**Some Applications of Superspace**

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Abstract of the Dissertation

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Supersymmetry is a very popular topic in recent high energy physics theories. Especially, in order to make sense of string theory, we have to impose supersymmetry. Superspace method is proven to be a very useful approach to supersymmetric theories by treating supersymmetry as part of the geometry. In this dissertation, we use superspace method to investigate superconformal field theory and string/brane theory.

The first part of the dissertation, we rewrite semi-shortening conditions using superspace approach. The rewritten expression is covariant under superconformal transformation. We found that all the known semi-shortening condition are part of the covariantized ones and can be generalized to weaker shortening conditions. We also give an example how one can find other constraints from the known ones, particularly in $\mathcal{N} = 4$ SYM in projective superspace.

The second part is focused on "**F**-theory", a theory that has manifestly **U**-duality. It can be reduced to **M**-theory, manifestly T-dual version string theory, and ordinary string theory. The theory is formulated on coset space $G/H$ where $G$ is the **U**-duality group and $H$ is the unbroken symmetry group. We modified coset space formalism to find a general algebra for the symmetry currents. And we give an explicit example on 10 dimensional **F**-theory which

can reduce to $3D$ string theory.

To my family.

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# Chapter 1

# Introduction

It is hard to overstate the role played by symmetry in fundamental physics today. From electromagnetism to gravity, all fundamental forces are consequences of symmetries. Ever since Maxwell unified electricity and magnetism in his 1865 paper "A dynamical theory of the electromagnetic field" [1], physicists are interested in searching for mechanisms that could unify forces. In the 1960's, Glashow, Salam, and Weinberg succefully unified electrodynamics and weak force [2–4]. In 1974, Georgi and Glashow proposed a unification theory [5] aimed to include electrodynamics, weak force, and strong force into one theory, called Grand Unified Theory (GUT) [6].

Although unifying all gauge theories into a single theory might seem possible, there's still one well-known force that doesn't seem to fit in these theories — gravity. To make things even worse, Coleman and Mandula proved a no-go theorem in their paper [7] that under certain assumptions, it is impossible to unify different spins into one single theory. Fortunately, the theorem is only as strong as its assumptions: Although the assumptions listed in the abstract of the paper [7] are hard to argue with, however, there is one assumption that's not listed in the "assumptions" — the no-go theorem is based on "Lie algebra". Therefore, to unify the gravity and the gauge theories, one have to seek for other "algebras" to work with. In 1974, Haag, Lopuszanski and Sohnius [8], inspired by Wess and Zumino's paper [9], found that in additional to the usual symmetries, supersymmetry (was called "supergauge symmetry") might also be a symmetry of nature. The supersymmetry theory is an unique theory based on graded Lie algebra.

We know that in quantum field theory, there are a lot of infinities that has to be taken care of. Since supersymmetry relates bonsons with fermions and vice versa, they sometimes provide the same "infinities" but with opposite sign. Therefore, with supersymmetry, the theories are "more" finite. However, with "usual" supersymmetric fields (called component fields, will be explained shortly), the cancellations need to be found by hand. Since supersymmetry transformation is a symmetry transformation, we can consider it, just like rotation in 3 dimensional space, as transforming from one coordinate to another. In order to do that, we have to put in more coordinates. Moreover, since supersymmetry transformation transforms Grassmann even field (commuting fields, classically) to Grassmann odd field (anticommuting field, again, before quantization) and vice versa, the coordinate should also have Grassmann parity. Since spacetime coordinates are Grassman even, the newly introduced coordinates should have some Grassmann odd coordinates. The result of this is called superspace (first invented by Salam and Strathdee [10]).

Instead of treating supersymmetry as an additional symmetry, superspace method treats supersymmetry as part of its geometry. Therefore, in some sense, superspace method is easier to picture. The fields live in superspace (called superfields), in general, depend on the ordinary spacetime coordinate as well as the additional Grassmann odd coordinates. The superfields, just like Taylor expansion, can be expanded in its coordinates. Because the Grassmann odd coordinates are nilpotent, the expansion in those coordinates gives only finite terms. When expanding only in the Grassmann coordinates, the "coefficients"/components are ordinary fields. As stated before, the purpose of superspace is to make supersymmetry transformation part of coordinate transformation, hence, those component fields can transform to each other through supersymmetry transformation or "rotation" in superspace. Since superfields contains all the supersymmetric fields, calculation with superfields should have the some infinity cancellations automatic, as opposed to component field approach. Another advantage of superspace method is that, since supersymmetry is geometrized, we can define derivatives that are covariant under supersymmetry transformation. All field equations and constraints in superspace are made of covariant derivatives, hence superspace makes supersymmetry explicit.

In Chapter 3 of this thesis, we include conformal transformation in superspace (hence is enhanced to superconformal symmetry). It is known that the usual free field equations ($p^2 = 0$) are not covariant under superconformal transformation. In the talk in 1987 [11], Siegel proposed an alternative form for free fields which is covariant under superconformal symmetry transformation. Since Poincaré symmetry is part of conformal symmetry, the form is also Poincaré covariant. We investigate the superconformal covariant field equation and find that it includes all the known second order semi-shortening conditions. It is not hard to generalize this form further to find weaker constraints and found that other semi-shortening conditions still lies in the generalized constraints. We, therefore, speculate that all the semi-shortening conditions can be found in the generalized constraints.

Back to unification theory. Another attempt to unify all the forces (and the matters) is string theory. Rather than a point like particle, the fundamental ingredient in string theory is an one dimensional object. However, string theory is not "realistic" without supersymmetry. We know that in nature, there are basically two kinds of particles — force carriers (boson) and matters (fermion). In early string theory, there was no fermions. In order to introduce fermions into the theory, we adapt the concept of supersymmetry — every bosons have their corresponding fermions (and vice versa). A consequence of supersymmetry is that theory predicts that the nature is 10 dimensional. However, there is something awkward, although string theory was anticipated to be "the" fundamental theory, there were five self-consistent string theories. It was once thought that one of them would be the correct theory of everything (TOE). In practice, we only observe 4 dimensional spacetime in our daily life rather than 10, there must be some reason why we do not observe the extra 6 dimensions. One of the possible reason is that the extra dimensions are compactified into a small region. By compactifying the extra dimensions, some dualities between some types of string theory appeared [12–14]. In 1995, Witten proposed a theory (M-theory, a theory with one dimensional higher than string theory [15, 16]) that links all types of string theories together, and, therefore, unifies string theories. An interesting difference between M-theory and string theory is that the fundamental objects are no longer strings but higher $p$ dimensional objects, called $p$-brane [17–22]. It's also known that M-theory would further imply U-duality [14, 23, 24], conjectured to be a discrete subgroup of $E_{n(n)}$ ($n$ is the dimension of M-theory), which is the most general duality (including S-duality and T-duality) of string theory. Although the full M-theory hasn't been found yet, many properties of M-theory can already be found (or guessed) using its low energy limit [16, 25, 26]. A draw back of M-theory is that, although claimed that all types of string theories are certain limit of M-theory, it does not reduce to every types of string theory directly. In the later year, Vafa proposed the idea of an even higher dimensional theory (called F-theory) [27] so that the reduction to different type of string becomes manifest.

On the other hand, before the invention of M-theory, a theory that has manifest T-duality was proposed [28–30] (the theory is later known as Double Field Theory [31]). We know that in a T-dual theory, coordinates can have its T-dual partner. To make the duality manifest, the theory includes all the coordinates manifestly (coordinates and its T-dual coordinates) and let them have $O(D,D)/O(D-1,1)^2$ symmetry (T-dual symmetry).

F-theory [32–38] is meant to take advantage of both M-theory and T-theory – having U-dualities manifest by including all of its coordinates together with their dual coordinates. The theory is basically what Vafa proposed in 1996. Therefore, we will call this theory F-theory. For the rest of the thesis, F-theory is not the 12 dimensional example in his original paper but the theory with manifest U-duality. It is worthwhile pointing out that there is an interesting property in F-theory — worldvolume indices are also spacetime indices.

In Chapter 4, since the idea is fairly new, we will give a brief introduction on the F-theory. Then we investigate the relations between all symmetry currents of F-theory including the symmetry currents of the vacuum. We found that even in "flat" background, the "global" symmetry currents are not compatible with the symmetry currents of the vacuum. The solution is to introduce "vielbeins" which depend only on the gauge parameters to localize the symmetry currents.

# Chapter 2

# Brief Review on Superspace

This chapter is aimed to provide the basic techniques and notions of superspace that will be used in this thesis. For more comprehensive introduction on the subject, we refer readers to [39–42].

## 2.1 Coordinates, Symmetry Generators, and Covariant Derivatives

We start with a simple example — a scalar field $\phi(x)$ living in a group space $G$. A scalar field in Hilbert space is defined as state $|\phi\rangle$ projecting on a coordinate state $|x\rangle$, i.e.

$$\phi(x) = \langle x|\phi\rangle.$$

To translate the coordinate from $x$ to $x+\epsilon$ (i.e. $\phi(x) \rightarrow \phi(x+\epsilon)$), we act a group element of $G$, $\hat{g}(\epsilon)$, on $|x\rangle$:

$$|x+\epsilon\rangle = \hat{g}(\epsilon+x)|0\rangle = \hat{g}(\epsilon)\hat{g}(x)|0\rangle = \hat{g}(\epsilon)|x\rangle,$$

where $|0\rangle$ is the vacuum state and $\hat{g}(\epsilon), \hat{g}(x) \in G$. The group elements above can be expressed in terms of exponential

$$\hat{g}(y) = \exp\left(iy\hat{G}\right), \tag{2.1}$$

the $i$ is there to make $\hat{G}$ hermitian so that $g^\dagger = g^{-1}$. Define $(|0\rangle)^\dagger = \langle 0|$, i.e. $\langle 0|$ is the dual or hermitian conjugate of $|0\rangle$, then

$$(|y\rangle)^\dagger = \langle y| = \langle 0|\exp\left(-iy\hat{G}\right).$$

$\hat{G}$ is known as infinitesimal symmetry generating operator (or symmetry operator for short) since for small $\epsilon$:

$$\delta_\epsilon|x+\epsilon\rangle = |x+\epsilon\rangle - |x\rangle = \exp\left(i\epsilon\hat{G}\right)|x\rangle - |x\rangle \approx i\epsilon\hat{G}|x\rangle.$$

For field $\phi(x)$, we don't always have to use $\hat{G}$ explicitly to generate transformation. Instead, we can simply act some derivative on $\hat{g}(x)$ to bring down $\hat{G}$ as follows:

$$\begin{cases} \hat{g}(x+\epsilon) \approx i\epsilon\hat{G}\hat{g}(x) \\ \hat{g}(x+\epsilon) \approx \epsilon\dfrac{\partial}{\partial x}\hat{g}(x) \end{cases} \Rightarrow -i\frac{\partial}{\partial x}\hat{g}(x) = \hat{G}\hat{g}(x),$$

therefore, $\hat{G}$ can be replaced by $\rhd \equiv -i\dfrac{\partial}{\partial x}$. In other word,

$$\begin{aligned} &\exp\left(i\epsilon\rhd\right)\hat{g}(x) = \exp\left(\epsilon\frac{\partial}{\partial x}\right)\hat{g}(x) = \exp\left(i\epsilon\hat{G}\right)\hat{g}(x) = \hat{g}(\epsilon)\hat{g}(x) \\ \Rightarrow &\exp\left(i\epsilon\rhd\right)\phi(x) = \langle 0|\exp\left(i\epsilon\rhd\right)\exp\left(-ix\hat{G}\right)|\phi\rangle = \langle 0|\exp\left(-ix\hat{G}\right)\exp\left(-i\epsilon\hat{G}\right)|\phi\rangle \\ &= \langle x|\exp\left[-i\left(x+\epsilon\right)\hat{G}\right]|\phi\rangle = \phi(x+\epsilon). \end{aligned}$$

A trivial consistency check:

$$\begin{aligned} g(\epsilon)\phi(x) = \exp\left(i\epsilon\rhd\right)\phi(x) &= \exp\left(\epsilon\frac{\partial}{\partial x}\right)\phi(x) \\ &= \phi(x) + \left(\epsilon\frac{\partial}{\partial x}\right)\phi(x) + \frac{1}{2}\left(\epsilon^2\frac{\partial^2}{\partial x^2}\right)\phi(x) + \cdots \\ &= \phi(x+\epsilon). \end{aligned}$$

However, when involving more than one symmetry operators, there might be some "ambiguities" if some of the $\hat{G}$'s don't commute. The first ambiguity comes from the ordering of the group elements since they don't commute. For example, if we have a field which depends on two coordinates, $\alpha_1$ and $\alpha_2$, whose symmetry operators do not commute, i.e.

$$\exp\left(\alpha_1\hat{G}_1\right)\exp\left(\alpha_2\hat{G}_2\right) \neq \exp\left(\alpha_2\hat{G}_2\right)\exp\left(\alpha_1\hat{G}_1\right);$$

then there are infinite many ways to define $|\alpha_1, \alpha_2\rangle$, e.g.

$$\begin{cases} |\alpha_1, \alpha_2\rangle & \equiv \exp\left(\alpha_1 \hat{G}_1\right) \exp\left(\alpha_2 \hat{G}_2\right) |0\rangle \\ |\alpha_1, \alpha_2\rangle & \equiv \exp\left(\alpha_2 \hat{G}_2\right) \exp\left(\alpha_1 \hat{G}_1\right) |0\rangle \\ |\alpha_1, \alpha_2\rangle & \equiv \exp\left(\alpha_1 \hat{G}_1 + \alpha_2 \hat{G}_2\right) |0\rangle \\ & \vdots \end{cases},$$

which, in general, are all different. Each configuration is called a basis. We can always choose a convenient basis to work with, just like choosing a convenient coordinates to work with when we have rotational symmetry or Lorentz symmetry.

To make things general, we do not choose any basis at the time being. We simply denote the basis to be $\hat{g}(\alpha)$, where $\alpha$ is the set of coordinates. Here comes the second "ambiguity": Since $\hat{G}_a$'s do not commute with $\hat{g}(\alpha)$ in general, there are two ways to bring down $\hat{G}_a$'s (without breaking $\hat{g}(\alpha)$)—$\hat{G}_a \hat{g}(\alpha)$ or $\hat{g}(\alpha)\hat{G}_a$.

Before "solving" the ambiguity, we first talk about how to bring down the symmetry operators using coordinate derivatives for both cases. Since we are looking for "infinitesimal" symmetry generators, we only have to consider infinitesimal change in coordinates. Coordinates are just numbers (functions, at most), they commute with each other. Therefore, there is no ambiguity to bring coordinates to the left or right, i.e.

$$d\hat{g}(\alpha) = d\alpha^i \partial_i \hat{g}(\alpha). \tag{2.2}$$

On the other hand, taking derivative on $\alpha$ brings down $\hat{G}$'s as well. It is always possible, in principle, to use Baker-Campbell-Hausdorff formula to move specific $\hat{G}_a$ to the most left or right. We can, therefore, move the "brought-down" $\hat{G}_a$'s to either side of $\hat{g}(\alpha)$:

$$d\hat{g}(\alpha) = \begin{cases} i d\alpha^i \left(e_L^{-1}\right)_i{}^a(\alpha) \hat{G}_a \hat{g}(\alpha) \\ i d\alpha^i \left(e_R^{-1}\right)_i{}^a(\alpha) \hat{g}(\alpha) \hat{G}_a \end{cases}. \tag{2.3}$$

The $e_L^{-1}$ and $e_R^{-1}$ come from the noncommutative property between $\hat{G}_a$'s.

Equating equation (2.2) and equation (2.3), we have the following equations:

$$\begin{cases} d\alpha^i \partial_i \hat{g}(\alpha) = i d\alpha^i \left(e_L^{-1}\right)_i{}^a(\alpha) \hat{G}_a \hat{g}(\alpha) \\ d\alpha^i \partial_i \hat{g}(\alpha) = i d\alpha^i \left(e_R^{-1}\right)_i{}^a(\alpha) \hat{g}(\alpha) \hat{G}_a \end{cases},$$

$$\Rightarrow \begin{cases} \partial_i \hat{g}(\alpha) = i \left(e_L^{-1}\right)_i{}^a(\alpha) \hat{G}_a \hat{g}(\alpha) \\ \partial_i \hat{g}(\alpha) = i \left(e_R^{-1}\right)_i{}^a(\alpha) \hat{g}(\alpha) \hat{G}_a \end{cases},$$

$$\Rightarrow \begin{cases} e_{La}{}^i(\alpha) \partial_i \hat{g}(\alpha) = i \hat{G}_a \hat{g}(\alpha) \\ e_{Ra}{}^i \partial_i \hat{g}(\alpha) = i \hat{g}(\alpha) \hat{G}_a \end{cases}.$$

Hence, we have found the derivatives that bring down the $\hat{G}_a$ to the left and the right of $\hat{g}(\alpha)$. We can now define

$$\begin{cases} \rhd_a(\alpha) \equiv -i e_{La}{}^i(\alpha) \partial_i \\ D_a(\alpha) \equiv -i e_{Ra}{}^i(\alpha) \partial_i \end{cases}.$$

The "$-i$" is just a convention. We can now check the commutation relation between $\rhd$'s and $D$'s. First, we write down the defining commutation relation between $\hat{G}$'s:

$$\left[\hat{G}_a, \hat{G}_b\right] = i f_{ab}{}^c \hat{G}_c.$$

The commutation relation between $\rhd$'s are

$$\begin{aligned} [\rhd_a, \rhd_b]\, \hat{g} &= (\rhd_a \rhd_b - \rhd_b \rhd_a)\, \hat{g} = \rhd_a \hat{G}_b \hat{g} - \rhd_b \hat{G}_a \hat{g} \\ &= \left(\hat{G}_b \hat{G}_a - \hat{G}_a \hat{G}_b\right) \hat{g} = \left[\hat{G}_b, \hat{G}_a\right] \hat{g} \\ &= i f_{ba}{}^c \hat{G}_c \hat{g} = -i f_{ab}{}^c \hat{G}_c \hat{g} \\ &= -i f_{ab}{}^c \rhd_c \hat{g} \\ \Rightarrow [\rhd_a, \rhd_b] &= -i f_{ab}{}^c \rhd_c. \end{aligned}$$

It can be shown that $D$'s satisfies the same commutation relation as $\hat{G}$'s:

$$\begin{aligned} [D_a, D_b]\, \hat{g} &= (D_a D_b - D_b D_a)\, \hat{g} = D_a \hat{g} \hat{G}_b - D_b \hat{g} \hat{G}_a \\ &= \hat{g} \left(\hat{G}_a \hat{G}_b - \hat{G}_b \hat{G}_a\right) = \hat{g} \left[\hat{G}_a, \hat{G}_b\right] \\ &= i f_{ab}{}^c \hat{g} \hat{G}_c = i f_{ab}{}^c D_c \hat{g} \\ \Rightarrow [D_a, D_b] &= i f_{ab}{}^c D_c. \end{aligned}$$

We now return to the "ambiguity". The definition of symmetry transformation is

$$\phi'(\alpha) = \langle\alpha|\phi'\rangle = \langle\alpha|\left(\hat{g}|\phi\rangle\right).$$

The infinitesimal symmetry transformation:

$$d_\epsilon \hat{g} \approx i d\epsilon^a \hat{G}_a \hat{g} = i d\epsilon^a \rhd_a \hat{g}.$$

Therefore, $\triangleright$ is the symmetry generator. We can now argue that $D$ is the covariant derivative. We first show that $\triangleright$ commutes with $D$:

$$\begin{aligned} \rhd_a D_b \hat{g} &= \rhd_a \hat{g} \hat{G}_b = \hat{G}_a \hat{g} \hat{G}_b \\ &= D_b \hat{G}_a \hat{g} = D_b \rhd_a \hat{g} \\ \Rightarrow [\rhd_a, D_b] = 0. \end{aligned}$$

Since $\triangleright$ and $D$ commutes, we have

$$g(\epsilon) D_a(\alpha)\phi(\alpha) = D_a(\alpha)\left(g(\epsilon)\phi(\alpha)\right),$$

where $g(\epsilon)$ is an element of symmetry group:

$$g(\epsilon) = \exp(i\epsilon^a \rhd_a).$$

Hence, $D$ is the covariant derivative.

We close this section by mentioning finite coordinate transformation. The difference between two coordinates, $\alpha_1$ and $\alpha_2$, is not $\alpha_{12} = \alpha_1 - \alpha_2$ in general. To see this, we have to come back to its definition of "coordinate". It is the "argument" of group element $\hat{g}(\alpha) \in G$. Hence, the difference between $\hat{g}(\alpha_{12})$ should involve $\hat{g}(\alpha_1)$ and $\hat{g}(-\alpha_2)$. By construction, $\hat{g}(-\alpha)\hat{g}(\alpha) = 1$ implies

$$\hat{g}(-\alpha) = \hat{g}^{-1}(\alpha).$$

We know the $|\alpha_{12}\rangle$ has to be either $\hat{g}^{-1}(\alpha_2)\hat{g}(\alpha_1)|0\rangle$ or $\hat{g}(\alpha_1)\hat{g}^{-1}(\alpha_2)|0\rangle$ but not both since they give different results. It is trivial to check that only $\hat{g}^{-1}(\alpha_2)\hat{g}(\alpha_1)$ is invariant under symmetry transformation, i.e. $\hat{g}(\alpha_{12})$ (which also implies $\alpha_{12}$) is invariant under symmetry transformation. We, therefore, claim that

$$\hat{g}(\alpha_{12}) = \hat{g}^{-1}(\alpha_2)\hat{g}(\alpha_1). \tag{2.4}$$

## 2.2 Superfields

Superspace is a very useful tool for understanding supersymmetry. It geometrizes supersymmetry by adding Grassmann coordinates to the normal spacetime in order to include supersymmetry transformation (since supersymmetry transformation is nilpotent). Therefore, the field in superspace (superfield, $\phi$) depends not only on spacetime coordinates ($x$) but also on the additional Grassmann coordinates ($\theta$) as well, i.e. $\phi = \phi(x, \theta)$. Since $\theta$ is Grassmann variable (nilpotent), the expansion on $\theta$ gives finite terms. For example, if a superfield that depends on two Grassmann variables,

$$\phi(x, \theta_1, \theta_2) = \phi_0(x) + \theta_1\phi_1(x) + \theta_2\phi_2(x) + \theta_1\theta_2\phi_{12}(x).$$

where $\phi_0 = \phi(x, \theta_1, \theta_2)|_{\theta_{1,2}=0}$, $\phi_1 = \frac{\partial}{\partial\theta_1}\phi(x, \theta_1, \theta_2)|_{\theta_{1,2}=0}$, $\phi_2 = \frac{\partial}{\partial\theta_2}\phi(x, \theta_1, \theta_2)|_{\theta_{1,2}=0}$, and $\phi_{12} = \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\phi(x, \theta_1, \theta_2)|_{\theta_{1,2}=0}$. Note that $\phi_0$ and $\phi_{12}$ have the same Grassmann parity as $\phi$, while $\phi_1$ and $\phi_2$ have the opposite. For simplicity, we will use "$|$" to mean evaluating the term with all fermionic (Grassmann odd) coordinates vanish for the rest of the thesis.

