-k-1}\bar{\psi}_{\dot{+}}\mathcal{D}^k\mathcal{F}_{++}\,. \quad \text{(B.28)}$$

For the $\mathcal{V}\times\bar{\tilde{\mathcal{X}}}$ primary we replace (Q,ψ) by $(\tilde{Q},\tilde{\psi})$ and interchange the order of the fields.

$\underline{\bar{\mathcal{X}} \times \bar{\mathcal{V}}}$**:**

$$\mathcal{B}_{-\frac{5}{3}(0,\frac{1}{2})} = \bar{Q}\bar{\lambda}_+\,, \quad \text{(B.29)}$$

$$\hat{\mathcal{C}}_{(\frac{q-1}{2},\frac{q+1}{2})} = \sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q-1}{k}\binom{q}{k}\mathcal{D}^k\bar{\psi}_{\dot{+}}\mathcal{D}^{q-k-1}\bar{\lambda}_{\dot{+}} + \sum_{k=0}^{q-1}\frac{(-1)^{q-k}}{q+1}\binom{q-1}{k}\binom{q+1}{k+2}\mathcal{D}^{q-k-1}\bar{Q}\mathcal{D}^k\bar{\mathcal{F}}_{\dot{+}\dot{+}}\,. \quad \text{(B.30)}$$

For the $\bar{\mathcal{V}}\times\bar{\tilde{\mathcal{X}}}$ primary we replace (Q,ψ) by $(\tilde{Q},\tilde{\psi})$ and interchange the order of the fields.

$\underline{\bar{\mathcal{X}} \times \mathcal{X}}$**:**

$$\hat{\mathcal{C}}_{(\frac{q}{2},\frac{q}{2})} = \sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q}{k}\mathcal{D}^{q-k}\bar{Q}\mathcal{D}^kQ + q\sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q-1}{k}\binom{q}{k}\mathcal{D}^{q-k-1}\bar{\psi}_{\dot{+}}\mathcal{D}^k\psi_+\,. \quad \text{(B.31)}$$

This primary is *gauge* contracted. For the *flavor* contracted $\bar{\tilde{\mathcal{X}}}\times\tilde{\mathcal{X}}$ primary we replace (Q,ψ) by $(\tilde{Q},\tilde{\psi})$.

$\underline{\tilde{\mathcal{X}} \times \bar{\tilde{\mathcal{X}}}}$**:**

$$\hat{\mathcal{C}}_{(\frac{q}{2},\frac{q}{2})} = \sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q}{k}\mathcal{D}^{q-k}\tilde{Q}\mathcal{D}^k\bar{\tilde{Q}} + q\sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q-1}{k}\binom{q}{k}\mathcal{D}^{q-k-1}\tilde{\psi}_+\mathcal{D}^k\bar{\tilde{\psi}}_{\dot{+}}\,. \quad \text{(B.32)}$$

This primary is *gauge* contracted. For the *flavor* contracted $\mathcal{X} \times \bar{\mathcal{X}}$ primary we replace $(\tilde{Q},\tilde{\psi})$ by (Q,ψ) .

<u>$\tilde{\mathcal{X}} \times \mathcal{X}$:</u>

$$\begin{aligned}
\bar{\mathcal{B}}_{\frac{4}{3}(0,0)} &= \tilde{Q}Q && \text{(B.33)}\\
\hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q}{2})} &= \sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q+1}{k+1}\mathcal{D}^{q-k}\tilde{Q}\mathcal{D}^k\psi_+ \\
&\quad -\sum_{k=0}^{q}(-1)^{q-k}\binom{q}{k}\binom{q+1}{k+1}\mathcal{D}^k\tilde{\psi}_+\mathcal{D}^{q-k}Q\,. && \text{(B.34)}
\end{aligned}$$

For the $\bar{\mathcal{X}}\times\bar{\tilde{\mathcal{X}}}$ primary we interchange $(\tilde{Q},\psi)$ by $(\bar{Q},\bar{\tilde{\psi}})$ (also for the conjugates).

<u>$\mathcal{V} \times \bar{\mathcal{V}}$:</u>

$$\begin{aligned}
\hat{\mathcal{C}}_{(\frac{q}{2},\frac{q}{2})} &= \sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q-1}{k}\binom{q}{k}\mathcal{D}^{q-k-1}\lambda_+\mathcal{D}^k\bar{\lambda}_{\dot{+}} \\
&\quad -\sum_{k=0}^{q-2}\frac{(-1)^{q-k}}{q}\binom{q1}{k}\binom{q}{k+2}\mathcal{D}^k\mathcal{F}_{++}\mathcal{D}^{q-k-2}\bar{\mathcal{F}}_{\dot{+}\dot{+}}\,. && \text{(B.35)}
\end{aligned}$$

For the $\bar{\mathcal{V}} \times \mathcal{V}$ primary we replace $(\lambda,\mathcal{F})$ by$(\bar{\lambda},\bar{\mathcal{F}})$.

Appendix C

Scalar conformal blocks

In this section we will use the method of [69] to obtain the scalar conformal blocks as eigenfunctions of the conformal Casimir operator. This procedure can be applied with no major changes to two-point functions involving tensor operators, and it was used succesfully in section 5.3 to decompose the two-point function of the stress tensor.