So far, the "superfield" or "superspace" does not have obvious supersymmetry in it yet. To include supersymmetry, we go back to the last section and realize what $x$ and $\theta$ means. As usual, $x$ is the "amount" that the spacetime differs from the identity in $\hat{P}$ direction. What about $\theta$? Its purpose is, as stated before, to include supersymmetry. Therefore, it is the amount it differs from the identity in $\hat{Q}$ direction. It is known that one of the defining properties of supersymmetry is that the supersymmetry generating operators do not anticommute with each other but to give a combination of spacetime generator, i.e.

$$\left\{\hat{Q}, \hat{Q}\right\} \sim \hat{P}.$$

From the discussion from last section, we know that, in general, $\frac{\partial}{\partial\theta}$'s are not covariant derivatives since they anticommute with each other. Therefore, expanding superfield using $\frac{\partial}{\partial\theta}$ is not the most convenient way to treat supersymmetry. An alternative way of expanding superfields is, instead of Taylor expanding it, to project it to each component using covariant derivative, $D_Q$, i.e.

$$\begin{aligned}\phi(x,\theta) &= \phi(x,\theta)| + \theta D_Q\phi(x,\theta)| + \theta\theta D_Q D_Q\phi(x,\theta)| + \cdots \\ &= \phi_0(x) + \theta\phi_1(x) + \theta\theta\phi_2(x) + \cdots .\end{aligned}$$

In this case, the components are manifestly covariant under supersymmetry transformation. We know that $D_Q$ qualitatively (an explicit derivation for $4D$ Minkowski $\mathcal{N}=1$ superspace is in Appendix A) looks like

$$D_Q \sim \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}.$$

For lower $\theta$ terms, they look exactly the same as the ones in Taylor expansion since the $\theta\dfrac{\partial}{\partial x}$ in $D_Q$ is killed by "|". However, for higher $\theta$ terms, there are some ambiguities the ordering of $D_Q$'s since they don't anticommute. These ambiguities are, in fact, just field redefinitions which does not effect physics and can be chosen the most convenient one to work with.

## 2.3 Coset Space

Many of the interesting physics theories live in coset spaces, $G/H$. In coset space, we split $\hat{G}$ into $\hat{T}_\iota \in \hat{H}$ and the rest of the generators into $\hat{T}_i$'s. It is usually convenient to choose the basis as

$$g(\alpha,\beta) = \exp\left(i\alpha^i\hat{T}_i\right)\exp\left(i\beta^\iota\hat{T}_\iota\right).$$

Similar to the definition in subsection 2.1, the coordinate state for the coset space is defined to be

$$|\alpha\rangle = \exp\left(i\alpha^i\hat{T}_i\right)|0\rangle,$$

however, the vacuum is invariant under $H$ transformation, i.e.

$$\hat{T}_\iota|0\rangle = 0.$$

As a side note, this is why choosing $g(\alpha,\beta) = \exp\left(i\alpha^i\hat{T}_i\right)\exp\left(i\beta^\iota\hat{T}_\iota\right)$ is convenient:

$$\begin{aligned}g(\alpha,\beta)|0\rangle &= \exp\left(i\alpha^i\hat{T}_i\right)\exp\left(i\beta^\iota\hat{T}_\iota\right)|0\rangle \\ &= \exp\left(i\alpha^i\hat{T}_i\right)|0\rangle \\ &= |\alpha\rangle.\end{aligned}$$

Wave functions in quantum mechanics is just a projection of an arbitrary state in Hilbert state to some complete set of states. In coset space, we can choose the arbitrary state, $|\psi\rangle$, projecting on the coordinate state we mentioned above:

$$\psi(\alpha) \equiv \langle\alpha|\psi\rangle = \langle 0|g^{-1}(\alpha, 0)|\psi\rangle.$$

A direct consequence of vacuum being $H$ invariant is that

$$\begin{aligned} D_\iota \psi(\alpha) &= D_\iota \langle 0|g^{-1}(\alpha)|\psi\rangle \\ &= -\langle 0|\hat{T}_\iota g^{-1}(\alpha)|\psi\rangle = 0. \end{aligned}$$

And the symmetry generators change $\psi$ as expected:

$$\begin{aligned} D_i \psi(\alpha) &= D_i \langle 0|g^{-1}(\alpha)|\psi\rangle \\ &= -\langle 0|g^{-1}(\alpha)\hat{T}_i|\psi\rangle \\ &= -\left(\hat{G}_i \psi\right)(\alpha). \end{aligned}$$

In quantum mechanics, besides ordinary coordinates, sometimes the fields carry some $H$ group indices which transforms under $H$ group (spin, for example). In such a case, we define the "vacuum" to transform under $H$ group as well:

$$\hat{T}_\iota |0,{}^A\rangle = |0,{}^B\rangle \left(H_\iota\right)_B{}^A.$$

When acting $g$ on the vacuum, we get

$$\begin{aligned} g(\alpha,\beta)|0,{}^A\rangle &= \exp\left(i\alpha^i \hat{T}_i\right) \exp\left(i\beta^\iota \hat{T}_\iota\right) |0,{}^A\rangle \\ &= \exp\left(i\alpha^i \hat{T}_i\right) |0,{}^M\rangle \left(\exp\left(i\beta^\iota H_\iota\right)\right)_M{}^A \\ &= \exp\left(i\alpha^i \hat{T}_i\right) |0,{}^M\rangle \left(e^{-1}\right)_M{}^A(\beta), \end{aligned}$$

where ${e^{-1}}_M{}^A$ is the "inverse vielbein". Then the projection of $|\psi\rangle$ on the state is

$$\begin{aligned} \langle{}_A, 0|g^{-1}(\alpha,\beta)|\psi\rangle &= e_A{}^M(\beta)\langle{}_M, \alpha|\psi\rangle \\ &= e_A{}^M(\beta)\psi_M(\alpha). \end{aligned}$$

Note that the vielbeins depend only on the gauge coordinates. The way we arrange the basis makes $\alpha$'s independent of gauge transformation, and $\beta$'s only reacts to gauge transformation.

# Chapter 3

# Covariant Semi-Shortening Condition

## 3.1 Introduction

In this chapter, we will re-derive semi-shortening conditions for four-dimensional superconformal field theory with a different approach. These conditions have similar patterns that can be generalized to weaker constraints, including all those of F. Dolan and H. Osborn [43]. In particular, for the case of $\mathcal{N} = 4$ super Yang-Mills theory, formulated in projective superspace, we find constraints for all BPS operators. We also give an example how constraints can be found from known ones. These constraints are a subset of our maximal set of semi-shortening conditions.

For a superconformal theory to be a valid quantum theory, it has to satisfy some unitarity bounds [44, 45]. When the bound is saturated, i.e., when the inequality becomes equality, the primary field loses some degrees of freedom. This implies the primary state can be annihilated by some combination of super charge and vice versa (Bogomol'nyi-Prasad-Sommerfield conditions). A supermultiplet satisfying a BPS condition will be truncated into a shorter supermultiplet [46–48], hence it is also called a shortening condition. Various short and semi-short representations for $\mathcal{N} = 2$ and $\mathcal{N} = 4$ in four dimension are discussed in [43].

We will first review how shortening conditions can be treated as defining coset superspaces [49]. We then show how most semi-shortening conditions in four dimensions can be obtained by superconformally transforming the massless field equation. The remaining known (semi-)shortening conditions can then be obtained by a simple generalization. Finally, we consider the example of $\mathcal{N} = 4$ SYM and apply the algorithm to find explicit expression for semi-shortening constraints.

## 3.2 Coset Superspace

In this section, we apply the techniques in Chapter 2 to specific case we are interested in. The covariant derivatives, $D$, as

$$D\phi(z) = \langle 0| \left(i\hat{G}\right) g^{-1}(z) \, |\phi\rangle \, .$$

Notice that there's an additional "$-i$" in the definition of covariant derivative. It will simplify the calculation a lot when dealing with large number of $D$'s.

In the usual supersymmetry theory, the generators are $\{\hat{P}, \hat{Q}, \hat{\bar{Q}}, \hat{M}, \hat{\bar{M}}, \hat{R}\}$ which correspond to translation, supersymmetry, rotation, and $R$-symmetry.

In superconformal field theory, in addition to the usual generators, there are also $\{\hat{K}, \hat{S}, \hat{\bar{S}}, \hat{D}\}$, known as the generators of special conformal transformations, superconformal transformations, and dilatation. In $D = 4$, the superconformal group is $(P)SU(2,2|\mathcal{N})$. We can wick rotate to $(P)SL(4|\mathcal{N})$ and treat not only "projective" $(P)$ but also "special" $(S)$ as gauge invariances. Then the group before gauge fixing is $GL(4|\mathcal{N})$. The coordinates of the full superspace, $z_{\mathcal{M}}{}^{\mathcal{A}}$, can be ordered as follows

$$\begin{aligned} z_{\mathcal{M}}{}^{\mathcal{A}} = {} & \begin{array}{c} \\ \beta \\ j \\ \dot\beta \end{array} \!\! \begin{array}{c} \begin{array}{ccc} \alpha \quad & \quad i \quad & \quad \dot\alpha \end{array} \\ \begin{pmatrix} z_\beta{}^\alpha & z_\beta{}^i & z_\beta{}^{\dot\alpha} \\ z_j{}^\alpha & z_j{}^i & z_j{}^{\dot\alpha} \\ z_{\dot\beta}{}^\alpha & z_{\dot\beta}{}^i & z_{\dot\beta}{}^{\dot\alpha} \end{pmatrix} \end{array} \\ = {} & \begin{pmatrix} \text{Lorentz+scale} & \text{supersymmetry} & \text{translation} \\ \text{superconformal} & \text{R-symmetry} & \text{supersymmetry} \\ \text{special conformal} & \text{superconformal} & \text{Lorentz}-\text{scale} \end{pmatrix} . \end{aligned} \tag{3.1}$$

Throughout this chapter, all the Greek indices are spinor (fermionic) indices, Latin indices stand for internal/R-symmetry (bosonic) indices, and calligraphic capital Latin indices can be both. The full superspace propagator can be written as $g(z) = \exp\left(iz\hat{G}\right)$, where $\hat{G}$ is the corresponding symmetry generator. If the ground state is invariant under some symmetries (with corresponding symmetry generating operators $\hat{H}_\iota$), we can divide symmetry generating operators into two groups, $\hat{G} = \{\hat{T}_i, \hat{H}_\iota\}$. Then the ground state propagates as

$$\begin{aligned}
&\exp\left(iz\hat{G}\right)|0\rangle = \exp\left(i\tilde{z}\hat{T}\right)\exp\left(i\zeta\hat{H}\right)|0\rangle = \exp\left(i\tilde{z}\hat{T}\right)|0\rangle \\
\Rightarrow &\tilde{U}(\tilde{z}) = \exp\left(i\tilde{z}\hat{T}\right) = \exp\left(iz\hat{G}\right)\ mod\ \hat{H}_\iota = g(z)\ mod\ \hat{H}_\iota .
\end{aligned}$$

In other words, the full superspace becomes a coset superspace. Therefore, we can set the coordinates corresponding to $H$ to zero. For example, to get the usual superspace, we gauge away the lower-left triangle and the diagonal parts of the coordinate matrix as

$$z_{\mathcal{M}}{}^{\mathcal{A}} \to \tilde{z}_{\mathcal{M}}{}^{\mathcal{A}} = \begin{pmatrix} 1 & \theta_{\alpha'}{}^{i} & x_{\alpha'}{}^{\dot{\alpha}} \\ 0 & 1 & \bar{\theta}_{i'}{}^{\dot{\alpha}} \\ 0 & 0 & 1 \end{pmatrix} .$$

We can also treat projective superspaces as coset superspaces by modding out some coordinates. Rearranging the full coordinate matrix as

$$z_{\mathcal{M}}{}^{\mathcal{A}} = \begin{array}{c c} & \begin{array}{cccc} \alpha & i & i' & \dot{\alpha} \end{array} \\ \begin{array}{c} \beta \\ j \\ j' \\ \dot{\beta} \end{array} & \begin{pmatrix} z_{\beta}{}^{\alpha} & z_{\beta}{}^{i} & z_{\beta}{}^{i'} & z_{\beta}{}^{\dot{\alpha}} \\ z_{j}{}^{\alpha} & z_{j}{}^{i} & z_{j}{}^{i'} & z_{j}{}^{\dot{\alpha}} \\ z_{j'}{}^{\alpha} & z_{j'}{}^{i} & z_{j'}{}^{i'} & z_{j'}{}^{\dot{\alpha}} \\ z_{\dot{\beta}}{}^{\alpha} & z_{\dot{\beta}}{}^{i} & z_{\dot{\beta}}{}^{i'} & z_{\dot{\beta}}{}^{\dot{\alpha}} \end{pmatrix} \end{array} ,$$

($i$ runs from 1 to $n$ and $i'$ from $n+1$ to $\mathcal{N}$), we again gauge away the lower-left and the diagonal blocks

$$z_{\mathcal{M}}{}^{\mathcal{A}} \to \begin{pmatrix} 1 & 0 & \theta_{\beta}{}^{i'} & x_{\beta}{}^{\dot{\alpha}} \\ 0 & 1 & u_{j}{}^{i'} & \bar{\theta}_{j}{}^{\dot{\alpha}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.2}$$

This is a consequence of setting the ground state to be annihilated by $\hat{Q}^{i}$'s and $\hat{\bar{Q}}^{i'}$'s in full superspace.

Take $\mathcal{N} = 4$ SYM field strength in projective superspace as an example, it can be expanded into component fields as follows:

$$
\begin{aligned}
\varphi = & (\phi + u_i{}^{i'}\phi_{i'}{}^i + \frac{1}{2}u^2\bar{\phi}) + \theta_\alpha{}^{i'}(\lambda_{i'}{}^\alpha + u^i{}_{i'}\lambda_i^\alpha) + \bar{\theta}_i{}^{\dot\alpha}(\bar{\lambda}_{\dot\alpha}{}^i + u^i{}_{i'}\bar{\lambda}_{\dot\alpha}{}^{i'}) \\
& + \theta^2_{\alpha\beta}f^{\alpha\beta} + \bar{\theta}^{2\dot\alpha\dot\beta}\bar{f}_{\dot\alpha\dot\beta} - i\theta_\alpha{}^{i'}\bar{\theta}_i^{\dot\alpha}\partial_{\dot\alpha}{}^\alpha(\phi_{i'}{}^i + u^i{}_{i'}\bar{\phi}) - i\theta^2_{\alpha\beta}\bar{\theta}_i{}^{\dot\alpha}\partial_{\dot\alpha}{}^\alpha\lambda^{i\beta} \\
& - i\bar{\theta}^{2\dot\alpha\dot\beta}\theta_\alpha{}^{i'}\partial_{\dot\alpha}{}^\alpha\bar{\lambda}_{\dot\beta i'} - \theta^2_{\alpha\beta}\bar{\theta}^{2\dot\alpha\dot\beta}\partial_{\dot\alpha}{}^\alpha\partial_{\dot\beta}{}^\beta\bar{\phi},
\end{aligned}
$$

where we have used the internal $SL(2)^2$ to raise and lower the indices. The "incomplete" expansion of $u$'s and $\theta$'s is explained at the beginning of subsection 3.7 (equation 3.15).

## 3.3 Shortening Conditions As Coset Space

As in section 3.2, the covariant derivatives for superconformal symmetry (the algebra is listed in appendix B.1.2) can be written as the following graded matrix (Note that the following algebras are modified ones, not the "usual" ones in appendix B.1.1):

$$
\begin{aligned}
D_{\mathcal{M}}{}^{\mathcal{N}} &= \begin{array}{c c} & \begin{array}{ccc} \alpha & i & \dot\alpha \end{array} \\ \begin{array}{c} \beta \\ j \\ \dot\beta \end{array} & \begin{pmatrix} D_\beta{}^\alpha & D_\beta{}^i & D_\beta{}^{\dot\alpha} \\ D_j{}^\alpha & D_j{}^i & D_j{}^{\dot\alpha} \\ D_{\dot\beta}{}^\alpha & D_{\dot\beta}{}^i & D_{\dot\beta}{}^{\dot\alpha} \end{pmatrix} \end{array} \\
&= \begin{array}{c c} & \begin{array}{ccc} \alpha & i & \dot\alpha \end{array} \\ \begin{array}{c} \beta \\ j \\ \dot\beta \end{array} & \begin{pmatrix} m_\beta{}^\alpha & s_\beta{}^i & k_\beta{}^{\dot\alpha} \\ q_j{}^\alpha & r_j{}^i & \bar{s}_j{}^{\dot\alpha} \\ p_{\dot\beta}{}^\alpha & \bar{q}_{\dot\beta}{}^i & m_{\dot\beta}{}^{\dot\alpha} \end{pmatrix} \end{array} \\
&= \begin{pmatrix} \text{Lorentz} & \text{superconformal} & \text{special conformal} \\ \text{supersymmetry} & \text{R-symmetry} & \text{superconformal} \\ \text{translation} & \text{supersymmetry} & \text{Lorentz} \end{pmatrix}
\end{aligned} \tag{3.3}
$$

In our conventions, the (anti)commutation relations are

$$
\left[D_{\mathcal{M}}{}^{\mathcal{N}}, D_{\mathcal{P}}{}^{\mathcal{Q}}\right\} = \delta^{\mathcal{N}}_{\mathcal{P}} D_{\mathcal{M}}{}^{\mathcal{Q}} - (-1)^{(\mathcal{M}+\mathcal{N})(\mathcal{P}+\mathcal{Q})}\,\delta^{\mathcal{Q}}_{\mathcal{M}} D_{\mathcal{P}}{}^{\mathcal{N}}, \tag{3.4}
$$

where in the exponent of $-1$

$$
\mathcal{A} = \begin{cases} 0 \quad , \quad \mathcal{A} \in \text{bosonic} \\ 1 \quad , \quad \mathcal{A} \in \text{fermionic} \end{cases} .
$$

The usual shortening conditions restrict some $D_i{}^\alpha\phi = 0$ or $D_{\dot\alpha}{}^i\phi = 0$ (“antichiral” or “chiral”). Together with superconformal symmetry, the shortening conditions imply the superfield also vanishes under some R-symmetry charges or Lorentz $\pm$ scale generators, by closure of the algebra. We can, therefore, set the left-bottom of the coordinate matrix (special conformal and superconformal coordinates) and some blocks at the right-top (“chiral” or “antichiral” invariant and the symmetries induced) to zero.

It is worth mentioning that the shortening conditions obtained from $D_i{}^\alpha\phi = 0$ (i.e., $q_i{}^\alpha\phi = 0$) form a closed set (as do $D_{\dot\alpha}{}^i\phi = 0$) that doesn't include other $D_j{}^\beta$ or $D_{\dot\beta}{}^j$. Derivation details are in appendix B.3.

Take projective superspace as an example: We first divide R-symmetry indices into two categories $(i,\, i')$. Some superspace coordinates vanish under some supercharges, $D_i{}^\alpha\phi = 0$ and $D_{\dot\alpha}{}^{i'}\phi = 0$. These conditions set some R-symmetry charges acting on the superfield to vanish (see appendix B.3). Therefore this gives the coordinate matrix shown in equation (3.2).