Bulk channel

The $SO(d+1,1)$ generators are,

$$L_{AB} = P_A \frac{\partial}{\partial P^B} - P_B \frac{\partial}{\partial P^A} \,, \tag{C.1}$$

where $P^A = (P^+, P^-, P^1, \dots P^d)$. To obtain the conformal blocks we solve the eigenvalue problem [69],

$$L^2 \langle \mathcal{O}_1(P_1) \mathcal{O}_2(P_2) \rangle = -C_{\Delta,0} \langle \mathcal{O}_1(P_1) \mathcal{O}_2(P_2) \rangle \,, \tag{C.2}$$

with $L^2 = \frac{1}{2}(L^{(1)}_{AB} + L^{(2)}_{AB})(L^{(1)\,AB} + L^{(2)\,AB})$ and $C_{\Delta,l} = \Delta(\Delta - d) + l(l + d - 2)$, where Δ and l are the dimension and spin of the internal operator. Because of Lorentz invariance no operators with spin can ever appear in the bulk conformal block decomposition, hence we set $l = 0$ in equation (C.2).

Once the asymptotic behavior of $f(\xi)$ is given, the conformal block is completely fixed. In the $\xi \to 0$ limit the bulk OPE dictates [61],

$$f(\xi) \sim \xi^{-\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta)} \,. \tag{C.3}$$

where Δ_1 and Δ_2 are the dimensions of the external operators. Stripping out this factor $f(\xi) = \xi^{-\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta)} g(\xi)$ and plugging in (C.2) we obtain a standard hypergeometric equation,

$$\xi(1+\xi)g''(\xi) + (c + (a+b+1)\xi)g'(\xi) + abg(\xi) = 0\,, \tag{C.4}$$

with,

$$a = \frac{1}{2}(\Delta + \Delta_1 - \Delta_2)\,, \quad b = \frac{1}{2}(\Delta - \Delta_1 + \Delta_2)\,, \quad c = \Delta - \frac{d}{2} + 1\,. \tag{C.5}$$

The conformal block for the bulk channel is then,

$$f(\xi) = \xi^{-\frac{1}{2}(\Delta_1+\Delta_2-\Delta)}{}_2F_1\Big(\frac{1}{2}(\Delta+\Delta_1-\Delta_2), \frac{1}{2}(\Delta-\Delta_1+\Delta_2), \Delta - \frac{d}{2}+1; -\xi\Big)\,, \tag{C.6}$$

in perfect agreement with [61].

Boundary channel

In this channel we consider the restricted conformal group. The $SO(d,1)$ generators are,

$$L_{ab} = P_a \frac{\partial}{\partial P^b} - P_b \frac{\partial}{\partial P^a}\,. \tag{C.7}$$

where $P^a = (P^+, P^-, P^1, \dots P^{d-1})$. To obtain the conformal blocks we act with the Casimir operator on one of the fields and solve the eigenvalue problem,

$$L^2 \langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)\rangle = -C_{\Delta,0}\langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)\rangle\,. \tag{C.8}$$

where $C_{\Delta,l} = \Delta(\Delta - d + 1) + l(l + d - 3)$ in this case. For this particular two point function only scalar blocks are present, so $l = 0$ again. However, this is no longer true for operators with indices (see subsection 5.3.4). The asymptotic behavior for $\xi \to \infty$ can be obtained from the bulk-to-boundary OPE [61],

$$f(\xi) \sim \xi^{-\Delta}\,. \tag{C.9}$$

Stripping out this factor and plugging in (C.8) we obtain another hypergeometric equation. The boundary block is,

$$f(\xi) = \xi^{-\Delta}{}_2F_1\Big(\Delta, \Delta - \frac{d}{2} + 1, 2\Delta + 2 - d; -\frac{1}{\xi}\Big)\,, \tag{C.10}$$

again in perfect agreement with [61].

Appendix D

Solutions to crossing symmetry for scalar operators

In this section we discuss a few solutions to the crossing symmetry equations for scalar two-point functions. We have:

$$\langle O(x_1)O(x_2)\rangle = \frac{1}{(2x_1^d)^\Delta(2x_2^d)^\Delta}\xi^{-\Delta}G(\xi) \tag{D.1}$$

where Δ is the conformal dimension of the operator O. The conformal block decomposition is,

$$G(\xi) = 1 + \sum_k \lambda_k a_k f_{\mathrm{bulk}}(\Delta_k;\xi) = \xi^\Delta \sum_l \mu_l^2 f_{\mathrm{bdy}}(\Delta_l;\xi) \tag{D.2}$$

with λ_k and μ_k three-point couplings and a_k the coefficient of the one-point function of the k'th operator.

D.1 Two-dimensional Ising model

In this section we will decompose several correlators for the two-dimensional Ising model. The basic fields of the theory, corresponding to the energy and spin operators, will be denoted by ε and σ respectively and have scaling dimensions $\Delta_\varepsilon = 1$ and $\Delta_\sigma = \frac{1}{8}$ respectively. As we discussed in section 5.2, there are three different conformally invariant boundary conditions (or boundary states), given in equations (6.1) and (6.2). The first two are related by the $\mathbb{Z}_2$ symmetry of the theory and result in the same two-point function of σ.

The two remaining possible two-point functions for the σ field are then [60],

$$G^{\pm}_{\sigma\sigma} = \xi^{1/8}\sqrt{\left(\frac{1+\xi}{\xi}\right)^{1/4} \pm \left(\frac{\xi}{1+\xi}\right)^{1/4}}\,. \tag{D.3}$$

As we shall see below, the $+$ sign corresponds to the extraordinary transition, *i.e.* the $|\mathbf{1}\rangle\!\rangle$ and $|\varepsilon\rangle\!\rangle$ Cardy boundary states, whereas the $-$ sign corresponds to the ordinary transition which is the $|\sigma\rangle\!\rangle$ Cardy boundary state.