We then consider the general case of superspaces with chiral, antichiral, or “achiral" fermionic coordinates. R-symmetry indices can be split into three parts $(i, i', i'')$, where $i$ is antichiral, $i'$ is achiral, and $i''$ is chiral. Then the generator matrix can be written as follows:

$$
D_{\mathcal{M}}{}^{\mathcal{N}} = \begin{array}{c c}
 & \begin{array}{ccccc} \alpha & i & i' & i'' & \dot\alpha \end{array} \\
\begin{array}{c} \beta \\ j \\ j' \\ j'' \\ \dot\beta \end{array} &
\begin{pmatrix}
D_\beta{}^\alpha & D_\beta{}^i & D_\beta{}^{i'} & D_\beta{}^{i''} & D_\beta{}^{\dot\alpha} \\
D_j{}^\alpha & D_j{}^i & D_j{}^{i'} & D_j{}^{i''} & D_j{}^{\dot\alpha} \\
D_{j'}{}^\alpha & D_{j'}{}^i & D_{j'}{}^{i'} & D_{j'}{}^{i''} & D_{j'}{}^{\dot\alpha} \\
D_{j''}{}^\alpha & D_{j''}{}^i & D_{j''}{}^{i'} & D_{j''}{}^{i''} & D_{j''}{}^{\dot\alpha} \\
D_{\dot\beta}{}^\alpha & D_{\dot\beta}{}^i & D_{\dot\beta}{}^{i'} & D_{\dot\beta}{}^{i''} & D_{\dot\beta}{}^{\dot\alpha}
\end{pmatrix}
\end{array}
= \begin{array}{c c}
 & \begin{array}{ccccc} \alpha & i & i' & i'' & \dot\alpha \end{array} \\
\begin{array}{c} \beta \\ j \\ j' \\ j'' \\ \dot\beta \end{array} &
\begin{pmatrix}
\times & \times & \times & \times & \times \\
\times & \otimes & \otimes & \otimes & \times \\
q_{j'}{}^\alpha & r_{j'}{}^i & r_{j'}{}^{i'} & \otimes & \times \\
q_{j''}{}^\alpha & r_{j''}{}^i & r_{j''}{}^{i'} & \otimes & \times \\
p_{\dot\beta}{}^\alpha & \bar q_{\dot\beta}{}^i & \bar q_{\dot\beta}{}^{i'} & \times & \times
\end{pmatrix}
\end{array},
$$

where “$\times$” mean it is zero by construction, “$\otimes$” is “induced” to zero. And therefore, the gauged coordinate matrix is

$$
z_{\mathcal{M}}{}^{\mathcal{N}} = \begin{array}{c} \\ \beta \\ j \\ j' \\ j'' \\ \dot\beta \end{array} \begin{array}{c} \begin{array}{ccccc} \alpha & i & i' & i'' & \dot\alpha \end{array} \\ \begin{pmatrix} 1 & 0 & \theta_\beta{}^{i'} & \theta_\beta{}^{i''} & x_\beta{}^{\dot\alpha} \\ 0 & 1 & u_j{}^{i'} & u_j{}^{i''} & \bar\theta_j{}^{\dot\alpha} \\ 0 & 0 & u_{j'}{}^{i'} & u_{j'}{}^{i''} & \bar\theta_{j'}{}^{\dot\alpha} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}
$$

## 3.4 On-shell Constraints

By definition, superconformal primary superfields must satisfy the conditions

$$s_\alpha{}^i \phi(z) = 0 \text{ and } \bar{s}_i{}^{\dot\alpha} \phi(z) = 0,$$

which also implies $k_{\dot\alpha}{}^\alpha \phi(z) = 0$. Note that these are covariant derivatives, not symmetry generators.

For a massless free field, the superfield has to satisfy the on-shell condition $p^2\phi(z) = 0$. However, this condition is not invariant under superconformal transformations. (These are not symmetry transformations, except on the vacuum. On the superfield, they are transformations generated by the coset constraints.) This can easily be seen from the following example:

$$
\begin{aligned}
0 &= p^2 \phi \\
\Rightarrow 0 &= s_\alpha^i p^2 \phi \\
&= \left( \left[ s_\alpha{}^i, p^2 \right] + p^2 s_i{}^\alpha \right) \phi \\
&= \left[ s_\alpha{}^i, p^2 \right] \phi \\
\Rightarrow 0 &= p^{\dot\alpha}{}_\alpha \bar{q}_{\dot\alpha}{}^i \phi.
\end{aligned} \tag{3.5}
$$

Therefore, a superconformal, massless, free field should also satisfy constraint eq. (3.5). One can keep applying $s$ or $\bar{s}$ to get more constraints on the massless superfield [11]. Since both $s$ and $\bar{s}$ are fermionic operators, the number of constraints on the field is finite. The constraints can be represented diagrammatically as follows:

$$
\begin{array}{c}
\{0\}\ p^2\phi = 0 \\
\swarrow \qquad \searrow \\
s \swarrow \qquad\qquad \searrow \bar{s} \\
\swarrow \qquad\qquad\qquad \searrow \\
\{1\} \qquad\qquad\qquad\qquad\qquad \{2\} \\
\swarrow \quad \searrow \qquad\qquad\qquad \swarrow \quad \searrow \\
\{3\} \qquad \{4\} \qquad\qquad \{5\} \qquad \{6\} \\
\swarrow \searrow \quad \swarrow \searrow \qquad \swarrow \searrow \quad \swarrow \searrow \\
\cancel{\{7\}}\nearrow 0 \quad \{8\} \quad \{9\} \quad \{10\} \qquad \{11\} \quad \{12\} \quad \{13\} \quad \cancel{\{14\}}\nearrow 0 \\
\swarrow \searrow \quad \swarrow \searrow \quad \swarrow \searrow \quad \swarrow \searrow \quad \swarrow \searrow \quad \swarrow \searrow \\
\cancel{\{15\}}\nearrow 0\,\{16\}\,\cancel{\{17\}}\nearrow 0\,\{18\}\,\{19\}\,\cancel{\{20\}}\nearrow 0\,\cancel{\{21\}}\nearrow 0\,\{22\}\,\{23\}\,\cancel{\{24\}}\nearrow 0\,\{25\}\,\cancel{\{26\}}\nearrow 0
\end{array}
$$

$\cancel{\{\#\}}\nearrow 0$ in the diagram means it is identically zero by the coset constraints $s = \bar{s} = k = 0$, hence doesn't imply any new constraints. All the semi-shortening conditions in the diagram are compatible with $p^2 = 0$. The full constraints obtained from $p^2 = 0$ are listed in appendix B.2. It is worth mentioning that this formalism is very general in that it automatically includes *all* semi-shortening conditions quadratic in covariant derivatives: Interacting cases will simply lack some of the higher-dimension conditions (e.g., $p^2 = 0$).

For example, we can translate the most well-known semi-shortening conditions $(\hat{Q}^i)^2 \left|0\right\rangle_{\dot{\alpha}_1\cdots\dot{\alpha}_{2\bar{j}}} = 0$ and $\epsilon^{\alpha\beta}\hat{Q}^i{}_\beta \left|0\right\rangle_{\alpha\alpha_2\cdots\alpha_{2j},\dot{\alpha}_1\cdots\dot{\alpha}_{2\bar{j}}} = 0$ into superspace language as $(q^i)^2\phi_{\dot{\alpha}_1\cdots\dot{\alpha}_{2\bar{j}}} = 0$ and $\epsilon^{\alpha\beta}q^i{}_\beta\phi_{\alpha\alpha_2\cdots\alpha_{2j},\dot{\alpha}_1\cdots\dot{\alpha}_{2\bar{j}}} = 0$ respectively. In the paper by F. Dolan and H. Osborn [43], there is another semi-shortening condition $(Q^i_2 - \frac{1}{2j+1}Q^i_1)\left|j,\bar{j}\right\rangle = 0$ which is, in fact, just another form of $\epsilon^{\alpha\beta}\hat{Q}^i{}_\beta \left|0\right\rangle_{\alpha\alpha_2\cdots\alpha_{2j},\dot{\alpha}_1\cdots\dot{\alpha}_{2\bar{j}}} = 0$. In terms of superfields, this condition is equivalent to

$$\left(q^{i\alpha}m_\alpha{}^+ + jq^{i+}\right)\phi_{\alpha_1\cdots\alpha_{2j},\dot{\alpha}_1\cdots\dot{\alpha}_{2\bar{j}}} = 0. \tag{3.6}$$

Equation (3.6) is a special case of constraint $\{13\}$, which is

$$q^{k\gamma}\left(\delta^i{}_j m_\gamma{}^\alpha - \delta^\alpha_\gamma r^i{}_j\right) + (k \leftrightarrow i) = 0$$

by taking $k = i$ and $\alpha = +$. These conditions all come from constraint $\{6\}$, $q^{j\alpha}q^{j}{}_{\alpha} = 0$. We can also obtain the complex conjugate semi-shortening conditions by using constraint $\{3\}$.

To conclude this section, we claim that the full set of possible semi-shortening conditions quadratic in covariant derivatives can be obtained by just analyzing field equations without using the unitarity condition.

## 3.5 Semi-shortening Conditions

We now generalize the method used in section 3.4. First we note that the full set of constraints quadratic in covariant derivatives can be expressed in manifestly covariant form as the equation [11]

$$D_{(\mathcal{M}}{}^{(\mathcal{N}}D_{\mathcal{P}]}{}^{\mathcal{Q}]} = 0, \tag{3.7}$$

where ( ] means it is antisymmetric when interchanging two fermionic indices and symmetric otherwise. We define the set $D^2$ as the collection of all quadratic generators of this form. This set includes the massless Klein-Gordan equation $p^2 = 0$ in 4D spacetime. Thus, the results of that section could be obtained by looking for the covariant expression containing $p^2 = 0$. This covariance is under transformations generated by covariant derivatives. (As for all covariant derivatives, these equations are *invariant* under superconformal symmetry transformations.) Thus, taking the (anti)commutator of almost any one of $D^2$ with $D_\alpha{}^i$ or $D_i{}^{\dot\alpha}$ gives other constraints in this set. In general,

$$\begin{aligned}
\left[D_{\mathcal{M}}{}^{\mathcal{N}}, D_{(\mathcal{P}}{}^{(\mathcal{Q}}D_{\mathcal{R}]}{}^{\mathcal{S}]}\right\} &= \delta_{\mathcal{P}}^{\mathcal{N}} D_{(\mathcal{M}}{}^{(\mathcal{Q}}D_{\mathcal{R}]}{}^{\mathcal{S}]} + (-1)^{\kappa((\mathcal{M}+\mathcal{N})(\mathcal{P}+\mathcal{Q}))}\delta_{\mathcal{R}}^{\mathcal{N}} D_{(\mathcal{P}}{}^{(\mathcal{Q}}D_{\mathcal{M}]}{}^{\mathcal{S}]} \\
&\quad - (-1)^{\kappa((\mathcal{M}+\mathcal{N})(\mathcal{P}+\mathcal{Q}))}\delta_{\mathcal{M}}^{\mathcal{Q}} D_{(\mathcal{P}}{}^{(\mathcal{N}}D_{\mathcal{R}]}{}^{\mathcal{S}]} \\
&\quad - (-1)^{\kappa((\mathcal{M}+\mathcal{N})(\mathcal{P}+\mathcal{Q}+\mathcal{R}+\mathcal{S}))}\delta_{\mathcal{M}}^{\mathcal{S}} D_{(\mathcal{P}}{}^{(\mathcal{Q}}D_{\mathcal{R}]}{}^{\mathcal{N}]}.
\end{aligned} \tag{3.8}$$

For example, if we start with $D_{(i}{}^{[\alpha}D_{i)}{}^{\beta]} = 0$ (i.e., $(q^i)^2 = 0$) together with superconformal generators leads to $D_{(i}{}^{(\gamma}D_{i)}{}^{j)} = 0$, $D_{[i}{}^{(\alpha}D_{\gamma]}{}^{\beta]} = 0$, $D_{(i}{}^{(k}D_{\gamma]}{}^{\rho]} = 0$, and $D_{(i}{}^{(k}D_{i)}{}^{j)} = 0$.

Of course, all the shortening conditions form a subset of the set of all generators $D^1$. Since the generators and the indices will increase rapidly as we go on and it is not important here to know what the indices and the coefficients are, we will only give qualitative expressions of the (anti)commutation relations unless otherwise needed. For example, we will write equation (3.8) as

$$\left[D, D^2\right\} \sim \delta D^2.$$

The next thing to check is the (anti)commutation relation of any two elements in $D^2$. It can be easily found by using the following identity:

$$\left[D_{(\mathcal{M}}{}^{(\mathcal{N}} D_{\mathcal{P}]}{}^{\mathcal{Q}]}, \mathcal{O}\right\} = 2D_{(\mathcal{M}}{}^{(\mathcal{N}} \left[D_{\mathcal{P}]}{}^{\mathcal{Q}]}, \mathcal{O}\right\} - \left[D_{(\mathcal{M}}{}^{(\mathcal{N}}, \left[D_{\mathcal{P}]}{}^{\mathcal{Q}]}, \mathcal{O}\right\}\right\}, \tag{3.9}$$

where $\mathcal{O}$ is an arbitrary operator. Therefore, by substituting $\mathcal{O}$ with some element in $D^2$ together with equation (3.7) we get

$$\left[D^2, D^2\right\} \sim \delta g\left(D^2\right) + \delta\delta D^2. \tag{3.10}$$

The $g\left(D^2\right)$ term means a symmetry generator "times" an element in $D^2$ that cannot be combined into $D^3$, the set of all cubic operator of the form $D_{(\mathcal{M}}{}^{(\mathcal{N}} D_{\mathcal{P}}{}^{\mathcal{Q}} D_{\mathcal{R}]}{}^{\mathcal{S}]}$. Equation (3.10) tells us that a superfield under some constraints in $D^2$ can only give constraints the same strength as or weaker than $D^2$, it never goes to $D^1$. In other words, no matter how many semi-shortening conditions there are, it won't imply any shortening conditions.

From the discussion above, we found that $D^1$ and $D^2$ have some nice features: They are closed under symmetry transformation and they don't give stronger constraints ($D^1$ is the strongest set of constraints other than making the field identically zero). The question now arises: Does $D^3$ have these properties? Before checking $[D^3, D^3\}$, we first derive an "intermediate step", $[D^2, D^3\}$, which is of the same importance as $[D^3, D^3\}$. By using equation (3.9), we have the following:

$$\left[D^2, D^3\right\} \sim \delta g(D^3) + \delta\delta(D^3). \tag{3.11}$$

Since we are interested in $[D^3, D^3\}$ at the first place, we will come back to the equation (3.11) later. With the aid of equation (3.11), we get the following:

$$\left[D^3, D^3\right\} \sim \delta(D^2)(D^3) + \delta\delta g(D^3) + \delta\delta\delta(D^3). \tag{3.12}$$

From the equation above, we can conclude that $[D^3, D^3\}$ won't imply any constraint stronger than $D^3$.

Back to the "intermediate step", equation (3.11). One may notice that the (anti)commutation relation between $D^2$ and $D^3$ gives constraints same as or weaker than $D^3$ (also $D^1$ with $D^3$ gives $D^3$). This means weak constraints always stay weak or even weaker, and it will not effect stronger constraints.

The above statements can be generalized to all $D^n$ with positive and finite integer $n$, where

$$D^n \equiv \left\{ D_{(\mathcal{M}_1}{}^{(\mathcal{N}_1} D_{\mathcal{M}_2}{}^{\mathcal{N}_2} \cdots D_{\mathcal{M}_n]}{}^{\mathcal{N}_n]} \right\}$$

The first thing to do is to find the (anti)commutation relations between elements in two arbitrary sets, $D^n$ and $D^m$. To find the (anti)commutation relation between the elements of these sets, we first generalize equation (3.9) to the $n^{\text{th}}$ power:

$$\begin{aligned}
[D^n, \mathcal{O}\} &= \binom{n}{1} D_{(\mathcal{A}_1}{}^{(\mathcal{B}_1} D_{\mathcal{A}_2}{}^{\mathcal{B}_2} \cdots \left[ D_{\mathcal{A}_n]}{}^{\mathcal{B}_n]}, \mathcal{O} \right\} \\
&\quad + (-1)^1 \binom{n}{2} D_{(\mathcal{A}_1}{}^{(\mathcal{B}_1} D_{\mathcal{A}_2}{}^{\mathcal{B}_2} \cdots \left[ D_{\mathcal{A}_{n-1}]}{}^{\mathcal{B}_{n-1}]}, \left[ D_{\mathcal{A}_n]}{}^{\mathcal{B}_n]}, \mathcal{O} \right\} \right\} + \cdots \\
&\quad + (-1)^{n-1} \binom{n}{n} \left[ D_{(\mathcal{A}_1}{}^{(\mathcal{B}_1}, \left[ D_{\mathcal{A}_2}{}^{\mathcal{B}_2}, \cdots \left[ D_{\mathcal{A}_{n-1}]}{}^{\mathcal{B}_{n-1}]}, \left[ D_{\mathcal{A}_n]}{}^{\mathcal{B}_n]}, \mathcal{O} \right\} \right\} \right\} \right\} \\
&= \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \tilde{D}^{n-i} \mathrm{ad}_{\tilde{D}}{}^{i} \mathcal{O},
\end{aligned} \tag{3.13}$$

where

$$\tilde{D}^{n-i} \mathrm{ad}_{\tilde{D}}{}^{i} = D_{(\mathcal{A}_1}{}^{(\mathcal{B}_1} \cdots D_{\mathcal{A}_{n-i}}{}^{\mathcal{B}_{n-i}} \left[ D_{\mathcal{A}_{n-i+1}}{}^{\mathcal{B}_{n-i+1}}, \cdots \left[ D_{\mathcal{A}_{n-1}}{}^{\mathcal{B}_{n-1}}, \left[ D_{\mathcal{A}_n]}{}^{\mathcal{B}_n]}, \mathcal{O} \right\} \right\} \right\}.$$

The proof is in appendix B.4.

Without loss of generality, we assume $m \geq n$ and substitute $\mathcal{O}$ with $D^m$. By using equation (3.13), the (anti)commutation relation between $D^n$ and $D^m$ is

$$[D^n, D^m\} \sim \delta D^{n-1} D^m + \delta\delta D^{n-2} D^m + \cdots + \underbrace{\delta\delta \cdots \delta}_{n} D^m. \tag{3.14}$$

From this relation, we conclude the stronger constraints transform weaker constraints into some other weaker constraints but not the other way around.

## 3.6 Comparison With The "Old" Results

In this section, we show that the semi-shortening conditions in Dolan and Osborn's paper [43] can be reproduced by using $D^2$ and $D^3$ constraints. As has been discussed in section 3.4,

$$(q_i)^2\phi_{\dot\alpha_1\cdots\dot\alpha_{2\bar j}} = 0 \quad \text{and} \quad q_i{}^{\alpha}\phi_{\alpha\alpha_2\cdots\alpha_{2j},\dot\alpha_1\cdots\dot\alpha_{2\bar j}} = 0$$

(and the complex conjugate of that) are just special cases of $D^2$ constraints. The rest of the semi-shortening conditions in the paper are listed in table 3.1. in superspace language. These are actually special cases of $D^3$-constraints

$$\begin{cases} p^{\dot\alpha\alpha}\phi_{\alpha\alpha_2\cdots\alpha_{2j},\dot\alpha\dot\alpha_2\cdots\dot\alpha_{2\bar j}} = 0 & \text{(with scale dimension} \quad \Delta = 2 + j + \bar j) \\ p^{\dot\alpha\alpha}q_{i\alpha}\phi_{\dot\alpha\dot\alpha_2\cdots\dot\alpha_{2\bar j}} = 0 & \text{(with} \quad \Delta = 2 + \bar j) \\ p^{\dot\alpha\alpha}\bar q_{\dot\alpha}{}^{i}\phi_{\alpha\alpha_2\cdots\alpha_{2j}} = 0 & \text{(with} \quad \Delta = 2 + j) \\ p^{\dot\alpha\alpha}\left[q_{i\alpha}, \bar q_{\dot\alpha}^{\ j}\right]\phi = 0 & \text{(with} \quad r = 0) \end{cases}$$

Table 3.1: $D^2$ semi-shortening conditions in Dolan and Osborn's.

acting on different superfields.

Take $p^{\dot\alpha\alpha}\left[q_{i\alpha}, \bar q_{\dot\alpha}^{\ j}\right]\phi = 0$ as example. It can be written as $D_{(\dot\alpha}^{\ (\alpha}D_{\dot\beta}^{\ \beta}D_{i]}^{\ j]}\phi = 0$ if $\phi$ satisfies $r = 0$. The detailed derivation is shown in the following:

$$\begin{aligned} 0 &= D_{(\dot\alpha}^{\ (\alpha}D_{\dot\beta}^{\ \beta}D_{i]}^{\ j]}\phi \\ &= -3\left(p_{[\dot\alpha}^{\ [\alpha}q_{|i|}^{\ \beta]}\bar q_{\dot\beta]}^{\ j} - p_{[\dot\alpha}^{\ [\alpha}\bar q_{\dot\beta]}^{\ |j|}q_i^{\ \beta]} - p_{[\dot\alpha}^{\ [\alpha}p_{\dot\beta]}^{\ \beta]}r_i^{\ j}\right)\phi \\ &= -3\left(p_{[\dot\alpha}^{\ [\alpha}q_{|i|}^{\ \beta]}\bar q_{\dot\beta]}^{\ j} - p_{[\dot\alpha}^{\ [\alpha}\bar q_{\dot\beta]}^{\ |j|}q_i^{\ \beta]}\right)\phi \\ \Rightarrow 0 &= p^{\dot\alpha\alpha}\left[q^i{}_{\alpha}, \bar q_{j\dot\alpha}\right]\phi. \end{aligned}$$

The relations between $D^3$ constraints and the semi-shortening conditions in their paper are listed in table 3.2.

| $D^3$ | Dolan and Osborn | Shortening conditions |
|---|---|---|
| $D_{(\dot\alpha}^{\ (\alpha}D_{\dot\beta}^{\beta}D_{i]}^{\ j]}$ | $p^{\dot\alpha\alpha}\left[q_{i\alpha}, \bar q_{\dot\alpha}^{\ j}\right]\phi = 0$ | R-symmetry eigenvalue $= 0$ |
| $D_{(\dot\alpha}^{\ (\alpha}D_{\dot\beta}^{\ \beta}D_{i]}^{\ \dot\gamma]}$ | $p^{\dot\alpha\alpha}q_{i\alpha}\phi_{\dot\alpha\dot\alpha_2\cdots\dot\alpha_{2\bar j}} = 0$ | $\Delta = 2 + \bar j$, $\dot\gamma = -$ |
| $D_{(\alpha}^{\ (\beta}D_{\dot\alpha}^{\gamma}D_{\dot\beta]}^{\ i]}$ | $p^{\dot\alpha\alpha}\bar q_{\dot\alpha}{}^{i}\phi_{\alpha\alpha_2\cdots\alpha_{2j}} = 0$ | $\Delta = 2 + j$, $\gamma = +$ |
| $D_{(\alpha}^{\ (\beta}D_{\dot\alpha}^{\gamma}D_{\dot\beta]}^{\ \dot\gamma]}$ | $p^{\dot\alpha\alpha}\phi_{\alpha\alpha_2\cdots\alpha_{2j},\dot\alpha\dot\alpha_2\cdots\dot\alpha_{2\bar j}} = 0$ | $\Delta = 2 + j + \bar j$, $\beta = +$, $\dot\gamma = -$ |

Table 3.2: $D^3$ semi-shortening conditions in Dolan and Osborn's paper.