The full conformal block decomposition can in principle be obtained from Virasoro representation theory. We content ourselves here with a simpler analysis where we expand the correlation function in the limits $\xi \to 0$ and $\xi \to \infty$ and match the coefficients of the expansion to conformal blocks. The bulk block decomposition becomes

$$G^{\pm}_{\sigma\sigma} = 1 \pm \frac{1}{2} f_{\text{bulk}}(1;\xi) + \frac{1}{64} f_{\text{bulk}}(4;\xi) + \frac{9}{40960} f_{\text{bulk}}(8;\xi) \pm \frac{1}{32768} f_{\text{bulk}}(9,\xi) + \ldots \tag{D.4}$$

The bulk spectrum corresponds to the identity $\mathbf{1}$ and the energy ε operators plus scalar Virasoro descendants. For example, we may identify the operator of dimension 4 with $L_{-2}\bar{L}_{-2}\mathbf{1}$ and the operator of dimension 9 with a level four descendant of ε. The absence of an operator of dimension 5 is in agreement with the fact that ε has a null descendant at level two, so $L_{-2}\bar{L}_{-2}\varepsilon$ is actually an $SO(2,2)$ descendant.

In the boundary channel we find that:

$$\begin{aligned}
\xi^{-\Delta_\sigma} G^{+}_{\sigma\sigma} &= \sqrt{2} + \frac{1}{32\sqrt{2}} f_{\text{bdy}}(2;\xi) + \frac{9}{20480\sqrt{2}} f_{\text{bdy}}(4;\xi) + \frac{25}{1835008\sqrt{2}} f_{\text{bdy}}(6;\xi) + \ldots \\
\xi^{-\Delta_\sigma} G^{-}_{\sigma\sigma} &= \frac{1}{\sqrt{2}} f_{\text{bdy}}\left(\frac{1}{2};\xi\right) + \frac{1}{16384\sqrt{2}} f\left(\frac{9}{2};\xi\right) + \frac{1}{327680\sqrt{2}} f_{\text{bdy}}\left(\frac{13}{2};\xi\right) + \ldots
\end{aligned} \tag{D.5}$$

The constant term in the $+$ case corresponds to a one-point function of σ and therefore the $\mathbb{Z}_2$ symmetry is broken by the boundary conditions. We can thus identify it with the extraordinary transition. As an additional check one may verify that the bulk block decomposition agrees with the decompositions (6.2) and (6.1).

For completeness, we present the conformal block decomposition for the energy two-point function. We have [60],

$$G^{\pm}_{\varepsilon\varepsilon} = \xi + \frac{1}{\xi+1}\,, \tag{D.6}$$

so this expression is valid for both boundary conditions. The decomposition in the bulk channel is,

$$G^{\pm}_{\varepsilon\varepsilon} = 1 + \sum_{n=1}^{\infty} \binom{2n-3}{n-2}^{-1} f_{\text{bulk}}(2n;\xi)\,. \tag{D.7}$$

For the boundary expansion we obtain,

$$\xi^{-\Delta_\varepsilon} G^{\pm}_{\varepsilon\varepsilon} = 1 + \sum_{n=1}^{\infty} \binom{4n-3}{2n-2}^{-1} f_{\text{bdy}}(2n;\xi). \tag{D.8}$$

From the expressions above we learn that the coefficients of the conformal blocks are positive both in the boundary and in the bulk channels.

D.2 The unitarity minimal models and their analytic continuation

Let us now generalize the results of the previous subsection to the whole series of the unitarity minimal models. Primary operators in the $(m, m+1)$ model, $m \geq 3$, are labeled by integers (r, s), with $1 \leq r \leq m-1$, $1 \leq s \leq m$ and the identification $(r, s) \sim (m-r, m+1-s)$. Denoting the $(1, 2)$ operator by σ and the $(1, 3)$ operator by ε, the relevant OPE and scaling dimensions are

$$\sigma \times \sigma = \mathbf{1} + \varepsilon\,, \qquad \Delta_\sigma = \frac{1}{2} - \frac{3}{2(m+1)}\,, \qquad \Delta_\varepsilon = 2 - \frac{4}{m+1}\,. \tag{D.9}$$

We can eliminate m to find

$$\Delta_\varepsilon = \frac{2}{3}(4\Delta_\sigma + 1)\,, \tag{D.10}$$

and we will work with Δ_σ rather than m as our independent variable from now on.

We are after the $\langle\sigma\sigma\rangle$ correlator with the Cardy boundary condition labelled by the identity. (Recall that in the Ising model this Cardy state is associated to the extraordinary transition, see equation (6.1)). This correlator can be obtained as a special case of a result obtained in the context of Liouville theory with ZZ boundary conditions [77], where the two-point function

$$\langle V_{-b/2}(x) V_\alpha(y)\rangle \tag{D.11}$$

was evaluated. Here $V_\alpha(x)$ denotes the usual Liouville vertex operator with scaling dimension $\Delta_\alpha = \alpha(Q-\alpha)$ with $Q = b + b^{-1}$. We will be interested in the case $\alpha = -b/2$ and b set to the minimal model values given by

$$c = 1 + 6Q^2 = 1 - \frac{6}{m(m+1)}\,. \tag{D.12}$$

One may verify that solving this equation for b results in a scaling dimension of $V_{-b/2}$ which is precisely Δ_σ given in (D.9).