In fact, we can get the whole list of constraints by starting with the first constraint ($D_{(\dot\alpha}^{(\alpha} D_{\dot\beta}^{\beta} D_{i]}^{j]} = 0$) and repetitively taking (anti)commutators with $s$ or $\bar{s}$. We can get the second constraint ($D_{(\dot\alpha}^{(\alpha} D_{\dot\beta}{}^{\beta} D_{i]}{}^{\dot\gamma]} = 0$) or the third constraint ($D_{(\alpha}^{(\beta} D_{\dot\alpha}^{\gamma} D_{\dot\beta]}{}^{i]} = 0$) by applying an $s$ or $\bar{s}$ on the first constraint. By applying both $s$ and $\bar{s}$ once, one can get $D_{(\alpha}^{(\beta} D_{\dot\alpha}^{\gamma} D_{\dot\beta]}{}^{\dot\gamma]} = 0$. The full constraints induced by $D_{(\dot\alpha}^{(\alpha} D_{\dot\beta}^{\beta} D_{i]}^{j]} = 0$ are listed in appendix B.5.

Here we should also mention that the constraints induced by $D_{(\dot\alpha}^{(\alpha} D_{\dot\beta}^{\beta} D_{i]}^{j]} = 0$ form a closed set. One might expect that some other constraints will be induced by the (anti)commutation relation between two arbitrary $D^3$-constraints. However, according to equation (3.14), the (anti)commutation relation between $D^3$-constraints will be "proportional" to $D^3$. In other words, since $D_{(\dot\alpha}^{(\alpha} D_{\dot\beta}^{\beta} D_{i]}^{j]} = 0$ already induced all possible $D^3$-constraints, the (anti-) commutation relation is "proportional" to some $D^3$-constraint. Hence, it will not give additional constraints.

As advertised, we have reproduced all the semi-shortening constraints by using $D^2$ and $D^3$ constraints. To this day, only $D^2$ and $D^3$ constraints have been considered in the literature. Our work shows that there can be infinite numbers of semi-shortening constraints (i.e. $D^n$'s) which we think are complete, in the sense that any set of semi-shortening conditions must be a subset of them. The following section is an explicit example of $D^{n+1}$ constraints satisfied by $\mathrm{tr}\varphi^n$. We expressed all the constraints on a multiplet, including those on the Lorentz and $SU(4)$ representations, as differential equations on coset space.

## 3.7 $\mathcal{N} = 4$ SYM In Projective Superspace

The generalized semi-shortening conditions ($D^n = 0$) can be used on the $\mathcal{N} = 4$ SYM field strength in projective superspace. In general, the field strength $\varphi$ obeys semi-shortening conditions

$$r_{(a}{}^{(b} r_{c)}{}^{d)} \varphi = 0 \quad (D_{(a}{}^{(b} D_{c)}{}^{d)} \varphi = 0). \tag{3.15}$$

In the free theory, this generalizes to all the $D^2$ constraints, but for the non-abelian case the derivatives must be generalized to gauge-covariant derivatives, and "nonminimal" field strength terms are needed. However, no non-minimal terms are needed for the above equation, since the $r$ derivatives have dimension 0, whereas field strengths have dimension of at least 1. (Furthermore, a gauge can be chosen where the gauge potential for $r$ vanishes.)

A direct consequence of this for the BPS operators is that

$$r^{n+1}\text{tr }\varphi^n = r_{(i_1}{}^{(j_1}\cdots r_{i_{n+1})}{}^{j_{n+1})}\text{tr }\varphi^n = 0$$

since at least one of the $\varphi$'s will be hit by two $r$'s. Also, note the $r$ derivatives always reduce to ordinary derivatives outside the trace, since it's a gauge singlet. Since we are working with projective superspace, we divide R-symmetry indices into two categories ($i'$, $i''$) where the primed ones are antichiral and the double primed ones are chiral. The field strength $\varphi$ vanishes when hit with $q_{i'}{}^{\alpha}$ and $\bar{q}_{\dot\alpha}{}^{i''}$ ($D_{i'}{}^{\alpha}$ and $D_{\dot\alpha}{}^{i''}$). However, the semi-shortening condition above is not invariant under some supersymmetry transformations. Therefore, we can apply the algorithm discussed in section 3.5 to find other semi-shortening conditions.

Take $n = 3$ as an example,

$$\begin{aligned}
0 &= D_{j'}{}^{\alpha}D_{(a}{}^{(b}D_c{}^dD_e{}^fD_{h)}{}^{i)}\text{tr }\varphi^3\\
&= \left[D_{j'}{}^{\alpha}, D_{(a}{}^{(b}D_c{}^dD_e{}^fD_{h)}{}^{i)}\right]\text{tr }\varphi^3 + D_{(a}{}^{(b}D_c{}^dD_e{}^fD_{h)}{}^{i)}D_{j'}{}^{\alpha}\text{tr }\varphi^3\\
&= \Big(\delta_{j'}{}^bD_{(a}{}^{(\alpha}D_c{}^dD_e{}^fD_{h)}{}^{i)} + \delta_{j'}{}^dD_{(a}{}^{(b}D_c{}^{\alpha}D_e{}^fD_{h)}{}^{i)} + \delta_{j'}{}^fD_{(a}{}^{(b}D_c{}^dD_e{}^{\alpha}D_{h)}{}^{i)}\\
&\quad +\delta_{j'}{}^iD_{(a}{}^{(b}D_c{}^dD_e{}^fD_{h)}{}^{\alpha)}\Big)\text{tr }\varphi^3,
\end{aligned} \tag{3.16}$$

where the unprimed Latin indices are arbitrary numbers from 1 to 4. It is obvious from equation (3.16) that $D_{(a}{}^{(\alpha}D_c{}^dD_e{}^fD_{h)}{}^{i)}\text{tr }\varphi^3 = 0$. Repeatedly applying $[D_{i'}{}^{\alpha},\ \cdot\ \}$, $\left[D_{\dot\alpha}{}^{i''},\ \cdot\ \right\}$, $[D_{\alpha}{}^{i},\ \cdot\ \}$, or $\left[D_i{}^{\dot\alpha},\ \cdot\ \right\}$ to all the constraints, we get the set of constraints induced by $D_{(a}{}^{(b}D_c{}^dD_e{}^fD_{h)}{}^{i)} = 0$, which is made of and only of all the positive scale dimension $D^4$ constraints.

One might expect that there are some weaker constraints implied by taking the (anti)commutator of two arbitrary constraints above. However, these weaker constraints can also be decomposed into three generators times some positive scale dimension constraints, therefore no additional constraints. For example, one of the constraints induced by

$$D_{(a}{}^{(b}D_c{}^dD_e{}^fD_{h)}{}^{i)} = 0 \quad\text{and}\quad D_{(j}{}^{(k}D_l{}^mD_n{}^oD_{p)}{}^{\alpha)} = 0$$

is

$$0 = D_{(a}{}^{(\alpha}D_b{}^cD_{d)}{}^{e)}D_{(f}{}^{(h}D_i{}^jD_k{}^lD_{m)}{}^{n)}\text{tr}(\varphi^3) = D_{(a}{}^{(\alpha}D_b{}^cD_{d)}{}^{e)}\left(D_{(f}{}^{(h}D_i{}^jD_k{}^lD_{m)}{}^{n)}\text{tr}(\varphi^3)\right)$$

which gives nothing but $0 = 0$. Therefore, the shortening and semi-shortening constraints in this case, $D^4$, form a closed set.

*A general rule for projective superspace: If there exists a particular constraint $D^m\phi = 0$, this would imply all the positive scale dimension elements in $D^m$ to be constraints on $\phi$; unless this $D^m$ has at least one R-symmetry index that is not arbitrary.*

The $n = 3$ discussion above is an example of this rule.

Since the constraint $r^{n+1}\mathrm{tr}\ \varphi^n = r_{(i_1}{}^{(j_1} \cdots r_{i_{n+1})}{}^{j_{n+1})}\mathrm{tr}\ \varphi^n = 0$ is always true for arbitrary $i$'s and $j$'s, by using the above mentioned rule,

$$\textit{all } D^{n+1}\mathrm{tr}\ \varphi^n = 0. \tag{3.17}$$

Therefore, we got constraints for tr $\varphi^n$ by using semi-shortening constraints satisfied by $\varphi$. One can get the explicit form of the constraint by simply expand it and rewrite everything in covariant derivatives.

We take $n = 3$ in equation (3.17) as an example. We can choose $D^4$ to be $D_{(\dot\rho}{}^{(\alpha} D_{\dot\sigma}{}^{\beta} D_i{}^j D_{k]}{}^{l]}$, which can be expanded as follows (together with equation (3.4)):

$$\begin{aligned}
0 &= D_{(\dot\rho}{}^{(\alpha} D_{\dot\sigma}{}^{\beta} D_i{}^j D_{k]}{}^{l]}\mathrm{tr}\varphi^3 \\
&= \Big[D_{\dot\rho}{}^{\alpha} D_{\dot\sigma}{}^{\beta}\left(6D_i{}^j D_k{}^l + \delta_i{}^j\delta_k{}^l\right) + 12 D_{\dot\rho}{}^{\alpha}\left[D_{\dot\sigma}{}^j, D_i{}^{\beta}\right] D_k{}^l - 3\left\{D_{\dot\rho}{}^j D_{\dot\sigma}{}^l, D_i{}^{\alpha} D_k{}^{\beta}\right\} \\
&\quad + (i \leftrightarrow k) + (j \leftrightarrow l) - (\dot\rho \leftrightarrow \dot\sigma) - (\alpha \leftrightarrow \beta)\Big]\mathrm{tr}\varphi^3.
\end{aligned}$$

Rewrite $D$ in terms of individual covariant derivatives (see equation (3.3)):

$$\begin{aligned}
0 &= \Big[p_{\dot\rho}{}^{\alpha} p_{\dot\sigma}{}^{\beta}\left(6r_i{}^j r_k{}^l + \delta_i{}^j\delta_k{}^l\right) + 12 p_{\dot\rho}{}^{\alpha}\left[\bar q_{\dot\sigma}{}^j, q_i{}^{\beta}\right] r_k{}^l - 3\left\{\bar q_{\dot\rho}{}^j \bar q_{\dot\sigma}{}^l, q_i{}^{\alpha} q_k{}^{\beta}\right\} \\
&\quad + (i \leftrightarrow k) + (j \leftrightarrow l) - (\dot\rho \leftrightarrow \dot\sigma) - (\alpha \leftrightarrow \beta)\Big]\mathrm{tr}\ \varphi^3 \\
&= -\epsilon^{\alpha\beta}\epsilon_{\dot\rho\dot\sigma}\Big[p^{\dot\gamma\delta}p_{\dot\gamma\delta}\left(6r_{(i}{}^{(j} r_{k)}{}^{l)} + \delta_{(i}{}^{(j}\delta_{k)}{}^{l)}\right) + 12 p^{\dot\gamma\delta}\left[\bar q_{\dot\gamma}{}^{(j}, q_{(i|\delta}\right] r_{|k)}{}^{l)} \\
&\quad - 3\left\{\bar q^{\dot\gamma(j}\bar q_{\dot\gamma}{}^{l)}, q_{(i}{}^{\delta} q_{k)\delta}\right\}\Big]\mathrm{tr}\varphi^3.
\end{aligned}$$

Therefore, we found tr$\varphi^3$ satisfies semi-shortening constraint:

$$\begin{aligned}
0 = &\Big[p^{\dot\gamma\delta}p_{\dot\gamma\delta}\left(6r_{(i}{}^{(j} r_{k)}{}^{l)} + \delta_{(i}{}^{(j}\delta_{k)}{}^{l)}\right) + 12 p^{\dot\gamma\delta}\left[\bar q_{\dot\gamma}{}^{(j}, q_{(i|\delta}\right] r_{|k)}{}^{l)} \\
&- 3\left\{\bar q^{\dot\gamma(j}\bar q_{\dot\gamma}{}^{l)}, q_{(i}{}^{\delta} q_{k)\delta}\right\}\Big]\mathrm{tr}\varphi^3.
\end{aligned}$$

We can also choose $D^5$ semi-shortening constraints on $\mathrm{tr}\varphi^4$. Here we choose $D^5$ as follows:

$$
\begin{aligned}
D_{(\dot\rho}^{\;(\alpha} D_{\dot\sigma}^{\;\beta} D_i^{\;j} D_k^{\;l} D_{m]}^{\;n]} = {}& 10 D_{[\dot\rho}^{\;[\alpha} D_{\dot\sigma]}^{\;\beta]} D_i^{\;j} D_k^{\;l} D_m^{\;n} - 60 D_{[\dot\rho|}^{\;[\alpha} D_i^{\;\beta]} D_{|\dot\sigma]}^{\;j} D_k^{\;l} D_m^{\;n} \\
& - 30 D_i^{\;[\alpha} D_k^{\;\beta]} D_{[\dot\rho}^{\;j} D_{\dot\sigma]}^{\;l} D_m^{\;n} + 30 D_{[\dot\rho}^{\;[\alpha} D_{\dot\sigma]}^{\;\beta]} D_i^{\;j} D_k^{\;l} \delta_m^{\;n} \\
& - 60 D_{[\dot\rho|}^{\;[\alpha} D_i^{\;\beta]} D_{|\dot\sigma]}^{\;j} D_k^{\;l} \delta_m^{\;n} + 35 D_{[\dot\rho}^{\;[\alpha} D_{\dot\sigma]}^{\;\beta]} D_i^{\;j} \delta_k^{\;l} \delta_m^{\;n} \\
& - 30 D_{[\dot\rho|}^{\;[\alpha} D_i^{\;\beta]} D_{|\dot\sigma]}^{\;j} \delta_k^{\;l} \delta_m^{\;n} + 15 D_{[\dot\rho}^{\;[\alpha} D_{\dot\sigma]}^{\;\beta]} \delta_i^{\;j} \delta_k^{\;l} \delta_m^{\;n} \\
& + (i \leftrightarrow k \leftrightarrow m) + (j \leftrightarrow l \leftrightarrow n),
\end{aligned}
\tag{3.18}
$$

which indicates $\mathrm{tr}\varphi^4$ satisfies the following constraint:

$$
\begin{aligned}
0 = \Big[ & p^{\dot\alpha\alpha} p_{\dot\alpha\alpha} \Big( 2 r_{(i}^{\;(j} r_k^{\;l} r_{m)}^{\;n)} + 6 r_{(i}^{\;(j} r_k^{\;l} \delta_{m)}^{\;n)} + 7 r_{(i}^{\;(j} \delta_k^{\;l} r \delta_{m)}^{\;n)} + 3 \delta_{(i}^{\;(j} \delta_k^{\;l} \delta_{m)}^{\;n)} \Big) \\
& - 6 p^{\dot\alpha\alpha} q_{(i|\alpha} \bar q_{\dot\alpha}^{\;(j} \Big( 2 r_{|k}^{\;l} r_{m)}^{\;n)} + 2 r_{|k}^{\;l} \delta_{m)}^{\;n)} + \delta_{|k}^{\;l} \delta_{m)}^{\;n)} \Big) - 6 q_{(i}^{\;\alpha} q_{k|\alpha} \bar q^{\dot\alpha(j} \bar q_{\dot\alpha}^{\;l} r_{|m)}^{\;n)} \Big] \mathrm{tr}\varphi^4 .
\end{aligned}
$$

The two examples above are satisfied on BPS representations.

## 3.8 Summary

In section 3.5, we proved operators $D^n \equiv \left\{ D_{(\mathcal{M}_1}^{\;(\mathcal{N}_1} D_{\mathcal{M}_2}^{\;\mathcal{N}_2} \cdots D_{\mathcal{M}_n]}^{\;\mathcal{N}_n]} \right\}$ transform covariantly (up to an overall coefficient) under $(P)SU(2,2|\mathcal{N})$ symmetry. From the discussions in sections 3.3 and 3.6, we found that the most well-known shortening and semi-shortening conditions form a subset of $D^1$, $D^2$, and $D^3$. Since the new method treat semi-shortening constraints as covariant operators $D^n$ (which are essentially derivatives), together with the algebras in section 3.5, it is easier to manipulate with and write down the explicit expressions of semi-shortening conditions. In particular, we found in subsection 3.7 for the case of $\mathcal{N}$=4 SYM that the full set of $D^{n+1}$ constraints apply to the BPS operators tr $\varphi^n$ and gave some examples with explicit forms.

# Chapter 4

# F-theory With Unbroken Symmetry Currents

As mentioned in Chapter 1, F-theory is not the 12 dimensional example in Vafa's paper [27] but a theory that has manifest U-duality. Because of some technical difficulty, rather than attacking the "full" F-theory (which gives all types of 10 dimensional string theory), we deal with only lower dimensional string theories and treat the rest of the dimension as scalars on the worldvolume. We use **F**-, **M**-, **T**-, and **S**-theory to distinguish from the full F-, M-, T-, and S-theory (where T-theory means T-duality manifest theory and S-theory means string theory). To understand **F**-theory, it is useful to understand **T**-theory. We, therefore, start with a review on **T**-theory.

## 4.1 Review On T-theory

In this section, we will use two different approaches to obtain the T-dual manifest theory. The first approach is the "field approach" [50–54]. It is the more "popular" approach, however, also the harder one. It starts with string theory, find the the field contents and symmetries, then play with the fields in particle limit (neglect worldsheet coordinates) and find a self consistent algebra. The second approach is the "worldsheet current approach" [28, 29, 55–60]. Also start with string theory, find its symmetry currents and their algebra, then drop its vibration modes. They will lead to the same algebra at the end of the day.

### 4.1.1 Field Approach

Unlike particle theories, string theory is a theory of extended objects. Because of this property, string theory has some symmetries or dualities particle theories doesn't have. One of the properties that particles doesn't have is that a (closed) string can wrap around compact dimensions which gives an extra "quantum number" — winding number. If the dimensions are compactified, the momentum in the dimensions are also quantized (an integer times of some number). The mass spectrum of a closed string on a circle is

$$M^2 = \frac{2}{\alpha'}\left(N_R + N_L - 2\right) + K^2\frac{1}{R^2} + W^2\frac{R^2}{\alpha'^2},$$

and the level matching condition is

$$N_R - N_L = KW.$$

The $N_R$ and $N_L$ in the above two equations are right moving and left moving excitation number respectively, $K$ and $W$ are momentum excitation number and winding number. The mass spectrum is invariant under the following "duality":

$$K \leftrightarrow W, \quad R \to \frac{\sqrt{\alpha'}}{R}.$$

If we push this a little further, compactify $d$ dimensions and turn on constant backgrounds, the string action becomes

$$S = -\frac{1}{2\pi\alpha'}\int d\sigma^2\left[\eta_{ab}\eta^{\alpha\beta}\partial_\alpha X^a\partial_\beta X^b + \left(G_{mn}\eta^{\alpha\beta} + B_{mn}\epsilon^{\alpha\beta}\right)\partial_\alpha X^m\partial_\beta X^n\right],$$

where $a, b = 0 \cdots D-d-2$ and $m, n = D-d-1 \cdots D-1$. The $\eta$'s above might be a little confusing, to clarify, $\eta^{\alpha\beta}$ is the worldsheet metric (a 2 dimensional metric) and $\eta_{ab}$ is the flat metric on the uncompactified dimensions (a $D-d$ dimensional metric). The canonical momentum densities on the compactified dimensions are

$$\Pi_m = \frac{1}{\pi\alpha'}\left(G_{mn}\dot{X} - B_{mn}X'\right).$$

We can define the left and right moving mode for the compactified dimensions as:

$$K_m = \int d\sigma \Pi_m = \frac{1}{2} G_{mn} \left(P_L^n + P_R^n\right) + \frac{1}{2} B_{mn} \left(P_L^n - P_R^n\right)$$

$$\Rightarrow \begin{cases} P_L^m = W^m + \left(G^{-1}\right)^{mn} \left(K_n - B_{np} W^p\right) \\ P_R^m = -W^m + \left(G^{-1}\right)^{mn} \left(K_n - B_{np} W^p\right) \end{cases} .$$

The mass formula becomes

$$\begin{aligned} M^2 &= \frac{2}{\alpha'} \left(N_R + N_L - 2\right) + G_{mn} \left(P_L^m P_L^n + P_R^m P_R^n\right) \\ &= \frac{2}{\alpha'} \left(N_R + N_L - 2\right) + (K \; W) \begin{pmatrix} G^{-1} & G^{-1} B \\ -B G^{-1} & G - B G^{-1} B \end{pmatrix} \begin{pmatrix} K \\ W \end{pmatrix} \\ &= \frac{2}{\alpha'} \left(N_R + N_L - 2\right) + P^T H P \end{aligned}$$

where

$$H = \begin{pmatrix} G^{-1} & G^{-1} B \\ -B G^{-1} & G - B G^{-1} B \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} K \\ W \end{pmatrix},$$

and the level matching condition is

$$N_R - N_L = \frac{1}{4} g_{mn} \left(P_L^m P_L^n - P_R^m P_R^n\right) = W^m K_m .$$

Define an $O(d,d)$ invariant metric

$$\eta = \begin{pmatrix} 0 & 1_{\mathbf{d}} \\ 1_{\mathbf{d}} & 0 \end{pmatrix},$$

we found that

$$H^{-1} = \eta^T H \eta. \tag{4.1}$$

Rewrite the above equations in terms of the $H$, $P$, and $\eta$:

$$\begin{aligned} M^2 &= \frac{2}{\alpha'} \left(N_R + N_L - 2\right) + P^T H P, \\ N_R - N_L &= \frac{1}{2} P^T \eta P. \end{aligned}$$

The theory is invariant under the following change:

$$P \to AP, \quad H \to A^T H A$$

does not affect the above equations provide

$$A^T \eta A = \eta.$$

Hence, $O(d,d)$ is a symmetry of the theory. Interchanging specific $W^m$ with $K_m$ is also an element of $O(d,d)$ by choosing

$$A_k = \begin{pmatrix} 1_{\mathbf{d}} - t_k & t_k \\ t_k & 1_{\mathbf{d}} - t_k \end{pmatrix}, \tag{4.2}$$

where $t_k = diag(\underbrace{0, 0, \cdots,}_{k-1} 1, 0, \cdots, 0)$. Therefore, T-duality is part of $O(d,d)$ transformation. Note that like usual (super)gravity theories, the (generalized) metric $H$ is, in fact, an element of $O(d,d)$ (can be checked from multiplying "$H\eta$" to equation (4.1) from the right and use $\eta^{-1} = \eta^T = \eta$ and symmetric property of $H$) and can be written as

$$H = VV^T,$$

where $V$ is another $O(d,d)$ element. $O(d,d)$ transformation on $H$ is

$$H' = O^T H O = O^T V^T V O = (VO)^T (VO) = V'^T V \Rightarrow V' = hVO,$$

provide $h^T h = 0$, i.e. $h \in O(d) \times O(d)$. Hence, the full symmetry is $\dfrac{O(d,d)}{O(d) \times O(d)}$. Note that if time dimension is compactified, the "unbroken symmetry" becomes $O(d-1,1) \times O(d-1,1)$ instead of $O(d) \times O(d)$. The same method mentioned above can be generalized to $D$ compactified dimensions.