The two-point function from [77] takes the form:

$$\begin{aligned} G_{\sigma\sigma}(\xi) = 2\sin\left(\frac{\pi}{6}(1+4\Delta_\sigma)\right)\xi^{(4\Delta_\sigma+1)/3}(1+\xi)^{-(\Delta_\sigma+1)/3} \\ \times {}_2F_1\left(\frac{1-2\Delta_\sigma}{3}, \frac{2+2\Delta_\sigma}{3}; \frac{2-4\Delta_\sigma}{3}; \frac{1}{\xi+1}\right)\,. \end{aligned} \tag{D.13}$$

The boundary conformal block decomposition of this correlation functions contains operators with even dimensions,

$$\begin{aligned} \xi^{-\Delta_\sigma} G_{\sigma\sigma}(\xi) = 2\sin\left(\frac{\pi}{6}(1+4\Delta_\sigma)\right)\Bigg(1 + \frac{\Delta_\sigma(1+\Delta_\sigma)}{2(5-4\Delta_\sigma)} f_{\rm bdy}(2,\xi) + \frac{\Delta_\sigma(1+\Delta_\sigma)^2(2+5\Delta_\sigma)}{40(11-4\Delta_\sigma)(5-4\Delta_\sigma)} f_{\rm bdy}(4,\xi) \\ + \frac{\Delta_\sigma(1+\Delta_\sigma)^2\left(20+106\Delta_\sigma+35\Delta_\sigma^2+21\Delta_\sigma^3\right)}{1008(17-4\Delta_\sigma)(11-4\Delta_\sigma)(5-4\Delta_\sigma)} f_{\rm bdy}(6,\xi) + \ldots\Bigg)\,, \end{aligned} \tag{D.14}$$

in agreement with the fact that the only boundary block is the identity Virasoro block. In the bulk channel we find the identity and the ε Virasoro blocks, leading to a decomposition into $SO(2,1)$ blocks with operators of dimension of $1+4n$ and $\Delta_\varepsilon + 4n$ with n a non-negative integer. For the first few coefficients we find

$$\begin{aligned} G_{\sigma\sigma}(\xi) = 1 + \frac{\Delta_\sigma(1+\Delta_\sigma)}{2(5-4\Delta_\sigma)} f_{\rm bulk}(4,\xi) + \frac{\Delta_\sigma(1+\Delta_\sigma)^2(2+5\Delta_\sigma)}{40(11-4\Delta_\sigma)(5-4\Delta_\sigma)} f_{\rm bulk}(8,\xi) + \ldots \\ - \frac{\Gamma\left(\frac{2-4\Delta_\sigma}{3}\right)\Gamma\left(\frac{2+2\Delta_\sigma}{3}\right)}{\Gamma(-2\Delta_\sigma)\Gamma\left(\frac{4+4\Delta_\sigma}{3}\right)}\left(f_{\rm bulk}(\Delta_\varepsilon,\xi) + \frac{(1+\Delta_\sigma)(2+5\Delta_\sigma)(-1+8\Delta_\sigma)}{6(7+4\Delta_\sigma)(5+8\Delta_\sigma)} f_{\rm bulk}(\Delta_\varepsilon+4,\xi) + \ldots\right)\,. \end{aligned} \tag{D.15}$$

Up to the normalization factor $2\sin(\frac{\pi}{6}(1+4\Delta_\sigma))$, the coefficients of the first series are the same as those of the boundary identity Virasoro block. Indeed, in either channel these blocks correspond to Virasoro descendants of an identity operator. Notice also that the coefficient of the block with dimension $\Delta_\varepsilon+4$ has a zero precisely when $\Delta_\sigma = \frac{1}{8}$, reflecting the aforementioned fact that $L_{-2}\varepsilon$ is actually an $SO(2,2)$ descendant in the two-dimensional Ising model. (Indeed in the Ising model the $(1,3)$ primary is identified with the $(2,1)$ primary which has a level-two null descendant.)

Remarkably, the coefficients of the boundary conformal blocks turn out to be positive for $0 < \Delta_\sigma < \frac{5}{4}$.[1] This implies that we have found a solution of the crossing symmetry equation that is consistent with the unitarity requirements for *any* value of Δ_σ in this interval, given simply by the analytic continuation of (D.13) away from the minimal model values for Δ_σ. Of course this does not imply that this correlator can always be embedded in a full-fledged unitary CFT – in fact we already know that this is only possible if Δ_σ has one of the minimal model values.

As we pointed out repeatedly in this paper, unitarity does not require the coefficients of the bulk channel conformal blocks to be positive. These coefficients however do turn out to be positive for the smaller range $\frac{1}{8} < \Delta_\sigma < 1$. The lower and upper endpoint of this range are determined by the zeroes of the blocks of dimension $\Delta_\varepsilon + 4$ and Δ_ε, respectively.

In summary, for the range $0 < \Delta_\sigma < \frac{5}{4}$ we have found an exact solution to the boundary crossing symmetry equation (5.21), with the dimension of the first bulk scalar primary ϵ in the $\sigma\sigma$ OPE given by (D.10). In the smaller range $\frac{1}{8} < \Delta_\sigma < 1$ the bulk expansion satisfies positivity.