Rather than compactifying just the extra dimensions, we consider all the spacetime dimensions are compactified and so that T-duality in every dimension is manifest. Then we will define its infinitesimal coordinate transformation (using generalized Lie derivative or D-bracket). We will later decompactify the dimension by applying the strong constraint.

Here, we assume that all the dimensions are compactified so that T-duality on every coordinates are manifest. From the above discussion, we know that the T-dual of canonical momentum, $K^m$, is $W_m$ which will be redefined (for the sake of simplicity) as

$$P \equiv \begin{pmatrix} p \\ \tilde{p} \end{pmatrix} \equiv \begin{pmatrix} K \\ W \end{pmatrix}.$$

And $p$ and $\tilde{P}$'s corresponding coordinates are $x$ and $\tilde{x}$ respectively, also, $X \equiv (x, \tilde{x})$. Although the theory has manifest T-duality by including the T-dual coordinate, we don't experience the dual coordinate in our real life experience. Therefore, there should be some constraint to kill the extra coordinates. The most direct way to achieve this is to impose

$$\frac{\partial}{\partial \tilde{x}_m} A(X) = 0 \quad \forall m,$$

where $A$ is an arbitrary function of doubled coordinates. However, $\dfrac{\partial}{\partial \tilde{x}_m}$ is not covariant under $O(D,D)$ transformation. The other possible solution is to impose

$$\frac{\partial}{\partial x^m} \frac{\partial}{\partial \tilde{x}_m} A(X) = 0,$$

where the solution is either $x^m$ or $\tilde{x}_m$ vanishes, i.e. for every coordinates, either it vanishes or its dual coordinate vanishes. This constraint not only kills half of the coordinates, it also goes well with T-duality since we can still perform T-dual transformation ($x \leftrightarrow \tilde{x}$) without violet this constraint. This constraint is called weak constraint (will be clear later when talking about strong constraint). One more thing to note is that this constraint is not only covariant, it is, in fact, invariant. This can be easily seen when writing down the constraint in doubled coordinates:

$$\begin{aligned}
0 =& \frac{\partial}{\partial x^m} \frac{\partial}{\partial \tilde{x}_m} A(X) \\
=& \frac{1}{2} \left( \frac{\partial}{\partial x^m} \; \frac{\partial}{\partial \tilde{x}_m} \right) \begin{pmatrix} 0 & \delta^n_m \\ \delta^m_n & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x^n} \\ \dfrac{\partial}{\partial \tilde{x}_n} \end{pmatrix} A(X) \\
=& \frac{1}{2} \partial_M \eta^{MN} \partial_N A(X) \\
\Rightarrow& \partial^M \partial_M A(X) = 0,
\end{aligned}$$

where $\partial_M = \left(\frac{\partial}{\partial x^m}, \frac{\partial}{\partial \tilde{x}_m}\right)$.

Finding coordinate transformation (diffeomorphism in the $2D$ manifold) is cumbersome and complicate, we will not derive it here (will be derived using current approach). The generalized Lie derivative acting on vector indices are defined as follows (generalization to higher rank tensor is straightforward):

$$\begin{aligned} \mathcal{L}_\Xi V^M &\equiv \Xi^N \partial_N V^M - V^N \partial_N \Xi^M + \left(\partial^M \Xi_N\right) V^N, \\ \mathcal{L}_\Xi V_M &\equiv \Xi^N \partial_N V_M - V^N \partial_N \Xi_M + \left(\partial_M \Xi_N\right) V^N \end{aligned} \tag{4.3}$$

where $\Xi^M = (\tilde{\xi}^m, \xi_m)$, and $M, N$ are $2D$ indices. We use this definition and act on $H_{MN}$:

$$\mathcal{L}_\Xi H_{MN} = \Xi^P \partial_P H_{MN} - \left(\partial^P \Xi_M\right) H_{PN} - \left(\partial^P \Xi_N\right) H_{MP} + \left(\partial_M \Xi^P\right) H_{PN} + \left(\partial_N \Xi^P\right) H_{MP}.$$

Assuming all fields in $H$ only depend on $x$ coordinate, i.e. $\tilde{\partial} H = 0$. We get

$$\begin{aligned} \mathcal{L}_\Xi g_{ij} &= \xi^k \partial_k g_{ij} + \left(\partial_i \xi^k\right) g_{kj} + \left(\partial_j \xi^k\right) g_{ik}, \\ \mathcal{L}_\Xi b_{ij} &= \xi^k \partial_k b_{ij} + \left(\partial_i \xi^k\right) b_{kj} + \left(\partial_j \xi^k\right) b_{ik} + \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i. \end{aligned}$$

The above reproduces diffeomorphism and gauge transformation on $g$ and $b$ in gravity theory where $\xi$ is the infinitesimal change in coordinate and $\tilde{\xi}$ is the infinitesimal change in gauge parameter. Not only is the generalized diffeomorphism consistent with the usual diffeomorphism after solving weak constraint, it also generates gauge transformation. As a side note, $\eta_{MN}$, $\eta^{MN}$, and $\delta_M^N$ are invariant tensors under generalized Lie derivative.

The form of the infinitesimal coordinate change (equation (4.3)) is also called D-bracket defined as

$$[\Xi, V]_D^M \equiv \mathcal{L}_\Xi V^M.$$

Another useful bracket is C-bracket:

$$[\Xi_1, \Xi_2]_C^M \equiv \frac{1}{2}\left(\mathcal{L}_{\Xi_1} X_2^M - \mathcal{L}_{\Xi_2} X_1^M\right),$$

which is the antisymmetric part of D-racket. We know what D-bracket is used for, here, we discuss what C-bracket is for. For any two successive coordinate transformations, the result should also be a coordinate transformation.

Therefore, the result of C-bracket is the difference of the two transformation, and the $\frac{1}{2}$ is just a normalization constant (also will be derived using current approach).

Therefore, the difference between acting consecutive coordinate transformations should equal to a coordinate transformation with the difference of the two transformation, i.e.

$$\left(\mathcal{L}_{\Xi_1}\mathcal{L}_{\Xi_2} - \mathcal{L}_{\Xi_1}\mathcal{L}_{\Xi_2}\right) V^M - \mathcal{L}_{[\Xi_1,\Xi_2]_C} V^M = 0.$$

However, doing this explicitly shows that left hand side of the equation is

$$LHS = \partial^N \Xi^M_{[1} \partial_N \Xi^P_{2]} V_P - \frac{1}{2} \Xi^N_{[1} \partial^P \Xi_{2]N} \partial_P V^M$$

To make LHS vanish, we have to seek for a constraint that kills it. We can make use of weak constraint — $\partial^M \partial_M A(X) = 0$, and "generalize" it into strong constraint — $\left(\partial^M A(X)\right)\left(\partial_M B(X)\right) = 0$ for any arbitrary $A$ and $B$.

List some of the results from this subsection is listed in Table 4.1.

i) T-dual manifest theory has $\dfrac{O(d,d)}{O(d)\times O(d)}$ symmetry and needs twice coordinate than usual.

ii) To kill the extra coordinates, we need weak constraint — $\partial^M \partial_M A(X) = 0$.

iii) Diffeomorphism is generated using D-bracket.

iv) The difference between two diffeomorphism is C-bracket.

v) To make the algebra consistent, we need strong constraint — $\left(\partial^M A(X)\right)\left(\partial_M B(X)\right) = 0$.

Table 4.1: T-dual manifest results.

### 4.1.2 Worldsheet Current Approach

This section we use another approach to get the same result in Table 4.1. We will not derive $\dfrac{O(D,D)}{O(D-1,1)\times O(D-1,1)}$ again since they are the same in both cases. All the derivation using this approach is more intuitive. The first quantized commutation relation for string theory:

$$[p_m(1), x_n(2)] = -i\eta_{mn}\delta(1-2),$$

where $p_m(1)$ is short for $p_m(\sigma_1)$, and the rest should be self-explanatory. Since string is an extended object, its solution can be decomposed into left-moving and right-moving mode. For the coordinates, T-duality changes the sign of right-moving mode:

$$\begin{cases} x = \frac{1}{2}(x_L + x_R) \\ \tilde{x} = \frac{1}{2}(x_L - x_R) \end{cases},$$

where $\tilde{x}$ is the T-dual of $x$. It is the same for momentum

$$\begin{cases} p = \frac{1}{2}(p_L + p_R) \\ \tilde{p} = \frac{1}{2}(p_L - p_R) \end{cases}.$$

For T-dual to really be an duality of the theory, the commutation relation must no be distinguished before and after T-duality transformation. Therefore, the commutation relation between the T-dualized momentum and coordinate should satisfy

$$[\tilde{p}_m(1), \tilde{x}_n(2)] = -i\eta_{mn}\delta(1-2).$$

The left-moving momentum and right-moving momentum can be written down explicitly using full momentum $p$ and $x$ as

$$\begin{cases} p_L = p + x' = x'_L \\ p_R = p - x' = -x'_R \end{cases},$$

where we have used

$$\begin{cases} p(\tau,\sigma) & = \partial_\tau x(\tau,\sigma) = \frac{1}{2}\partial_\tau\left(x_R(\tau-\sigma) + x_L(\tau+\sigma)\right) \\ & = \frac{1}{2}\partial_\sigma\left(-x_R(\tau-\sigma) + x_L(\tau+\sigma)\right) \\ & = \tilde{x}'(\tau,\sigma) \\ \tilde{p}(\tau,\sigma) & = \partial_\sigma x(\tau,\sigma) = \frac{1}{2}\partial_\sigma\left(x_R(\tau-\sigma) + x_L(\tau+\sigma)\right) \\ & = x'(\tau,\sigma) \end{cases}. \tag{4.4}$$

Although $p$'s commutes with each other, $p_L$'s doesn't commute with each other (either do $p_R$). The commutation relations between $p_L$'s, $p_R$'s are

$$\begin{cases} [p_{Lm}(1), p_{Ln}(2)] = -i2\eta_{mn}\delta'(1-2) \\ [p_{Rm}(1), p_{Rn}(2)] = i2\eta_{mn}\delta'(1-2) \\ [p_{Lm}(1), p_{Rn}(2)] = 0 \end{cases}.$$

To make contact with field approach, we use $p$ and $\tilde{p}$ rather than $p_L$ and $p_R$, we can, again, define

$$P_M = (p_m, \tilde{p}^m) = (p_m, \eta^{mn}\tilde{p}_n) = \left(\frac{1}{2}\left(p_{Lm} + p_{Rm}\right), \eta^{mn}\left(p_{Ln} - p_{Rn}\right)\right).$$

The commutation relation between $P$'s is

$$[P_M(1), P_N(2)] = i\eta_{MN}\delta'(1-2),$$

where

$$\eta_{MN} = \begin{pmatrix} 0 & \delta_m^n \\ \delta_n^m & 0 \end{pmatrix},$$

which is an $O(D,D)$ metric. Using this metric together with equation (4.4), we found that

$$P_M = \eta_{MN}X'^N. \tag{4.5}$$

Since $X$ commutes with itself, we have

$$\begin{aligned}[P_M(1), F(X(2))] &= \left[P_M(1), F(X)|_{X=0} + X^N(2)\partial_N F(X)|_{X=0} + \cdots\right] \\ &= -i\partial_M F(X(1))\delta(1-2).\end{aligned}$$

The infinitesimal coordinate transformation in the $2D$ spacetime on a vector $V = V^M P_M$ is

$$\begin{aligned}&\left[\int d\sigma_1 \Xi(1)^M P_M(1), V^N(2)P_N(2)\right] \\ &= -i\int d\sigma_1 \big(\Xi^M\left(\partial_M V^N\right)P_N\delta(1-2) - V^M\left(\partial_M \Xi^N\right)P_N\delta(1-2) \\ &- \Xi^M(1)V^N(2)\eta_{MN}\delta'(1-2)\big) \\ &= -i\big(\Xi^M\left(\partial_M V^N\right)P_N - V^M\left(\partial_M\Xi^N\right)P_N + V^N\partial^P\Xi^M P_P\eta_{MN}\big)(1),\end{aligned}$$

where the last equality comes from equation (4.5) and implicit normal ordering. Notice that everything we have here has an extra $-i$ comparing to field approach. The reason is that what we are doing here is at the level of first quantization. To compare with the "classical" result, we only have to multiply the result by $i$ and drop the oscillating modes (i.e. let the functions do not depend on $\sigma$, which goes back to particle limit) after the calculation. Therefore, the generalized Lie derivative or D-bracket is

$$[\Xi, V]_D^M \equiv \mathcal{L}_\Xi V^M \equiv \Xi^M \left(\partial_M V^N\right) P_N - V^M \left(\partial_M \Xi^N\right) P_N + V^N \partial^P \Xi^M P_P \eta_{MN}.$$

We can also calculate the difference between two coordinate transformations:

$$\begin{aligned}
&\left[\int d\sigma_1 \Xi_1(1)^M P_M(1), \int d\sigma_2 \Xi_1(2)^M P_M(2)\right] \\
&= \int d\sigma_1 d\sigma_2 \left[\Xi_1(1)^M P_M(1), \Xi_1(2)^M P_M(2)\right] \\
&= -i \int d\sigma_1 d\sigma_2 \big(\Xi^M \left(\partial_M \Xi_2^N\right) P_N \delta(1-2) - \Xi_2^M \left(\partial_M \Xi_1^N\right) P_N \delta(1-2) \\
&- \Xi_1^M(1) \Xi_2^N(2) \eta_{MN} \delta'(1-2)\big) \\
&= -i \int d\sigma \big(\Xi^M \left(\partial_M \Xi_2^N\right) P_N - \Xi_2^M \left(\partial_M \Xi_1^N\right) P_N \\
&+ \frac{1}{2} \partial^P \Xi_1^M \Xi_2^N \eta_{MN} P_P - \frac{1}{2} \Xi_1^M \partial^P \Xi_2^N \eta_{MN} P_P\big) \\
&= -i \int d\sigma \Xi_{[12]}^M P_M,
\end{aligned}$$

Hence, C-bracket is:

$$\begin{aligned}
[\Xi_1, \Xi_2]_C^M \equiv\ & \Xi^M \left(\partial_M \Xi_2^N\right) P_N - \Xi_2^M \left(\partial_M \Xi_1^N\right) P_N \\
& + \frac{1}{2} \partial^P \Xi_1^M \Xi_2^N \eta_{MN} P_P - \frac{1}{2} \Xi_1^M \partial^P \Xi_2^N \eta_{MN} P_P\big).
\end{aligned}$$

Before closing this subsection, we would like to derive "weak" and "strong" constraints from this current approach. As a standard starting point, we write down the Lagrangian with background fields:

$$\mathcal{L} = \frac{-1}{2\pi\alpha'} \left(\sqrt{-h} h^{\alpha\beta} G_{mn} + \epsilon^{\alpha\beta} B_{mn}\right) \partial_\alpha X^m \partial_\beta X^n. \tag{4.6}$$

The Virasoro constraint [56](the worldsheet energy-momentum tensor) is

$$\left(\partial_\alpha X^m\right) \left(\partial_\beta X^n\right) G_{mn} - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \left(\partial_\gamma X^m\right) \left(\partial_\delta X^n\right) G_{mn} = 0$$

After choosing conformal gauge, the above constraint becomes two parts (we only choose $\alpha = 0$ case since it is the symmetry charge)

$$\left\{ \begin{array}{l} \left(\partial_0 X^m\right)\left(\partial_0 X^n\right) G_{mn} - \frac{1}{2}\eta^{\gamma\delta}\left(\partial_\gamma X^m\right)\left(\partial_\delta X^n\right) G_{mn} = 0 \\ \left(\partial_0 X^m\right)\left(\partial_1 X^n\right) G_{mn} = 0 \end{array} \right. , \tag{4.7}$$

Using the definition of canonical momentum density we get

$$\begin{aligned} &\left\{ \begin{array}{l} p_m = \frac{1}{\pi\alpha'}\left(G_{mn}\dot{x}^n - B_{mn}x'^n\right) \\ \tilde{p}^m = \frac{1}{\pi\alpha'}x'^m \end{array} \right. \\ \Rightarrow &\left\{ \begin{array}{l} \dot{x}^m = \pi\alpha\left(G^{mn}p_n + G^{mn}B_{no}\tilde{p}^o\right) \\ x'^m = \pi\alpha\tilde{p}^m \end{array} \right. . \end{aligned}$$

Plug this back into equation (4.7), we get

$$\left\{ \begin{array}{l} \frac{1}{2}p_m G^{mn}p_n - \frac{1}{2}\tilde{p}^p B_{pm}G^{mn}p_n + \frac{1}{2}p_m G^{mn}B_{no}\tilde{p}^o - \tilde{p}^m p_m + \frac{1}{2}\tilde{p}^m G_{mn}\tilde{p}^n = 0 \\ \tilde{p}^m p_m = 0 \end{array} \right. .$$

In terms of $2D$ spacetime tensors

$$\begin{aligned} &\left\{ \begin{array}{l} \frac{1}{2}P_M H^{MN}P_N - \frac{1}{2}P_M\eta^{MN}P_N = 0 \\ \frac{1}{2}P_M\eta^{MN}P_N = 0 \end{array} \right. \\ \Rightarrow &\left\{ \begin{array}{l} \frac{1}{2}P_M H^{MN}P_N = 0 \\ \frac{1}{2}P_M\eta^{MN}P_N = 0 \end{array} \right. , \end{aligned} \tag{4.8}$$

where

$$H^{MN} = \left( \begin{array}{cc} \left(G^{-1}\right)^{mn} & \left(G^{-1}\right)^{mo}B_{on} \\ -B_{mo}\left(G^{-1}\right)^{on} & G_{mn} - B_{mo}\left(G^{-1}\right)^{op}B_{pn} \end{array} \right),$$

just like the one in the last section. We now focus on the second equation of equation (4.8). Again, if we only care about the particle limit, then we can drop all the dependence on $\sigma$'s. In such limit, the representation of $P$'s in coordinate basis is $P_M = -i\frac{\partial}{\partial X^M}$. Since all the fields in the theory should satisfy Virasoro constraints (equation (4.8)):

$$0 = P_M \eta^{MN} P_N A(X) = -\partial^M \partial_M A(X),$$

which gives the weak constraint. If we further assume that this is also true for composite fields, say $A(X)B(X)$, we get

$$\begin{aligned} 0 &= -\partial^M \partial_M \left(A(X)B(X)\right) \\ &= -\partial^M \partial_M \left(A(X)\right) B(X) - \partial^M \partial_M \left(B(X)\right) A(X) - 2\left(\partial^M A(X)\right)\left(\partial_M B(X)\right) \\ &= -2\left(\partial^M A(X)\right)\left(\partial_M B(X)\right), \end{aligned}$$

which is the strong constraint.

Therefore, we showed that both approaches gives the same result. The differences is that in field approach is that it start with string first quantization (to find its level matching condition and mass formula), drop the dependence on $\sigma$ then do the rest of the calculation, while in worldsheet current approach, we do all the calculations first, and find its classical correspondence and drop the $\sigma$ dependence at the end.

### 4.1.3 U-duality and F-theory

This subsection, we discuss why U-dualities are needed (in addition to T-theory). In the above discussion, we start with one compact dimension which gives us T-duality between some strings (type IIA and type IIB, for example). Then we work the up to $\frac{O(D,D)}{O(D-1,1)\times O(D-1,1)}$. However, type IIB string has an additional "symmetry" or a duality (dual to itself), namely $SL(2)$ or S-duality, which is not part of $O(D,D)$. In contrast to $O(D,D)$, this $SL(2)$ symmetry /duality has a very geometrical explanation since toroidal compactify 2 of the dimensions always gives $SL(2)$ symmetry (it is a modular group of $T^2$).

Since the birth of M-theory, people start to think that the most fundamental theories might live in eleven dimensions and all string theories might just be different limit of the eleven dimensional theory. To reduce to string theories, we need to compactify one of the dimensions into small size. To get even lower dimensional theories, we must compactify more dimensions on this eleven dimensional theory. Under this assumption, $d$ dimensional theories lives in $R^d \times M^{11-d}$ space, where $M^{11-d}$ is an $11-d$ dimensional manifold. To discuss U-duality, we only limit ourselves in toroidal compactification cases, i.e. $M^{11-d} = T^{11-d}$.

For $T^n$ compactification, the symmetry group will always have $SL(n)$ as its subgroup since $SL(n)$ is a geometric symmetry group of $T^n$. However, for extended objects, there might be some non-geometrical symmetry group shows up, e.g. T-duality group. Therefore, for string theories, there should also be $O(n-1, n-1)$ symmetry. Hence, from these arguments, we can conclude that the symmetry group with $n$ dimensions compactified into $T^n$ will have symmetry group $G$ such that

$$\begin{cases} SL(n) \subset G \\ O(n-1, n-1) \subset G \end{cases} .$$

We don't need to G to be the direct product of the two groups since the two groups might intertwine nontrivially so that direct product is way larger than what we need. We will give some examples later to show how groups intertwine with each other and gives smaller group than direct product. In fact, there is a series of groups that could achieve the above two requirement, which is $E_{n(n)}$, the maximal noncompact form of $E_n$.