D.3 $\langle \phi^2 \phi^2 \rangle$ correlator

In this section we will decompose $\langle \phi^2 \phi^2 \rangle$ in free field theory. This expansion complements the order ϵ expression for the scalar two-point function of section 5.2. The ϕ^2 two-point function is,

$$G^{\pm}_{\phi^2\phi^2} = \left(1 \pm \left(\frac{\xi}{\xi+1}\right)^{\frac{1}{2}d-1}\right)^2 + \frac{N}{2}\xi^{d-2}\,, \tag{D.16}$$

where the plus/minus sign corresponds to Neumann/Dirichlet boundary conditions, and N is the number of scalars. The conformal block expansion in the bulk channel is

$$G^{\pm}_{\phi^2\phi^2} = 1 + \lambda a_{\phi^2} f_{\text{bulk}}(d-2;\xi) + \sum_{n=0}^{\infty} \lambda a_{\phi^4,n} f_{\text{bulk}}(2d-4+2n;\xi)\,, \tag{D.17}$$

[1] We have verified this statement to high order and believe that it is generally true although we currently cannot offer a rigorous proof.

with

$$\lambda a_{\phi^2} = \pm 2\,,$$
$$\lambda a_{\phi^4,n} = \frac{\left((-1)^n 2^d \Gamma(\frac{d-1}{2})\Gamma(\frac{1}{2}d+n-1) + 4N\sqrt{\pi}\Gamma(d+n-2)\right)\Gamma(d+n-2)\Gamma(\frac{3}{2}d+n-4)}{8\sqrt{\pi}\Gamma(d-2)^2\Gamma(n+1)\Gamma(\frac{3}{2}d+2n-4)}\,. \tag{D.18}$$

The Neumann expansion exhibits positivity while the Dirichlet case has one negative coefficient. In the boundary channel we have,

$$\xi^{-d+2} G^{\pm}_{\phi^2\phi^2} = \frac{N}{2} + \sum_{n=0}^{\infty} \mu_n^2 f_{\text{bdy}}(d-2+2n;\xi)\,, \tag{D.19}$$

with

$$\mu_n^2 = (1 \pm \delta_{n,0}) \frac{4^{1-n}}{(2n)!} \frac{\Gamma(\frac{d-1}{2}+n)\Gamma(\frac{1}{2}d+n-1)\Gamma(d+2n-3)}{\Gamma(\frac{1}{2}d-1)\Gamma(d-2)\Gamma(\frac{d+4n-3}{2})}\,, \tag{D.20}$$

with positivity in both cases.

D.4 The extraordinary transition

There is no extraordinary transition in 4 dimensions since the conformally invariant one-point function of a free field is not compatible with its equation of motion. In $4-\epsilon$ dimensions the equation of motion is however modified to:

$$\Box\phi = \frac{\lambda_*}{6}\phi^3 \tag{D.21}$$

with $\lambda_* = 48\pi^2\epsilon/(N+8)$ and $N = 1$ in our case. On the half-space this equation admits the solution:

$$\langle\phi(x)\rangle = \sqrt{\frac{12}{\lambda_*}} \frac{1}{x^d} \tag{D.22}$$

which to leading order is consistent with boundary conformal invariance. This solution is our starting point for the analysis of the extraordinary transition in the ϵ expansion.

Let us compute the two-point function of the scalar field ϕ. We may shift the field ϕ by its classical one-point function,

$$\phi(x) = \langle\phi(x)\rangle + \chi(x) \tag{D.23}$$

and find the propagator $G(x,y) = \langle \chi(x)\chi(y)\rangle$ by solving the linearized equation of motion around this solution,

$$\left(\Box - \frac{6}{(x^d)^2}\right) G(x,y) = \delta^d(x-y) \tag{D.24}$$

The solution compatible with the boundary conditions at $x^d = 0$ takes the form:

$$\begin{aligned} G(x,y) &= \frac{1}{(2x^d)(2y^d)}\xi^{-1}\left(\frac{1}{4\pi^2}G^0(\xi)\right) \\ G^0(\xi) &= \frac{1}{1+\xi} + 12\xi + 6\xi(1+2\xi)\log\left(\frac{\xi}{1+\xi}\right) \end{aligned} \tag{D.25}$$

with $\xi = (x-y)^2/(4x^d y^d)$, as before. On the first line we recognize the familiar form of a scalar two-point function for a CFT with a boundary. Taking the limit $\xi \to 0$ in (D.25) we see that the properly normalized operator is actually $2\pi\chi$ rather than χ, and similarly $2\pi\phi$ rather than ϕ. We will henceforth work with these rescaled operators. This implies that from now on $\langle\phi(x)\rangle = 3/(\sqrt{\epsilon}\,x^d)$ and we can drop the $4\pi^2$ on the first line of (D.25).

We will now expand the two-point function of ϕ in conformal blocks. It is important to note that OPE statements always refer to full correlation functions, *i.e.* including any disconnected contributions. In our case the disconnected part $\langle\phi(x)\rangle\langle\phi(y)\rangle$ is of order $1/\epsilon$ which makes it the leading-order term. Our first task is thus to decompose $\langle\phi(x)\rangle\langle\phi(y)\rangle$ in conformal blocks. In the boundary channel we of course find precisely the block corresponding to the identity operator and nothing else. In the bulk channel we find:

$$\langle\phi(x)\rangle\langle\phi(y)\rangle = \frac{36/\epsilon}{(2x^d)(2y^d)} = \frac{36/\epsilon}{(2x^d)(2y^d)}\xi^{-1}\left(f_{\text{bulk}}(2,\xi) + \sum_{n=1}^{\infty}\frac{2(n!)^2}{(2n)!}f_{\text{bulk}}(2+2n,\xi)\right) \tag{D.26}$$

Interestingly, the product of two one-point functions decomposes into an infinite set of bulk blocks with dimensions given by the even integers and with positive coefficients. However, as expected for a totally disconnected correlator, the *bulk* identity operator is missing at this order.