Here we give some examples where $SL(n)$ and $O(n, n)$ intertwine nontrivially and gives a small group than $SL(n) \times O(n, n)$. The first case is $n = 3$, where $E_{3(3)} = SL(3) \times SL(2)$. Obviously, this group contains $SL(3)$. Next we check $E_{3(3)}$ has a subgroup $O(2, 2)$. There is an isomorphism between $O(2, 2)$ and $SL(2) \times SL(2)$. Hence we have found that

$$\begin{cases} SL(3) \subset SL(3) \times SL(2) = E_{3(3)} \\ O(3,3) \cong SL(2) \times SL(2) \subset SL(3) \times SL(2) = E_{3(3)} \end{cases} .$$

Another one is $n = 5$, in this case $G = E_{5(5)} = SO(5, 5)$. This group contains $O(4, 4)$ (up to a $Z_2$ reflection). The $SL(5)$ subgroup can be seen from its Dynkin diagram, where if we remove one of the bifurcating points, we get the Dynkin diagram of $SL(5)$. Therefore, $E_{5(5)}$ is a group that fulfills both above mentioned requirements. So when compactifying all eleven dimensions in M-theory, following the pattern, we should get $E_{11(11)}$ symmetry. Therefore, it is conjectured that the full symmetry of the full M-theory enjoys $E_{11(11)}$ symmetry.

In 1996, Vafa realized that M-theory (or eleven dimensional theory) is not enough to reduce to type IIB string without having a zero area $T^2$ compactification in order to have $SL(2, \mathbb{Z})$. In some sense, type IIB string dimensional reduced from M-theory gives an effective nine dimensional theory rather than the full ten dimensional string. He, therefore, add in one more dimension to the theory so that compactifying a zero area two-torus will still give us a proper ten dimensional type IIB string theory. Many people thought that F-theory the Vafa proposed is a twelve dimensional theory. However, just as his title showed, the twelve dimensional theory he gave in the paper is an example and an evidence that a higher (than eleven) dimensional theory should exist so that all string theories can be manifestly included in the F-theory.

### 4.1.4 F-theory From Worldvolume Currents Approach

As shown in subsection 4.1.1 and 4.1.2, in order to have manifest T-duality, we need to double the coordinates. The extra dimensions can be killed by introducing some constraints. We follow the same route here to find a candidate of F-theory Vafa proposed. We assume that the symmetry of F-theory is governed by M-theory/S-theory. Therefore, as argued in the last subsection, F-theory should have manifest $E_{11(11)}$ symmetry. However, attacking $E_{11(11)}$ symmetry is hard since the generator is infinite dimensional. We, therefore, start with lower dimensional **F**-theory such that is reduces to lower dimensional **S**-theory (lower dimensional string theory). Instead of treating **S**-theory ten dimensional, we start with three dimensional **S**-theory.

We found that worldsheet current approach is easier to work with since it only involves the maths we use in quantum mechanics. However, there is one difference, since U-duality is a consequence of M-theory, rather than string, we need brane. Hence, rather than worldsheet, we use worldvolume current approach. A direct consequence is that worldvolume indices becomes spacetime indices.

As an example, we start with the well-studied example [36] — $10D$ **F**-theory. It all started with a 5-brane (a six dimensional object). This theory can be reduced to $4D$ **M**-theory, $6D$ **T**-theory (with $O(3,3)$ symmetry), and $3D$ **S**-theory. Hence, this **F**-theory should have manifest $E_{4(4)} = SL(5)$ symmetry. The action (after choosing conformal gauge) is:

$$S = \frac{1}{12}\int d^6\sigma F_{MNO}F^{MNO},$$

where $M, N, O = -2\cdots 3$ and

$$F_{MNO} = \frac{1}{2}\partial_{[M}X_{NO]}.$$

$X^{MN}$'s are spacetime coordinate, but they are worldvolume gauge fields. Therefore, not all of them are physically present. As we will see later, some of them can be gauged away, some of them can be killed by generalized Virasoro constraints. The interesting thing here is that this spacetime coordinate carries two world volume indices. This shows something "abnormal", the first one is that the "spacetime index" is in fact two indices. However, this has already been seen in exceptional field theory which is a theory also trying to include manifest U-duality. The really new and exotic thing here is that worldvolume indices are also spacetime indices. We can find its canonical momentum density for $X^{mn}$ (where $m, n, = -1 \cdots 3$, assuming $-2$ direction is the time direction, $\tau$) is $P_{mn}$. We found that $x^{\tau m}$'s have no canonical momentum, they are Lagrange multipliers like in $U(1)$ gauge field theory. These can be seen using first order formalism

$$S = -\int d\tau d^5\sigma \frac{1}{2} P_{mn} \partial_\tau X^{mn} + \int d\tau H,$$

where

$$H = \int d^5\sigma \left( \frac{1}{4} P_{mn} P^{mn} + \frac{1}{12} F_{mno} F^{mno} + X^{\tau m} \partial^n P_{mn} \right). \tag{4.9}$$

The $F_{mno}$ in the above is $\frac{1}{2}\partial_{[o} X_{mn]}$ and $P_m n = F_{\tau mn}$. Since the field content is not irreducible, we should use (anti)self-dual form. Although a covariant action with right number of fields can be formulated [61, 62], but we still use the "traditional" way to treat (anti)self-dual form for 5-brane — impose (anti)self-dual condition by hand.

We define the self-dual momentum as

$$\overset{\circ}{\triangleright}_{mn} \equiv P_{mn} + \frac{1}{2} \epsilon_{mnopq} \partial^q X^{op}.$$

The canonical commutation relation is, by construction,

$$[P_{mn}(1), X^{op}(2)] = -i \delta^{op}_{mn} \delta(1-2),$$

where

$$\delta^{op}_{mn} = \frac{1}{2} \left( \delta^o_m \delta^p_n - \delta^o_n \delta^p_m \right).$$

The commutation relation between two $\overset{\circ}{\triangleright}$'s is then

$$\left[ \overset{\circ}{\triangleright}_{mn}(1), \overset{\circ}{\triangleright}_{op}(2) \right] = 2i \epsilon_{mnopq} \partial^q \delta(1-2).$$

We found that the "metric" is $\epsilon_{mnopq}$. The first two indices can be regarded are "one" spacetime index, the following two are "one" spacetime index as well. However, there is one more worldvolume index left. We pause here for a second. Recall that in subsection 4.1.2, one of the Virasoro constraints is

$$P_M \eta^{MN} P_N = 0,$$

i.e. inner product of two currents (the other Virasoro constraint has to do with spacetime background, which we assume to be flat here). Rather than deriving Virasoro constraints from the Lagrangian (as shown in paper [36]), we directly generalize the result in T-theory — we sum over all inner product of symmetry currents

$$S^q \equiv \mathring{\triangleright}_{mn} \epsilon^{mnopq} \mathring{\triangleright}_{op} = 0.$$

Hence, the extra index in the metric gives extra copies of Virasoro constraints. Other than Virasoro constraint, there are additional constraint coming from the Lagrangian

$$\partial^n P_{mn} = 0,$$

which has its own name — U-constraint. Note that these constraints all involves worldvolume coordinate, $\sigma$. To find its particle/massless limit, we can simply drop the oscillating modes. In other word, the constraint so far is not limited to the particle/massless modes. At this point, we just mention some earlier results without deriving them:

a) Solving Virasoro constraints reduces to **M**-theory.

b) Solving U-constraints reduces to **T**-theory.

c) Solving both constraints will further reduce to **S**-theory.

In the following we will generalize the above 5-brane algebra for $10D$ **F**-theory to any brane. We first assume that we already have the symmetry currents corresponds to the canonical momentum of $X^M$, $\mathring{\triangleright}_M$, and the commutation relations between them is

$$\left[\mathring{\triangleright}_{P_1}(1), \mathring{\triangleright}_{P_2}(2)\right] = 2i\eta_{P_1P_2r}\partial^r \delta(1-2).$$

One thing to note here is that $\overset{\circ}{\rhd}$ can be any symmetry current, for example, for supersymmetry theories, supersymmetry generator is also a current. General commutation relation is

$$\left[\overset{\circ}{\rhd}_M(1), \overset{\circ}{\rhd}_N(2)\right] = if_{MN}{}^O \overset{\circ}{\rhd}_O \delta(1-2) + 2i\eta_{MNr}\partial_1^r \delta(1-2).$$

Then the generalized Virasoro constraints are

$$S^r = \overset{\circ}{\rhd}_M \eta^{MNr} \overset{\circ}{\rhd}_N = 0.$$

And the U-constraints comes from the consistencies of the algebra (no need to have Lagrangian description), which is, in general,

$$U_M^r \equiv \left(\delta_M^N \delta_r^q - \frac{1}{2}\eta_{OMr}\eta^{ONq}\right)\partial^r P_N = 0.$$

For higher dimensional branes, more constraints (V-constraints, W-constraints, $\cdots$) will show up. For general treatment, we recommend the paper by Linch and Siegel [37].

The above techniques is pretty much the same as the technique mentioned in subsection 4.1.2 except that now rather than worldsheet, it is generalized to worldvolume. In the following, we will now review something that is not used in T-theory.

From the last subsection, we know that for $d$-dimensional **S**-theory, the corresponding **F**-theory enjoys $E_{d+1(d+1)}$ symmetry. However, like in **T**-theory: Although the theory should have $O(d, d)$ symmetry, the "vacuum" of **T**-theory is $O(d-1, 1) \times O(d-1, 1)$ invariant, i.e. the true symmetry of **T**-theory is, in fact, $\dfrac{O(d,d)}{O(d-1,1)\times O(d-1,1)}$. We call the symmetry of vacuum unbroken symmetry. To determine what the unbroken symmetry is for **F**-theory, we use the fact that **F**-theory can be reduced to **M**-theory and **T**-theory. Therefore, the unbroken symmetry of **F**-theory should contain both the unbroken symmetry of **M**- and **T**-theory. Unlike the global symmetry which has a general pattern, unbroken symmetry can only be treated case by case.

For example, in $3D$ **S**-theory case, the corresponding **T**-theory has $O(2,1)\times O(2,1)$ unbroken symmetry; **M**-theory has $SO(3,1)$ unbroken symmetry (since it is 4 dimensional supergravity symmetry, the flat limit is just the Lorentz symmetry). We use the fact that $SO(2,1)\times SO(2,1) \cong SO(2,2)$, the smallest group that contains both $SO(3,1)$ and $SO(2,2)$ is $SO(3,2)$. Since we want to include supersymmetry into **F**-theory, rather than $SO(3,2)$ unbroken symmetry, we should double the group space, which is $Spin(3,2) = Sp(4)$. Hence, for supersymmetric **S**-theory, the unbroken symmetry of the corresponding **F**-theory is $Sp(4)$.

## 4.2 Unbroken Symmetry Currents In F-theory

As mentioned in the above subsection, although the unbroken symmetry (“$H$”) of **F**-theory should be determined case by case, we know that it certainly exists. However, since the worldvolume indices are also spacetime indices and there is an additional index in the metric. These make the global brane current algebra incompatible with $H$ symmetry currents (will be elaborated in the coming section). The solution is to introduce worldvolume covariant derivatives, which depend on the $H$ coordinates even in a “flat" background.

The currents of the **F**-theories have been found [37], it is usually easier to work with by having $H$ symmetry manifest instead of gauging them to zero directly. We therefore use the group element $g$ to make $H$ symmetry local even in “flat” spacetime. The group coordinates are then included with the other “spacetime" coordinates as worldvolume fields. This requires that the derivatives of $\delta$-functions that appear in the Schwinger terms be covariantized with $g$. These derivatives were found previously to need covariantization in nontrivial backgrounds, but when gauging $H$ even “flat" spacetime has a vielbein that is not constant.

Later in this chapter we start by showing the global brane current algebra does not go along with $H$ symmetry currents. Then we modify the theory and give a very general construction for arbitrary finite dimensional current algebras and check its consistency with Jacobi identities. We will give an explicit example for the 5-brane case, where the $H$ group is $Spin(3,2)$.

## 4.3 General Construction

### 4.3.1 Problem With Naive Approach

We start this subsection with an observation, and then work the way to the general case, showing why the naive current algebra is not compatible with global $H$ symmetry.

As described in [36], we know that in general the worldvolume current $\mathring{\triangleright}_M$ obeys the following algebra:

$$\left[\mathring{\triangleright}_N(1), \mathring{\triangleright}_N(2)\right] = if_{MN}{}^{O}\mathring{\triangleright}_O\delta(1-2) + 2i\eta_{MNr}\partial_1^r\delta(1-2).$$

We could naively introduce additional $H$-group worldvolume currents $\mathring{\triangleright}_S$ and force

$$\left[\mathring{\triangleright}_S(1), \mathring{\triangleright}_M(2)\right] = if_{SM}{}^{M'}\mathring{\triangleright}_{M'}\delta(1-2).$$

Then a symmetric part of Jacobi identities gives

$$f_{SM}{}^{M'}\eta_{M'Nr} + f_{SN}{}^{N'}\eta_{MN'r} = 0. \tag{4.10}$$

However, this doesn't work because spacetime indices in F-theory are also worldvolume indices. If we want spacetime indices to transform under $H$ group, then worldvolume should transform as well. Since $\eta$ is a "constant" under group $G$-transformations, it should also be invariant under $H$-transformation. And we're led to the following identity:

$$f_{SM}{}^{M'}\eta_{M'Nr} + f_{SN}{}^{N'}\eta_{MN'r} + f_{Sr}{}^{q}\eta_{MNq} = 0,$$

which contradicts equation (4.10).

To find a solution, it is useful to go back to the general construction of symmetry generators of $H$ group. The generalized symmetry generator method from particle to brane is listed in table 4.2.

| Particle | Brane |
|---|---|
| $(dg)g^{-1} = d\alpha^i(e^{-1})_i{}^s g_s$ | $(\delta g)g^{-1} = \delta\alpha^I(e^{-1})_I{}^S G_S$ |
| $\triangleright_s = ie_s{}^i\frac{\partial}{\partial x^i}$ | $\mathring{\triangleright}_S = ie_S{}^I\frac{\delta}{\delta\alpha^I}$ |
| | $\mathring{\triangleright}^{Sr} = \left(\partial^r\alpha^I\right)(e^{-1})_I{}^S$ |

Table 4.2: Symmetry generator difference between particle and brane.

Here $G$ is the symmetry generator of the group. The last brane current is the "dual" current of $\triangleright_S$, which does not have a particle analog. The relations between structure constants and generalized metric are listed as follows:

$$\begin{aligned}
\left[\mathring{\rhd}_{S_1}(1), \mathring{\rhd}_{S_2}(2)\right] &= -e_{[S_1|}{}^I \left(\frac{\delta}{\delta\alpha^I} e_{|S_2)}{}^J\right) (e^{-1})_J{}^O e_O{}^L \frac{\delta}{\delta\alpha^L}\delta(1-2) \\
&= ie_{[S_1|}{}^I \left(\frac{\delta}{\delta\alpha^I} e_{|S_2)}{}^J\right) (e^{-1})_J{}^O \mathring{\rhd}_O \delta(1-2) \\
&= if_{S_1 S_2}{}^O \mathring{\rhd}_O \delta(1-2), \\
\left[\mathring{\rhd}_{S_1}(1), \mathring{\rhd}^{S_2 r}(2)\right] &= ie_P{}^I \left(\frac{\delta}{\delta\alpha^I} e_{S_1}{}^J\right) (e^{-1})_J{}^O e_O{}^{S_2} \left(\partial^r \alpha^L\right) (e^{-1})_L{}^P \delta(1-2) \\
&+ i\delta_{S_1}^{S_2} \partial_2^r \delta(1-2) \\
&= if_{S' S_1}{}^{S_2} \mathring{\rhd}^{S' r} \delta(1-2) + i\eta_{S_1}{}^{S_2 r}{}_q \partial_2^q \delta(1-2), \\
\left[\mathring{\rhd}^{S_1 r}(1), \mathring{\rhd}^{S_2 q}(2)\right] &= 0.
\end{aligned}$$

It's useful to inspect the simplest case with metric only (we neglect $f$ term):

$$\left[\mathring{\rhd}_{S_1}(1), \mathring{\rhd}^{S_2 r}(2)\right] \sim i\eta_{S_1}{}^{S_2 r}{}_q \partial_2^q \delta(1-2).$$

In **F**-theory, again, worldvolume indices are also spacetime indices. They both have to transform the same way under $H$ group. However, by construction, $\partial^r$ is not a function of $\alpha$, therefore, both $r$ and $q$ don't transform under $\mathring{\rhd}_S$. We are led to an impasse.

For the rest of the thesis we will denote $\mathring{\rhd}_\Sigma$ instead of $\rhd^{Sr}$ for simplicity.

### 4.3.2 Solution

To include the group $H$ in the theory, we introduce a set of $H$ group coordinates as worldvolume fields $\alpha^I(\sigma)$, and the corresponding group elements $g(\alpha(\sigma))) \in H$, and their inverses. By definition, they obey the following commutation relation:

$$\begin{aligned}
\left[\rhd_S(1), g_A{}^M(2)\right] &= if_{SA}{}^B g_B{}^M \delta(1-2), \\
\left[\rhd_S(1), (g^{-1})_M{}^A(2)\right] &= i(g^{-1})_M{}^B f_{BS}{}^A \delta(1-2),
\end{aligned}$$

where $\rhd_S$ is the symmetry generator of group $H$. We also define new sets of currents by multiplying the old ones with $g's$, i.e.

$$\rhd_A \equiv g_A{}^M \mathring{\rhd}_M,$$

so that all the indices transform under $H$-group as well.

We should point out that since we have introduced a current $\triangleright_S$, we should also introduce its dual current $\mathring{\triangleright}_\Sigma$:

$$\mathring{\triangleright}_\Sigma \equiv \mathring{\triangleright}^{Sr} = \left(\partial^r \alpha^I\right)(e^{-1})_I{}^S.$$

It can be shown that the original metrics are unaffected if the worldvolume derivative is also multiplied by $g$:

$$\begin{aligned}
&\left[\mathring{\triangleright}_M(1), \mathring{\triangleright}_N(2)\right\} = i f_{MN}{}^O \mathring{\triangleright}_O \delta(1-2) + 2i\eta_{MNr}\partial_1^r \delta(1-2) \qquad (4.11)\\
&\Rightarrow \left[\triangleright_A(1), \triangleright_B(2)\right\} = \left[g_A{}^M \mathring{\triangleright}_M(1), g_B{}^N \mathring{\triangleright}_N(2)\right\}\\
&= i g_A{}^M g_B{}^N f_{MN}{}^O \mathring{\triangleright}_O \delta(1-2) + 2i g_A{}^M(1) g_B{}^N(2)\eta_{MNr}\partial_1^r \delta(1-2)\\
&= i g_A{}^M g_B{}^N f_{MN}{}^O (g^{-1})_O{}^C \triangleright_C \delta(1-2) + 2i g_A{}^M(1) g_B{}^N(2)\eta_{MNr}\partial_1^r \delta(1-2)\\
&= i g_A{}^M g_B{}^N f_{MN}{}^O (g^{-1})_O{}^C \triangleright_C \delta(1-2) + i g_{[A|}{}^M \partial^r g_{|B)}{}^N \eta_{MNr}\delta(1-2)\\
&\quad + i\left(g_A{}^M g_B{}^N \eta_{MNr}\right)((1)+(2))\partial_1^r \delta(1-2)\\
&= i g_A{}^M g_B{}^N f_{MN}{}^O (g^{-1})_O{}^C \triangleright_C \delta(1-2) + i g_{[A|}{}^M \partial^r g_{|B)}{}^N \eta_{MNr}\delta(1-2)\\
&\quad + i\left(g_A{}^M g_B{}^N g_a{}^p (g^{-1})_r{}^a \eta_{MNp}\right)((1)+(2))\partial_1^r \delta(1-2)\\
&= i f_{AB}{}^C \triangleright_C \delta(1-2) + i g_{[A|}{}^M \partial^r g_{|B)}{}^N \eta_{MNr}\delta(1-2)\\
&\quad + i\eta_{ABa}(g^{-1})_r{}^a((1)+(2))\partial_1^r \delta(1-2)\\
&= i f_{AB}{}^C \triangleright_C \delta(1-2) + i g_{[A|}{}^M \partial^r g_{|B)}{}^N \eta_{MNr}\delta(1-2)\\
&\quad + i\eta_{ABa}\triangleright^a((1)-(2))\delta(1-2),
\end{aligned}$$

where $f_{AB}{}^C = g_A{}^M g_B{}^N f_{MN}{}^O (g^{-1})_O{}^C$, $\eta_{ABa} = g_A{}^M g_B{}^N g_a{}^r \eta_{MNr}$, and

$$\triangleright^a = (g^{-1})_r{}^a \partial^r$$

Since both $f$'s and $\eta$'s are invariant under $H$-group, $f_{AB}{}^C$ and $\eta_{ABa}$ are numerically equal to $f_{MN}{}^O$ and $\eta_{MNr}$ respectively. The term $g_{[A|}{}^M \partial^r g_{|B)}{}^N \eta_{MNr}\delta(1-$

2) is in fact a torsion term:

$$\begin{aligned}
g_{[A|}{}^{M}\partial^{r}g_{|B)}{}^{N}\eta_{MNr} &= g_{[A|}{}^{M}\partial^{r}g_{|B)}{}^{O}(g^{-1})_{O}{}^{C}g_{C}{}^{N}\eta_{MNr}\\
&= g_{[A|}{}^{M}\left(\partial^{r}\alpha^{I}\right)\left(\frac{\delta}{\delta\alpha^{I}}g_{|B)}{}^{O}\right)(g^{-1})_{O}{}^{C}g_{C}{}^{N}\eta_{MNr}\\
&= g_{[A|}{}^{M}\left(\partial^{r}\alpha^{I}\right)(e^{-1})_{I}{}^{S}\left(G_{S}\right)_{|B)}{}^{C}g_{C}{}^{N}\eta_{MNr}\\
&= -\left(\triangleright^{a}\alpha^{I}\right)(e^{-1})_{I}{}^{S}f_{S[A|}{}^{C}\eta_{C|B)a}\\
&= \triangleright^{Sa}f_{S[A|}{}^{C}\eta_{C|B)a}\\
&= \triangleright_{\Sigma}f_{AB}{}^{\Sigma},
\end{aligned}$$

where

$$\triangleright_{\Sigma} = \triangleright^{Sa} \equiv (g^{-1})_{r}{}^{a}\mathring{\triangleright}^{Sr}.$$

The third equality comes from $(\delta g)g^{-1} = (\delta\alpha)e^{-1}G$. For the fourth equality we use the fact that $(G_{a})_{b}{}^{c} = f_{ab}{}^{c}$ in adjoint representation. The $\left(\triangleright^{a}\alpha^{I}\right)(e^{-1})_{I}{}^{S}$ in the fourth line is the covariant "dual" of $\triangleright_{S}$ ($\triangleright_{\Sigma} = \triangleright^{\widetilde{S}a}$). Using that fact that $\eta_{S\Sigma a} = \eta_{S}{}^{\widetilde{Sb}}{}_{a} = -\frac{1}{2}\delta_{S}^{\widetilde{S}}\delta_{a}^{b}$ (the $-\frac{1}{2}$ comes from the definition of equation (4.11)), we get

$$f_{AB}{}^{\Sigma}\eta_{\Sigma Sa} = \frac{1}{2}f_{S[A|}{}^{C}\eta_{C|B)a}. \tag{4.12}$$

We close this section by calculating the commutation relation between $\triangleright_{S}$

and $\triangleright_\Sigma$:

$$\begin{aligned}
&\left[\triangleright_S(1), \triangleright_\Sigma(2)\right] = \left[\triangleright_{S_1}(1), \triangleright^{S_2 a}(2)\right] \\
&= \left[ie_{S_1}{}^I \frac{\delta}{\delta\alpha^I}(1), (g^{-1})_r{}^a \left(\partial^r\alpha^J\right) e_J{}^{S_2}(2)\right] \\
&= if_{bS_1}{}^a (g^{-1})_r{}^b \left(\partial^r\alpha^I\right) e_I{}^{S_2}\delta(1-2) + i\left(e_{S_1}{}^I \frac{\partial}{\partial\alpha^I}(e^{-1})_J{}^{S_2}\right)(g^{-1})_r{}^a \left(\partial^r\alpha^J\right)\delta(1-2) \\
&\quad + ie_{S_1}{}^I(1)(g^{-1})_r{}^a(2)(e^{-1})_I{}^{S_2}(2)\partial_2^r\delta(1-2) \\
&= if_{bS_1}{}^a (g^{-1})_r{}^b \left(\partial^r\alpha^I\right) e_I{}^{S_2}\delta(1-2) + i\left(e_{S_1}{}^I \frac{\partial}{\partial\alpha^I}(e^{-1})_J{}^{S_2}\right)(g^{-1})_r{}^a \left(\partial^r\alpha^J\right)\delta(1-2) \\
&\quad + i\left(\partial^r e_{S_1}{}^I\right)(g^{-1})_r{}^a(e^{-1})_I{}^{S_2}\delta(1-2) + i\left(e_{S_1}{}^I(g^{-1})_r{}^a(e^{-1})_I{}^{S_2}\right)(2)\partial_2^r\delta(1-2) \\
&= if_{bS_1}{}^a (g^{-1})_r{}^b \left(\partial^r\alpha^I\right) e_I{}^{S_2}\delta(1-2) \\
&\quad - ie_{S_1}{}^I(e^{-1})_J{}^{S_3}\left(\frac{\partial}{\partial\alpha^I}e_{S_3}{}^K\right)(e^{-1})_K{}^{S_2}(g^{-1})_r{}^a\left(\partial^r\alpha^J\right)\delta(1-2) \\
&\quad + ie_{S_3}{}^I(e^{-1})_J{}^{S_3}\left(\frac{\partial}{\partial\alpha^I}e_{S_1}{}^K\right)(e^{-1})_K{}^{S_2}(g^{-1})_r{}^a\left(\partial^r\alpha^J\right)\delta(1-2) + i\delta_{S_1}^{S_2}\triangleright^a(2)\delta(1-2) \\
&= if_{bS_1}{}^a (g^{-1})_r{}^b \left(\partial^r\alpha^I\right) e_I{}^{S_2}\delta(1-2) + if_{S_3S_1}{}^{S_2}(e^{-1})_J{}^{S_3}\left(\partial^r\alpha^J\right)\delta(1-2) \\
&\quad + i\delta_{S_1}^{S_2}\triangleright^a(2)\delta(1-2) \\
&= if_{bS_1}{}^a\triangleright^{S_2 b}\delta(1-2) + if_{S_3S_1}{}^{S_2}\triangleright^{S_3 a}\delta(1-2) + i\delta_{S_1}^{S_2}\triangleright^a(2)\delta(1-2) \\
&= if_{S\Sigma}{}^{\Sigma'}\triangleright_{\Sigma'}\delta(1-2) + i\eta_{S\Sigma a}\triangleright^a(2)\delta(1-2).
\end{aligned}$$

## 4.4 Jacobi Identity

In the last section, we have constructed a very general mechanism to find all the currents. We will now check the Jacobi identity between currents and check if they are consistent with the method described above. Here we mention some results in [37]: The worldvolume currents are $\{\mathring{\triangleright}_D,\ \mathring{\triangleright}_P,\ \mathring{\triangleright}_\Omega\}$ with the nonvanishing commutation relations listed in the following:

$$
\begin{aligned}
&\left\{\overset{\circ}{\rhd}_{D_1}(1),\overset{\circ}{\rhd}_{D_2}(2)\right\}=if_{D_1D_2}{}^{P}\overset{\circ}{\rhd}_{P}\delta(1-2),\\
&\left[\overset{\circ}{\rhd}_{D}(1),\overset{\circ}{\rhd}_{P}(2)\right]=if_{D_1P}{}^{\Omega}\overset{\circ}{\rhd}_{\Omega}\delta(1-2),\\
&\left[\overset{\circ}{\rhd}_{P_1}(1),\overset{\circ}{\rhd}_{P_2}(2)\right]=2i\eta_{P_1P_2r}\partial_1^r\delta(1-2)\\
&\left\{\overset{\circ}{\rhd}_{D}(1),\overset{\circ}{\rhd}_{\Omega}(2)\right\}=2i\eta_{D\Omega r}\partial_2^r\delta(1-2).
\end{aligned}
$$

We now generalize the above commutation relations using the method mentioned in Section 4.3 to include $H$ group; we get:

$$
\begin{aligned}
&\left\{\rhd_{D_1}(1),\rhd_{D_2}(2)\right\}=if_{D_1D_2}{}^{P}\rhd_{P}\delta(1-2),\\
&\left[\rhd_{D}(1),\rhd_{P}(2)\right]=if_{D_1P}{}^{\Omega}\rhd_{\Omega}\delta(1-2),\\
&\left[\rhd_{P_1}(1),\rhd_{P_2}(2)\right]=if_{P_1P_2}{}^{\Sigma}\rhd_{\Sigma}+i\eta_{P_1P_2a}\rhd^{a}((1)-(2))\delta(1-2)\\
&\left\{\rhd_{D}(1),\rhd_{\Omega}(2)\right\}=if_{D\Omega}{}^{\Sigma}\rhd_{\Sigma}+i\eta_{D\Omega a}\rhd^{a}((1)-(2))\delta(1-2),\\
&\left[\rhd_{S}(1),\rhd_{D}(2)\right]=if_{SD}{}^{D'}\rhd_{D'}\delta(1-2),\\
&\left[\rhd_{S}(1),\rhd_{P}(2)\right]=if_{SP}{}^{P'}\rhd_{P'}\delta(1-2),\\
&\left[\rhd_{S}(1),\rhd_{\Omega}(2)\right]=if_{S\Omega}{}^{\Omega'}\rhd_{\Omega'}\delta(1-2),\\
&\left[\rhd_{S}(1),\rhd_{S}(2)\right]=if_{SS}{}^{S'}\rhd_{S'}\delta(1-2),\\
&\left[\rhd_{S}(1),\rhd_{\Sigma}(2)\right]=if_{S\Sigma}{}^{\Sigma'}\rhd_{\Sigma'}\delta(1-2)+2i\eta_{S\Sigma a}\rhd^{a}(2)\delta(1-2).
\end{aligned}
$$

We can find all the relations between $f$'s and $\eta$'s by plugging in the above commutation relations into the Jacobi identities, which are listed in Appendix C.2. Here we point out some of the interesting ones.

The first example is combining equation (C.19) and (C.16),

$$
\left\{\begin{array}{l}
0=f_{SP_1}{}^{P'}\eta_{P'P_2a}+f_{SP_2}{}^{P'}\eta_{P_1P'a}+f_{Sa}{}^{b}\eta_{P_1P_2b}\\
0=f_{P_1S}{}^{P'}\eta_{P'P_2a}+f_{P_1P_2}{}^{\Sigma'}\eta_{S\Sigma'a}-\frac{1}{2}f_{Sa}{}^{b}\eta_{P_1P_2b}
\end{array}\right.
$$

gives

$$
f_{P_1P_2}{}^{\Sigma}\eta_{\Sigma Sa}=\frac{1}{2}f_{S[P_1|}{}^{P'}\eta_{P'|P_2]a},
$$

which is exactly the result in equation (4.12). It is worthwhile to point out that equation (C.17) or (C.21) together with (C.15) gives the same result as equation (4.12).

And another interesting result is equation (C.14),

$$0 = f_{S_1 S_2}{}^{S'} \eta_{S' \Sigma a} + f_{S_1 \Sigma}{}^{\Sigma'} \eta_{S_2 \Sigma' a} + f_{S_1 a}{}^{b} \eta_{S_2 \Sigma b}.$$

If we write the above equation explicit in $\widetilde{S}$ and $b$ (worldvolume) indices, we get:

$$\begin{aligned}
0 &= f_{S_1 S_2}{}^{S'} \eta_{S'}{}^{\widetilde{S} b}{}_{a} + f_{S_1}{}^{\widetilde{S} b}{}_{\widetilde{S}' b'} \eta_{S_2}{}^{\widetilde{S}' b'}{}_{a} + f_{S_1 a}{}^{a'} \eta_{S_2}{}^{\widetilde{S} b}{}_{a'} \\
&= f_{S_1 S_2}{}^{S'} \eta_{S_2}{}^{\widetilde{S} b}{}_{a} + f_{\widetilde{S}' S_1}{}^{\widetilde{S}} \eta_{S_2}{}^{\widetilde{S}' b}{}_{a} + f_{b' S_1}{}^{b} \eta_{S_2}{}^{\widetilde{S} b'}{}_{a} + f_{S_1 a}{}^{a'} \eta_{S_2}{}^{\widetilde{S} b}{}_{a'} \\
&= f_{S_1 S_2}{}^{S'} \eta_{S'}{}^{\widetilde{S}} \delta^b_a + f_{\widetilde{S}' S_1}{}^{\widetilde{S}} \eta_{S_2}{}^{\widetilde{S}'} \delta^b_a + f_{b' S_1}{}^{b} \eta_{S_2}{}^{\widetilde{S}} \delta^{b'}_a + f_{S_1 a}{}^{a'} \eta_{S_2}{}^{\widetilde{S}} \delta^b_{a'} \\
&= f_{S_1 S_2}{}^{S'} \eta_{S'}{}^{\widetilde{S}} \delta^b_a + f_{\widetilde{S}' S_1}{}^{\widetilde{S}} \eta_{S'}{}^{\widetilde{S}'} \delta^b_b \\
\Rightarrow 0 &= f_{S_1 S_2}{}^{S'} \eta_{S'}{}^{\widetilde{S}} + f_{\widetilde{S}' S_1}{}^{\widetilde{S}} \eta_{S_2}{}^{\widetilde{S}'},
\end{aligned}$$

i.e. $\eta_S{}^{\widetilde{S}}$ is $H$-invariant by itself.

## 4.5 Example: 5-brane

**F**-theory on the 5-brane has been investigated quite intensively, e.g. [35–38,63]. We go along with the trend and apply the above method to this case. It is shown in [37] that the bosonic sector lives in $SL(5)/SO(3,2)$. In order to be generalized to supersymmetry, rather than choosing $H = SO(3,2)$, we look for its double covering group $H = Spin(3,2) \cong Sp(4)$ and impose

$$\left\{ \mathring{\triangleright}_{D_1}, \mathring{\triangleright}_{D_2} \right\} = i f_{D_1 D_2}{}^{P} \mathring{\triangleright}_{P}.$$

The only invariant tensors that are symmetric in the two symmetric spinor indices are the Dirac $\gamma$-matrices with two antisymmetric vector indices, i.e. $(\gamma^{mn})_{\alpha\beta}$. Hence,

$$\left\{ \mathring{\triangleright}_{D_1}(1), \mathring{\triangleright}_{D_2}(2) \right\} = \left\{ \mathring{\triangleright}_{\alpha_1}(1), \mathring{\triangleright}_{\alpha_2}(2) \right\} = i \left(\gamma^{mn}\right)_{\alpha_1 \alpha_2} \mathring{\triangleright}_{mn} \delta(1-2),$$

i.e. $\mathring{\triangleright}_{P} = \mathring{\triangleright}_{mn} = -\mathring{\triangleright}_{nm}$. This then leads to

$$\left[ \mathring{\triangleright}_{P_1}(1), \mathring{\triangleright}_{P_2}(2) \right] = \left[ \mathring{\triangleright}_{m_1 n_1}(1), \mathring{\triangleright}_{m_2 n_2}(2) \right] = 2 i \eta_{m_1 n_1 m_2 n_2 r} \partial^r_1 \delta(1-2).$$

Again, $SL(5)$ invariant tensors are proportional to 5 dimensional Levi-Civita tensors or their combinations. Since $\mathring{\triangleright}_P = \frac{1}{2}\mathring{\triangleright}_{[mn]}$ and $\eta_{P_1P_2r}$ should be symmetric in $P_1$ and $P_2$, it can be chosen to be

$$\eta_{P_1P_2r} = \eta_{m_1n_1m_2n_2r} = \epsilon_{m_1n_1m_2n_2r}.$$

By construction,

$$\begin{aligned}\left\{\mathring{\triangleright}_D, \mathring{\triangleright}_\Omega\right\} = \left\{\mathring{\triangleright}_\alpha, \mathring{\triangleright}^{\beta r}\right\} = 2i\delta_\alpha{}^\beta \partial_1^r \delta(1-2)\\ \Rightarrow \eta_{D\Omega q} = \eta_\alpha{}^{\beta r}{}_q = \delta_\alpha{}^\beta \delta_q^r.\end{aligned}$$

As explained in Section 4.3, the above structure constants and metrics are numerically the same as before and after introducing $H$ group element $g$. All we have to put in is $\eta_{S\Sigma a}$, and the rest of the structure constants and the metrics can be found by using the equations in Appendix C.2. By construction, $\triangleright_S$ transforms all the indices the same way as usual $Spin(3,2)$ indices.

We first use equation C.18 to find the only one that doesn't involve $\Sigma$ that's left, $f_{DP}{}^\Omega$:

$$\begin{aligned}0 &= f_{D_1D_2}{}^{P'}\eta_{P'Pa} + f_{D_1P}{}^{\Omega'}\eta_{D_2\Omega' a}\\ &= f_{\alpha_1\alpha_2}{}^{c'd'}\eta_{c'd'cda} + f_{\alpha_1 cd\beta b}\,\eta_{\alpha_2}{}^{\beta b}{}_a\\ \Rightarrow f_{DP}{}^\Omega &= f_{\alpha cd\beta a} = \left(\gamma^{ef}\right)_{\alpha\beta}\epsilon_{efcda}.\end{aligned}$$

We now determine what $\eta_{S\Sigma a}$ is. As explained in Section 4.3, $\eta_{S\Sigma a} = \eta_S{}^{\widetilde{S}b}{}_a = \eta_S{}^{\widetilde{S}}\delta_a^b$. In the 5-brane case, $\triangleright_S$ is $Spin(3,2)$ generators, hence $\mathbf{S}$ is antisymmetric in its two indices. Using this property, we can conclude $\eta_S{}^{\widetilde{S}} = \frac{1}{2}\delta_{cd}^{ef}$. Using equation (4.12) we found the rest of two unsolved structure constants:

$$\begin{aligned}&f_{P_1P_2}{}^\Sigma\eta_{\Sigma Sa} = \frac{1}{2}f_{S[P_1|}{}^{P'}\eta_{P'|P_2]a}\\ \Rightarrow\ &f_{P_1P_2}{}^\Sigma = f_{c_1d_1c_2d_2efa} = -\left(\mathring{\eta}_{[c_1|[e}\epsilon_{f]|d_1]c_2d_2a} - \mathring{\eta}_{[c_2|[e}\epsilon_{f]|d_2]c_1d_1a}\right),\\ &f_{D\Omega}{}^\Sigma\eta_{\Sigma Sa} = \frac{1}{2}f_{SD}{}^{D'}\eta_{D'\Omega a} + \frac{1}{2}f_{S\Omega}{}^{\Omega'}\eta_{\Omega' Da}\\ \Rightarrow\ &f_{D\Omega}{}^\Sigma = f_\alpha{}^{\beta b}{}_{efa} = \frac{1}{2}\left(\gamma_{ef}\right)_\alpha{}^\beta\delta_a^b.\end{aligned}$$

The $\mathring{\eta}$ above is $SO(3,2)$ metric.

The full commutation relations are listed in Appendix C.3.

## 4.6 Conclusion

The method presented in this chapter gives a consistent mathematical structure for higher dimensional brane current algebra (higher than 1) by construction, as opposed to the usual Jacobi identity method used in string theory [57,58]. The main reason is that for 1-brane or string, on the "metric" the additional worldvolume index can take only one value, which is inert under $H$ transformation. For (higher dimensional) brane current algebra, the additional worldvolume indices can have more than one choice. However, in **F**-theory worldvolume indices are also spacetime indices, therefore they all have to react to $H$-group transformations the same way spacetime indices do. This property makes the original construction unsuitable for the higher dimensional brane algebra (mentioned in the beginning of Section 4.3). The method presented in this chapter can be used in any finite dimensional brane. We've worked out the 5-brane case in detail.

The method used in this chapter is not just interesting by itself but also can be utilized for the following subjects:

i) Generalize the method to curved spacetime (**F**-gravity).

ii) Analyze massive modes.

iii) Understand string field theory.

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# Appendices

# Appendix A

# Appendix: $4D$ Minkowski $\mathcal{N}=1$ Superspace generators and covariant derivatives

We derive the covariant derivatives using the method in the Section 2.1 for four dimensional superspace.