At the next order we should take into account that the one-point function of ϕ a priori has subleading corrections,

$$\langle\phi(x)\rangle = \frac{3}{\sqrt{\epsilon}\,x^d}\left(1+\epsilon\,a + \frac{\epsilon}{2}\log(2x^d)\right) \tag{D.27}$$

with an unknown coefficient a and with the logarithm originating from the correction to the scaling dimension of ϕ in $4-\epsilon$ dimensions. The full two-point function to order ϵ^0 becomes:

$$\begin{aligned} \langle\phi(x)\phi(y)\rangle &= \frac{1}{(2x^d)^{\Delta_\phi}(2y^d)^{\Delta_\phi}}\xi^{-\Delta_\phi}G^{\text{ext}}(\xi) \\ G^{\text{ext}}(\xi) &= \frac{36}{\epsilon}(1+2\epsilon\, a)\xi - 18\xi\log(\xi) + G^0(\xi) \end{aligned} \tag{D.28}$$

with $\Delta_\phi = 1-\epsilon/2$ the free-field dimension of ϕ in $4-\epsilon$ dimensions.

In the boundary channel the conformal block decomposition of this corrected correlator is again straightforward. The corrections to the disconnected part of course simply become corrections to the boundary identity block, whilst for the connected part we find that:

$$\xi^{-1}G^0(\xi) = \frac{1}{10}f_{\text{bdy}}(4;\xi) \tag{D.29}$$

so we find a single boundary block of dimension $d=4$. This is completely as expected. In particular, the existence of a gap of size d was an essential assumption in the numerical bootstrap for the bulk bounds.

In the bulk channel we find subleading corrections to the infinite series of blocks in (D.26) but no new blocks. The first few terms take the form:

$$\begin{aligned} G^{\text{ext}}(\xi) = 1 &+ \left(\frac{36}{\epsilon}+11+72a\right)f_{\text{bulk}}\left(2-\frac{2}{3}\epsilon;\xi\right) \\ &+ \left(\frac{36}{\epsilon}-12+72a\right)f_{\text{bulk}}\left(4,\xi\right) + \left(\frac{12}{\epsilon}-18+24a\right)f_{\text{bulk}}(6+2\epsilon,\xi) \\ &+ \left(\frac{18}{5\epsilon}+\frac{1}{20}(-241+144a)\right)f_{\text{bulk}}\left(8+\frac{16}{3}\epsilon,\xi\right)+\dots \end{aligned} \tag{D.30}$$

where it is understood that the blocks are evaluated in $4-\epsilon$ spacetime dimensions. The order $1/\epsilon$ terms in (D.30) of course coincide with (D.26). The identity operator is now present in the bulk channel, and the dimension of the next operator (which is $2-\frac{2}{3}\epsilon$) is precisely the one-loop dimension of ϕ^2 in the epsilon expansion. It would be interesting to compute a so we can get an idea of positivity of the coefficients for $\epsilon=1$.

D.5 A trivial solution

A particularly simple solution of (5.21) is obtained by assuming that the bulk channel only contains the identity operator, so all the non-trivial one-point functions are set to zero. In that case there is effectively no boundary at all and the two-point function is just $(x_1 - x_2)^{-2\Delta}$. This two-point function still has a boundary conformal block decomposition of the form:

$$\xi^{-\Delta} = \sum_{m=0}^{\infty} \mu_m^2 f_{\text{bdy}}(\Delta + m; \xi) \tag{D.31}$$

with

$$\mu_m^2 = \begin{cases} \frac{1}{2^m m!}(\Delta)_m(\Delta - \frac{d}{2} + 1)_{m/2}(\Delta - \frac{d-1}{2} + m)_{-m/2} & m \text{ even} \\ \frac{1}{2^m m!}(\Delta)_m(\Delta - \frac{d}{2} + 1)_{(m-1)/2}(\Delta - \frac{d-1}{2} + m)_{(1-m)/2} & m \text{ odd} \end{cases} \tag{D.32}$$

All the coefficients are positive for Δ greater than the unitarity bound and the boundary spectrum begins with an operator of dimension Δ.

D.6 Generalized free field theory

As a simple generalization of the free field theory result we define generalized free field (or gff) two-point functions in the presence of a boundary as:

$$\begin{aligned} \langle O(x_1)O(x_2)\rangle &= \frac{1}{(x_1 - x_2)^{2\Delta}} \pm \frac{1}{\left((x_1 - x_2)^2 + 4x_1^d x_2^d\right)^\Delta} \\ &= \frac{1}{(2x_1^d)^\Delta (2x_2^d)^\Delta}\xi^{-\Delta} G_{\text{gff}}^{\pm}(\xi) \qquad G_{\text{gff}}^{\pm}(\xi) = 1 \pm \left(\frac{\xi}{\xi+1}\right)^\Delta \end{aligned} \tag{D.33}$$

The conformal block decomposition in the bulk takes the form

$$G_{\text{gff}}^{\pm}(\xi) = 1 \pm \sum_{n=0}^{\infty} \frac{(-1)^n (\Delta)_n \left(2\Delta - \frac{d}{2} + 2n\right)_{-n}}{\left(\Delta - \frac{d}{2} + n + 1\right)_{-n} n!} f_{\text{bulk}}(2\Delta + 2n; \xi) \tag{D.34}$$

which has the expected 'double trace' infinite operator spectrum and coefficients with alternating signs. On the boundary we find that:

$$\begin{aligned}\xi^{-\Delta}G^{+}_{\text{gff}}(\xi) &= \sum_{n=0}^{\infty}\frac{(\Delta)_{2n}\left(\Delta-\frac{d-1}{2}+2n\right)_{-n}}{2^{2n-1}(2n)!\left(\Delta-\frac{d}{2}+n+1\right)_{-n}}f_{\text{bdy}}(\Delta+2n;\xi)\\ \xi^{-\Delta}G^{-}_{\text{gff}}(\xi) &= \sum_{n=0}^{\infty}\frac{(\Delta)_{2n+1}\left(\Delta-\frac{d-3}{2}+2n\right)_{-n}}{2^{2n}(2n+1)!\left(\Delta-\frac{d}{2}+n+1\right)_{-n}}f_{\text{bdy}}(\Delta+2n+1;\xi)\end{aligned} \tag{D.35}$$

and we find two 'single trace' operator spectra on the boundary, both with positive coefficients.

D.7 $O(N)$ model at large N

For the the $O(N)$ model with Neumann boundary conditions the scalar two-point function is given by [61],

$$G_{O(N)} = \left(\frac{1}{1+\xi}\right)^{\frac{1}{2}d-1}(1+2\xi)\,. \tag{D.36}$$

The bulk channel expansion is,

$$G_{O(N)} = 1+\sum_{n=0}^{\infty}\lambda a_n f_{\text{bulk}}(2n+2;\xi)\,, \tag{D.37}$$

with

$$\lambda a_n = (-1)^{2n}\frac{(d^2-4d(n+2)+8(1+n)^2+4)\Gamma(1-\frac{1}{2}d+n)\Gamma(2-\frac{1}{2}d+n)^2}{4\Gamma(2-\frac{1}{2}d)^2\Gamma(n+2)\Gamma(2-\frac{1}{2}d+2n)}\,. \tag{D.38}$$

As in all the expansions with Neumann boundary conditions studied in this appendix, the bulk channel coefficients are positive. Finally, the boundary channel expansion is,

$$\xi^{-\frac{1}{2}d+1}G_{O(N)} = 2f_{\text{bdy}}(d-3;\xi)\,. \tag{D.39}$$

It is somewhat unexpected that in this channel we have a single block.

Appendix E

Conformal block decompositions for $T_{\mu\nu}$

In this appendix we present a few explicit examples of conformal block decompositions of the form (5.40) for the two-point function of the stress tensor.

E.1 Two bulk dimensions

The conformal block decomposition of the stress tensor two-point function in two dimensions is a bit subtle, see [59] for details. First of all, the residual Virasoro symmetry plus the absence of energy flow across the boundary completely determines the two-point function. Furthermore, the number of independent tensor structures decreases to two and the two functions $f(\xi)$ and $g(\xi)$ have to be replaced with the single function $2\xi f(\xi) + (1+\xi)g(\xi)$. With our unit normalization we find that the resulting two-point function is given precisely by the boundary scalar block, with a coefficient that is equal to 4. In the bulk channel we find the identity plus a single block of dimension 4 with unit coefficient.

E.2 Free field theory for general d

The two-point function of the stress tensor in free field theory for $d > 2$ decomposes into infinitely many blocks in either channel. Without presenting all the formulas, we have presented the first few operators and their associated coefficients in both the bulk and the boundary channel in the tables. Notice that the coefficients in the bulk channel are not positive for either boundary condition.

Δ	$\lambda a_{\mathcal{O}}$
0	1
$d-2$	$\pm\left(\frac{(-2+d)d(1+d)}{4(-1+d)}\right)$
$2d$	$+1$
$2d+2$	$-\left(\frac{(-2+d)d}{(2+3d)}\right)$
$2d+4$	$+\left(\frac{(-2+d)d^2(1+d)}{6(2+d)(4+3d)}\right)$
$2d+6$	$-\left(\frac{(-2+d)d^2(1+d)(2+d)}{18(8+3d)(10+3d)}\right)$
$2d+2m$	$\ldots$

Table E.1: Bulk conformal block decomposition of the two-point function of the stress tensor in free field theory. The first block corresponds to the identity operator and its coefficient sets the overall normalization. The plus/minus sign corresponds to the special/ordinary transition, i.e. Neumann/Dirichlet boundary conditions.

Δ	l	μ^2
d	0	$\frac{2d}{(-1+d)}$
d	2	$2^{1-2d}(1\pm 1)$
$d+2$	2	$\frac{2^{-2-2d}d(-1+d)(2+d)}{(1+d)}$
$d+4$	2	$\frac{2^{-6-2d}d(-1+d)2+d)^2(4+d)}{3(7+d)}$
$d+2m$	2	$\ldots$

Table E.2: Boundary conformal block decomposition of the two-point function of the stress tensor in free field theory.

E.3 Extraordinary transition

In this subsection we compute the two-point function of the stress tensor in the extraordinary transition.

The classical stress tensor for the $\lambda\phi^4$ theory with a curvature coupling z

takes the form:

$$T_{\mu\nu}(x) = \frac{2}{\sqrt{3}}\left(\left(2z - \frac{1}{2}\right)\partial_\mu\phi\partial_\nu\phi + 2z\,\phi\partial_\mu\partial_\nu\phi + g_{\mu\nu}\left(\frac{\lambda}{48}\phi^4 + \left(\frac{1}{4} - 2z\right)\partial_\rho\phi\partial^\rho\phi - 2z\phi\Box\phi\right)\right) \tag{E.1}$$

One may easily verify that it is traceless in $d = 4$ for $z = 1/12$ which therefore corresponds to the conformally coupled scalar. We will henceforth use $z = 1/12$. In that case $T_{\mu\nu}$ is unit normalized in free field theory, more precisely $\langle T_{\mu\nu}(x)T_{\rho\sigma}(y)\rangle = \frac{H_{12}^2}{(4\xi)^4(x^d)^4(y^d)^4}$ provided $\langle\phi(x)\phi(y)\rangle = \frac{1}{(x-y)^2}$.

The correlation functions of the scalar ϕ were computed to leading order in subsection D.4. Upon substituting the solution $\langle\phi(x)\rangle = 3/(\sqrt{\epsilon}\,x^d)$ in (E.1) we find that the one-point function of $T_{\mu\nu}$ vanishes, in agreement with the requirements of boundary conformal invariance. At the next order we substitute $\phi(x) = \langle\phi(x)\rangle + \chi(x)$ and expand in ϵ to find an expression of the form:

$$T_{\mu\nu}(x) = \frac{1}{\sqrt{\epsilon}}\mathcal{T}_{\mu\nu}[x^d,\partial_x]\chi(x) + \ldots \tag{E.2}$$

where $\mathcal{T}_{\mu\nu}[x^d,\partial_x]$ is a linear differential operator which explicitly depends on x^d. To leading order we therefore obtain that

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(y)\rangle = \frac{1}{\epsilon}\mathcal{T}_{\mu\nu}[x^d,\partial_x]\mathcal{T}_{\rho\sigma}[y^d,\partial_y]\langle\chi(x)\chi(y)\rangle \tag{E.3}$$

We can now substitute the solution $G(x,y) = \langle\chi(x)\chi(y)\rangle$, which is equation (D.25) without the factor of $4\pi^2$, work out the action of the differential operators $\mathcal{T}$ and collect various terms to eventually find a two-point function of the form:

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(y)\rangle = \frac{f^{\text{ext}}(\xi)H_{12}^2 + g^{\text{ext}}(\xi)H_{12}Q_{12} + h^{\text{ext}}(\xi)Q_{12}^2}{(4\xi)^4(x^d)^4(y^d)^4} \tag{E.4}$$

where the tensor structures H_{12} and Q_{12} are defined (in the projective cone notation) in (5.45) and (5.56) and

$$f^{\text{ext}}(\xi) = \frac{16\xi}{\epsilon(1+\xi)^3} \qquad g^{\text{ext}}(\xi) = \frac{64\xi^2(2+5\xi)}{\epsilon(1+\xi)^4} \qquad h^{\text{ext}}(\xi) = \frac{64\xi^3\left(1+5\xi+10\xi^2\right)}{\epsilon(1+\xi)^5} \tag{E.5}$$

Upon comparing (E.4) with the last equation in (5.55) we see that this correlation function has exactly the right tensor structure to be consistent with boundary conformal invariance. Furthermore, the functions $(f^{\text{ext}}, g^{\text{ext}}, h^{\text{ext}})$ also satisfy the Ward identities (5.57). These are rather non-trivial checks of our result.

The conformal block decomposition of (E.4) turns out to be remarkably simple. In the boundary we find only a scalar block (which must have dimension d by the Ward identities) with coefficient $640/\epsilon$. In the bulk we find three blocks,

$$\begin{pmatrix} f^{\text{ext}}(\xi) \\ g^{\text{ext}}(\xi) \\ h^{\text{ext}}(\xi) \end{pmatrix} = \frac{160}{\epsilon}\begin{pmatrix} f_{\text{bulk}}(2;\xi) \\ g_{\text{bulk}}(2;\xi) \\ h_{\text{bulk}}(2;\xi) \end{pmatrix} + \frac{480}{\epsilon}\begin{pmatrix} f_{\text{bulk}}(4;\xi) \\ g_{\text{bulk}}(4;\xi) \\ h_{\text{bulk}}(4;\xi) \end{pmatrix} + \frac{320}{\epsilon}\begin{pmatrix} f_{\text{bulk}}(6;\xi) \\ g_{\text{bulk}}(6;\xi) \\ h_{\text{bulk}}(6;\xi) \end{pmatrix} \tag{E.6}$$

all with positive coefficients. Notice that the identity operator is absent at this order.

Closer inspection of (E.6) leads to a subtlety that we would like to clarify. We easily identify the bulk block with dimension 2 as the operator ϕ^2. It appears in the TT OPE with an order one coefficient and its one-point function is $\langle \phi^2 \rangle = \langle \phi \rangle^2 \sim \epsilon^{-1}$ so altogether it appears at the right order in ϵ. The counting for the operator of dimension 4 is however a bit different. The only scalar primary of that dimension is ϕ^4 but its one-point function is of order ϵ^{-2}. Our result can therefore only be consistent if ϕ^4 appears in the stress tensor OPE only at order ϵ. It is in fact easy to see that the leading-order Feynman diagram for the $\langle TT\phi^4 \rangle$ tree-point function (which would be of order ϵ^0) has to vanish. This is because it factorizes into a product of two Feynman diagrams that each correspond to the $\langle T\phi^2 \rangle$ two-point function, which in turn vanishes by conformal invariance. This is also consistent with the fact that no dimension 4 block appears in the bulk conformal block decomposition of the stress tensor two-point function in free-field theory, cf. table E.1. From these tables we may also deduce that a similar cancellation should occur for the dimension 6 operator.

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