In four dimensional Minkowski spacetime, the supersymmetry generators in the full superspace are $\left\{\hat{P}, \hat{Q}, \hat{\bar{Q}}\right\}$, and the only nonvanishing (anti)commutation relations are

$$\left\{\hat{Q}_\alpha, \hat{\bar{Q}}_{\dot{\beta}}\right\} = i\left(\sigma^m\right)_{\alpha\dot{\beta}} \hat{P}_m,$$

where $\sigma$ is a Pauli matrix. Here we choose the basis to be

$$g(x,\theta,\bar{\theta}) = \exp\left(ix^m \hat{P}_m + i\theta^\alpha \hat{Q}_\alpha + i\bar{\theta}^{\dot{\alpha}} \hat{\bar{Q}}_{\dot{\alpha}}\right).$$

Using equation (2.3), we find

$$
\begin{aligned}
& dg(x,\theta,\bar{\theta}) = d\left[\exp\left(ix^m\hat{P}_m + i\theta^\alpha\hat{Q}_\alpha + i\bar{\theta}^{\dot{\alpha}}\hat{\bar{Q}}_{\dot{\alpha}}\right)\right] \\
& = dx^n\frac{\partial}{\partial x_n}\exp\left(ix^m\hat{P}_m\right)\exp\left(i\theta^\alpha\hat{Q}_\alpha + i\bar{\theta}^{\dot{\alpha}}\hat{\bar{Q}}_{\dot{\alpha}}\right) \\
& + d\theta^\beta\frac{\partial}{\partial\theta^\beta}\exp\left(i\theta^\alpha\hat{Q}_\alpha + \frac{1}{2}\theta^\alpha\bar{\theta}^{\dot{\beta}}\left\{\hat{Q}_\alpha,\hat{\bar{Q}}_{\dot{\beta}}\right\}\right)\exp\left(ix^m\hat{P}_m + i\bar{\theta}^{\dot{\alpha}}\hat{\bar{Q}}_{\dot{\alpha}}\right) \\
& + d\bar{\theta}^{\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}}\exp\left(i\bar{\theta}^{\dot{\alpha}}\hat{\bar{Q}}_{\dot{\alpha}} + \frac{1}{2}\bar{\theta}^{\dot{\alpha}}\theta^\beta\left\{\hat{\bar{Q}}_{\dot{\alpha}},\hat{Q}_\beta\right\}\right)\exp\left(ix^m\hat{P}_m + i\theta^\alpha\hat{Q}_\alpha\right) \\
& = idx^n\hat{P}_n\exp\left(ix^m\hat{P}_m\right)\exp\left(i\theta^\alpha\hat{Q}_\alpha + i\bar{\theta}^{\dot{\alpha}}\hat{\bar{Q}}_{\dot{\alpha}}\right) \\
& + id\theta^\beta\left(\hat{Q}_\beta + \frac{1}{2}\bar{\theta}^{\dot{\beta}}\left(\sigma^n\right)_{\beta\dot{\beta}}\hat{P}_n\right)\exp\left(i\theta^\alpha\hat{Q}_\alpha + \frac{i}{2}\theta^\alpha\bar{\theta}^{\dot{\gamma}}\left(\sigma^n\right)_{\alpha\dot{\gamma}}\hat{P}_n\right)\exp\left(ix^m\hat{P}_m + i\bar{\theta}^{\dot{\alpha}}\hat{\bar{Q}}_{\dot{\alpha}}\right) \\
& + id\bar{\theta}^{\dot{\beta}}\left(\hat{\bar{Q}}_{\dot{\beta}} + \frac{1}{2}\theta^\beta\left(\sigma^n\right)_{\beta\dot{\beta}}\hat{P}_n\right)\exp\left(i\bar{\theta}^{\dot{\alpha}}\hat{\bar{Q}}_{\dot{\alpha}} + \frac{i}{2}\bar{\theta}^{\dot{\alpha}}\theta^\gamma\left(\sigma^n\right)_{\gamma\dot{\alpha}}\hat{P}_n\right)\exp\left(ix^m\hat{P}_m + i\theta^\alpha\hat{Q}_\alpha\right) \\
& = i\left[dx^n\hat{P}_n + d\theta^\beta\left(\hat{Q}_\beta + \frac{1}{2}\bar{\theta}^{\dot{\beta}}\left(\sigma^n\right)_{\beta\dot{\beta}}\hat{P}_n\right) + d\bar{\theta}^{\dot{\beta}}\left(\hat{\bar{Q}}_{\dot{\beta}} + \frac{1}{2}\theta^\beta\left(\sigma^n\right)_{\beta\dot{\beta}}\hat{P}_n\right)\right] \\
& \times\exp\left(ix^m\hat{P}_m + i\theta^\alpha\hat{Q}_\alpha + i\bar{\theta}^{\dot{\alpha}}\hat{\bar{Q}}_{\dot{\alpha}}\right).
\end{aligned}
$$

We can read off $\left(e_L^{-1}\right)_i{}^a$ from the above equation and, therefore, find $e_{La}{}^i$:

$$
\left(e_L^{-1}\right)_i{}^a = \begin{array}{c} \\ m \\ \alpha \\ \dot{\alpha} \end{array}
\begin{array}{c} \begin{array}{ccc} n & \beta & \dot{\beta} \end{array} \\
\begin{pmatrix} \delta_m^n & 0 & 0 \\ \frac{1}{2}\bar{\theta}^{\dot{\alpha}}\left(\sigma^n\right)_{\alpha\dot{\alpha}} & \delta_\alpha{}^\beta & 0 \\ \frac{1}{2}\theta^\alpha\left(\sigma^n\right)_{\alpha\dot{\alpha}} & 0 & \delta_{\dot{\alpha}}{}^{\dot{\beta}} \end{pmatrix} \end{array}
$$

$$
\Rightarrow e_{La}{}^i = \begin{array}{c} \\ m \\ \alpha \\ \dot{\alpha} \end{array}
\begin{array}{c} \begin{array}{ccc} n & \beta & \dot{\beta} \end{array} \\
\begin{pmatrix} \delta_m^n & 0 & 0 \\ -\frac{1}{2}\bar{\theta}^{\dot{\alpha}}\left(\sigma^n\right)_{\alpha\dot{\alpha}} & \delta_\alpha{}^\beta & 0 \\ -\frac{1}{2}\theta^\alpha\left(\sigma^n\right)_{\alpha\dot{\alpha}} & 0 & \delta_{\dot{\alpha}}{}^{\dot{\beta}} \end{pmatrix} \end{array}.
$$

The covariant generators are then

$$\begin{cases} \triangleright_P = \triangleright_m = -i\dfrac{\partial}{\partial x^m} \\ \triangleright_Q = \triangleright_\alpha = -i\dfrac{\partial}{\partial \theta^\alpha} + \dfrac{i}{2}\bar{\theta}^{\dot{\alpha}} \left(\sigma^m\right)_{\alpha\dot{\alpha}} \dfrac{\partial}{\partial x^m} \\ \triangleright_{\bar{Q}} = \triangleright_{\dot{\alpha}} = -i\dfrac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + \dfrac{i}{2}\theta^{\alpha} \left(\sigma^m\right)_{\alpha\dot{\alpha}} \dfrac{\partial}{\partial x^m} \end{cases}.$$

Which it is easy to check that they satisfy the same commutation relation as their corresponding $\hat{G}$'s as expected.

We can use the same method above and find ${e_{Ra}}^i$. However, we can use a little trick rather than go through all the painful calculation above. We first note that

$$\begin{cases} \left(\dfrac{\partial}{\partial \alpha^i} g(\alpha)\right) g^{-1}(\alpha) = i\left(e_L^{-1}\right)_i{}^a(\alpha)\hat{G}_a \\ g^{-1}(\alpha)\left(\dfrac{\partial}{\partial \alpha^i} g(\alpha)\right) = i\left(e_R^{-1}\right)_i{}^a(\alpha)\hat{G}_a \end{cases}.$$

After replacing $\alpha$ with $-\alpha$

$$\begin{aligned} i\left(e_L^{-1}\right)_i{}^a(-\alpha)\hat{G}_a &= -\left(\frac{\partial}{\partial \alpha^i} g(-\alpha)\right) g^{-1}(-\alpha) \\ &= -\left(\frac{\partial}{\partial \alpha^i} g^{-1}(\alpha)\right) g(\alpha) \\ &= g^{-1}(\alpha)\left(\frac{\partial}{\partial \alpha^i} g(\alpha)\right) g^{-1}(\alpha) g(\alpha) \\ &= g^{-1}(\alpha)\left(\frac{\partial}{\partial \alpha^i} g(\alpha)\right) \\ &= i\left(e_R^{-1}\right)_i{}^a(\alpha)\hat{G}_a. \end{aligned}$$

In other word, $\left(e_R^{-1}\right)_i{}^a(\alpha) = \left(e_L^{-1}\right)_i{}^a(-\alpha)$ or

$${e_{Ra}}^i(\alpha) = {e_{La}}^i(-\alpha).$$

Making use of the above trick, covariant derivatives are

$$\begin{cases} D_P = D_m = -i\dfrac{\partial}{\partial x^m} \\ D_Q = D_\alpha = -i\dfrac{\partial}{\partial \theta^\alpha} - \dfrac{i}{2}\bar{\theta}^{\dot{\alpha}} \left(\sigma^m\right)_{\alpha\dot{\alpha}} \dfrac{\partial}{\partial x^m} \\ D_{\bar{Q}} = D_{\dot{\alpha}} = -i\dfrac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - \dfrac{i}{2}\theta^{\alpha} \left(\sigma^m\right)_{\alpha\dot{\alpha}} \dfrac{\partial}{\partial x^m} \end{cases} .$$

One thing to notice is that in superspace, since some generators don't commute with each other ($\hat{Q}$, to be exact), the "difference" between two coordinates is not just the subtraction of the two. We have to use the definition in section 2.1 (equation (2.4)) to find its form for the chosen basis.

$$\begin{aligned} &\exp\left(ix_{12}^m\hat{P}_m + i\theta_{12}^\alpha\hat{Q}_\alpha + i\bar{\theta}_{12}^{\dot{\alpha}}\hat{\bar{Q}}_{\dot{\alpha}}\right) \\ &= g(x_{12}, \theta_{12}, \bar{\theta}_{12}) \\ &= g^{-1}(x_2, \theta_2, \bar{\theta}_2) g(x_1, \theta_1, \bar{\theta}_1) \\ &= \exp\left(-ix_2^m\hat{P}_m - i\theta_2^\alpha\hat{Q}_\alpha - i\bar{\theta}_2^{\dot{\alpha}}\hat{\bar{Q}}_{\dot{\alpha}}\right) \exp\left(ix_1^n\hat{P}_n + i\theta_1^\beta\hat{Q}_\beta + i\bar{\theta}_1^{\dot{\beta}}\hat{\bar{Q}}_{\dot{\beta}}\right) \\ &= \exp\left[i(x_1^m - x_2^m)\hat{P}_m + i(\theta_1^\alpha - \theta_2^\alpha)\hat{Q}_\alpha + i(\bar{\theta}_1^{\dot{\alpha}} - \bar{\theta}_2^{\dot{\alpha}})\hat{\bar{Q}}_{\dot{\alpha}} - \frac{1}{2}\theta_2^\alpha\bar{\theta}_1^{\dot{\alpha}}\left\{\hat{Q}_\alpha, \hat{\bar{Q}}_{\dot{\alpha}}\right\} - \frac{1}{2}\bar{\theta}_2^{\dot{\alpha}}\theta_1^\alpha\left\{\hat{Q}_\alpha, \hat{\bar{Q}}_{\dot{\alpha}}\right\}\right] \\ &= \exp\left[i(x_1^m - x_2^m - \frac{1}{2}\theta_2^\alpha\bar{\theta}_1^{\dot{\alpha}}\left(\gamma^m\right)_{\alpha\dot{\alpha}} - \frac{1}{2}\bar{\theta}_2^{\dot{\alpha}}\theta_1^\alpha\left(\gamma^m\right)_{\alpha\dot{\alpha}})\hat{P} + i(\theta_1^\alpha - \theta_2^\alpha)\hat{Q}_\alpha + i(\bar{\theta}_1^{\dot{\alpha}} - \bar{\theta}_2)^{\dot{\alpha}}\hat{\bar{Q}}_{\dot{\alpha}}\right]. \end{aligned}$$

Hence

$$\begin{cases} x_{12}^m = x_1^m - x_2^m - \frac{1}{2}\theta_2^\alpha\bar{\theta}_1^{\dot{\alpha}}\left(\gamma^m\right)_{\alpha\dot{\alpha}} - \frac{1}{2}\bar{\theta}_2^{\dot{\alpha}}\theta_1^\alpha\left(\gamma^m\right)_{\alpha\dot{\alpha}} \\ \theta_{12}^\alpha = \theta_1^\alpha - \theta_2^\alpha \\ \bar{\theta}_{12}^{\dot{\alpha}} = \bar{\theta}_1^{\dot{\alpha}} - \bar{\theta}_2 \end{cases} .$$

# Appendix B

# Covariant Semi-Shortening

## B.1 Superconformal Algebra

In this appendix, we first show the superconformal algebra. We will later modify the algebra to give commutation relations that satisfy

$$\left[\hat{G}_{\mathcal{M}}{}^{\mathcal{N}}, \hat{G}_{\mathcal{P}}{}^{\mathcal{Q}}\right\} = i\delta^{\mathcal{N}}_{\mathcal{P}}\hat{G}_{\mathcal{M}}{}^{\mathcal{Q}} - i\,(-1)^{(\mathcal{M}+\mathcal{N})(\mathcal{P}+\mathcal{Q})}\,\delta^{\mathcal{Q}}_{\mathcal{M}}\hat{G}_{\mathcal{P}}{}^{\mathcal{N}}.$$

### B.1.1 Superconformal Algebra

The usual superconformal commutation relations between $\hat{G}$'s are:

- $\left[\hat{K}_{\alpha}{}^{\dot{\alpha}}, \hat{M}_{\beta}{}^{\gamma}\right] = -i\delta_{\alpha}{}^{\gamma}\hat{K}_{\beta}{}^{\dot{\alpha}} + \frac{i}{2}\delta_{\beta}{}^{\gamma}\hat{K}_{\alpha}{}^{\dot{\alpha}}$
- $\left[\hat{K}_{\alpha}{}^{\dot{\alpha}}, \hat{\bar{M}}_{\dot{\beta}}{}^{\dot{\gamma}}\right] = i\delta_{\dot{\beta}}{}^{\dot{\alpha}}\hat{K}_{\alpha}{}^{\dot{\gamma}} - \frac{i}{2}\delta_{\dot{\beta}}{}^{\dot{\gamma}}\hat{K}_{\alpha}{}^{\dot{\alpha}}$
- $\left[\hat{K}_{\alpha}{}^{\dot{\alpha}}, \hat{D}\right] = -\hat{K}_{\alpha}{}^{\dot{\alpha}}$
- $\left[\hat{K}_{\alpha}{}^{\dot{\alpha}}, \hat{Q}_{i}{}^{\beta}\right] = -i\delta_{\alpha}{}^{\beta}\hat{\bar{S}}_{i}{}^{\dot{\alpha}}$
- $\left[\hat{K}_{\alpha}{}^{\dot{\alpha}}, \hat{\bar{Q}}_{\dot{\beta}}{}^{i}\right] = i\delta_{\dot{\beta}}{}^{\dot{\alpha}}\hat{S}_{i}{}^{\dot{\alpha}}$
- $\left[\hat{K}_{\alpha}{}^{\dot{\alpha}}, \hat{P}_{\dot{\beta}}{}^{\beta}\right] = i\left(-\delta_{\alpha}{}^{\beta}\hat{M}_{\dot{\beta}}{}^{\dot{\alpha}} + \delta_{\dot{\beta}}{}^{\dot{\alpha}}\hat{M}_{\alpha}{}^{\beta} - i\delta_{\alpha}{}^{\beta}\delta_{\dot{\beta}}{}^{\dot{\alpha}}\hat{D}\right)$
- $\left\{\hat{S}_{\alpha}{}^{i}, \hat{\bar{S}}_{j}{}^{\dot{\alpha}}\right\} = i\delta^{i}_{j}\hat{K}_{\alpha}{}^{\dot{\alpha}}$

- $\left[\hat{S}_\alpha{}^i, \hat{M}_\beta{}^\gamma\right] = -i\delta_\alpha{}^\gamma \hat{S}_\beta{}^i + \frac{i}{2}\delta_\beta{}^\gamma \hat{S}_\alpha{}^i$
- $\left[\hat{S}_\alpha{}^i, \hat{D}\right] = -\frac{1}{2}\hat{S}_\alpha{}^i$
- $\left[\hat{S}_\alpha{}^i, \hat{R}_j{}^k\right] = i\delta_j^i \hat{S}_\alpha{}^k - \frac{i}{4}\delta_j^k \hat{S}_\alpha{}^i$
- $\left\{\hat{S}_\alpha{}^i, \hat{Q}_j{}^\beta\right\} = i\delta_j^j \left(\hat{M}_\alpha{}^\beta - \frac{i}{2}\delta_\alpha{}^\beta \hat{D}\right) - i\delta_\alpha{}^\beta \hat{R}_j{}^i$
- $\left[\hat{S}_\alpha{}^i, \hat{P}_{\dot{\beta}}{}^\beta\right] = -i\delta_\alpha{}^\beta \hat{\bar{Q}}_{\dot{\beta}}{}^i$
- $\left[\hat{\bar{S}}_i{}^{\dot{\alpha}}, \hat{D}\right] = -\frac{1}{2}\hat{\bar{S}}_i{}^{\dot{\alpha}}$
- $\left[\hat{\bar{S}}_i{}^{\dot{\alpha}}, \hat{R}_j{}^k\right] = i\delta_j^i \hat{\bar{S}}_i{}^{\dot{\alpha}} - \frac{i}{4}\delta_j^k \hat{\bar{S}}_i{}^{\dot{\alpha}}$
- $\left\{\hat{\bar{S}}_i{}^{\dot{\alpha}}, \hat{\bar{Q}}_{\dot{\beta}}{}^j\right\} = -i\delta_i^j \left(\hat{M}_{\dot{\beta}}{}^{\dot{\alpha}} + \frac{i}{2}\delta_{\dot{\beta}}{}^{\dot{\alpha}} \hat{D}\right) - i\delta_{\dot{\beta}}{}^{\dot{\alpha}} \hat{R}_i{}^j$
- $\left[\hat{\bar{S}}_i{}^{\dot{\alpha}}, \hat{P}_{\dot{\beta}}{}^\beta\right] = i\delta_{\dot{\beta}}{}^{\dot{\alpha}} \hat{Q}_i{}^\beta$
- $\left[\hat{M}_\alpha{}^\beta, \hat{M}_\gamma{}^\delta\right] = i\delta_\gamma{}^\beta \hat{M}_\alpha{}^\delta - i\delta_\alpha{}^\delta \hat{M}_\gamma{}^\beta$
- $\left[\hat{M}_\alpha{}^\beta, \hat{Q}_i{}^\gamma\right] = -i\delta_\alpha{}^\gamma \hat{Q}_i{}^\beta + \frac{i}{2}\delta_\alpha{}^\beta \hat{Q}_i{}^\gamma$
- $\left[\hat{M}_\alpha{}^\beta, \hat{P}_{\dot{\gamma}}{}^\delta\right] = -i\delta_\alpha{}^\delta \hat{P}_{\dot{\gamma}}{}^\beta + \frac{i}{2}\delta_\alpha{}^\beta \hat{P}_{\dot{\gamma}}{}^\delta$
- $\left[\hat{\bar{M}}_{\dot{\alpha}}{}^{\dot{\beta}}, \hat{\bar{M}}_{\dot{\gamma}}{}^{\dot{\delta}}\right] = i\delta_{\dot{\gamma}}{}^{\dot{\beta}} \hat{\bar{M}}_{\dot{\alpha}}{}^{\dot{\delta}} - i\delta_{\dot{\alpha}}{}^{\dot{\delta}} \hat{\bar{M}}_{\dot{\gamma}}{}^{\dot{\beta}}$
- $\left[\hat{\bar{M}}_{\dot{\alpha}}{}^{\dot{\beta}}, \hat{\bar{Q}}_{\dot{\gamma}}{}^i\right] = i\delta_{\dot{\gamma}}{}^{\dot{\beta}} \hat{\bar{Q}}_{\dot{\alpha}}{}^i - \frac{i}{2}\delta_{\dot{\alpha}}{}^{\dot{\beta}} \hat{\bar{Q}}_{\dot{\gamma}}{}^i$
- $\left[\hat{\bar{M}}_{\dot{\alpha}}{}^{\dot{\beta}}, \hat{P}_{\dot{\gamma}}{}^\delta\right] = i\delta_{\dot{\gamma}}{}^{\dot{\beta}} \hat{P}_{\dot{\alpha}}{}^\delta - \frac{i}{2}\delta_{\dot{\alpha}}{}^{\dot{\beta}} \hat{P}_{\dot{\gamma}}{}^\delta$
- $\left[\hat{D}, \hat{Q}_i{}^\alpha\right] = -\frac{1}{2}\hat{Q}_i{}^\alpha$

- $\left[\hat{D}, \hat{\bar{Q}}_{\dot{\alpha}}{}^{i}\right] = -\frac{1}{2}\hat{\bar{Q}}_{\dot{\alpha}}{}^{i}$
- $\left[\hat{D}, \hat{P}_{\dot{\alpha}}{}^{\beta}\right] = -\hat{P}_{\dot{\alpha}}{}^{\beta}$
- $\left[\hat{R}_{i}{}^{j}, \hat{R}_{k}{}^{l}\right] = i\delta_{k}^{j}\hat{R}_{i}{}^{l} - i\delta_{i}^{l}\hat{R}_{k}{}^{j}$
- $\left[\hat{R}_{i}{}^{j}, \hat{Q}_{k}{}^{\alpha}\right] = i\delta_{k}^{j}\hat{Q}_{i}{}^{\alpha} - i\frac{1}{4}\delta_{i}^{j}\hat{Q}_{k}{}^{\alpha}$
- $\left[\hat{R}_{i}{}^{j}, \hat{\bar{Q}}_{\dot{\alpha}}{}^{k}\right] = -i\delta_{i}^{k}\hat{\bar{Q}}_{\dot{\alpha}}{}^{j} + i\frac{1}{4}\delta_{i}^{j}\hat{\bar{Q}}_{\dot{\alpha}}{}^{k}$
- $\left\{\hat{Q}_{i}{}^{\alpha}, \hat{\bar{Q}}_{\dot{\alpha}}{}^{j}\right\} = i\delta_{i}^{j}\hat{P}_{\dot{\alpha}}{}^{\alpha}$

### B.1.2 Modified Superconformal Algebra

To simplify the calculation, rather than the "normal" commutation relations above, we modify the algebra. We redefine algebra in the following table:

$$
\begin{aligned}
&\hat{k}_{\alpha}{}^{\dot{\alpha}} \equiv \hat{K}_{\alpha}{}^{\dot{\alpha}}, \quad \hat{s}_{\alpha}{}^{i} \equiv \hat{S}_{\alpha}{}^{i}, \quad \hat{\bar{s}}_{i}{}^{\dot{\alpha}} \equiv \hat{\bar{S}}_{i}{}^{\dot{\alpha}}, \\
&\hat{p}_{\dot{\alpha}}{}^{\alpha} \equiv \hat{P}_{\dot{\alpha}}{}^{\alpha}, \quad \hat{q}_{i}{}^{\alpha} \equiv \hat{Q}_{i}{}^{\alpha}, \quad \hat{\bar{q}}_{\dot{\alpha}}{}^{j} \equiv \hat{\bar{Q}}_{\dot{\alpha}}{}^{j}, \\
&\hat{r}_{i}{}^{j} \equiv \hat{R}_{i}{}^{j} + \frac{1}{4-\mathcal{N}}\delta_{i}^{j}\hat{R}, \\
&\hat{m}_{\alpha}{}^{\beta} \equiv \hat{M}_{\alpha}{}^{\beta} - \frac{i}{2}\delta_{\alpha}{}^{\beta}\hat{D} + \frac{1}{4}\delta_{\alpha}{}^{\beta}\hat{R} \\
&\hat{\bar{m}}_{\dot{\alpha}}{}^{\dot{\beta}} \equiv \hat{M}_{\dot{\beta}}{}^{\dot{\alpha}} + \frac{i}{2}\delta_{\dot{\beta}}{}^{\dot{\alpha}}\hat{D} - \frac{1}{4}\delta_{\alpha}{}^{\beta}\hat{R}.
\end{aligned}
$$

Table B.1: Modified superconformal algebra.

For covariant derivatives, we use capitalized Latin letters to denote the "unmodified" ones, lower case Latin letters correspond to the modified ones in this appendix.

## B.2 Appendix: Constraints from $p^2 = 0$

To compare with other literatures ( [43] for example). Rather than using the modified algebra that we use in chapter 4, we switch back to the "usual" covariant derivatives (capitalized, corresponding to the symmetry generators in appendix B.1.1).

$\{\ 0\ \}$ $P^2 = 0$

$\{\ 1\ \}$ $P^{\dot\alpha\alpha}\bar{Q}_{i\dot\alpha} = 0$

$\{\ 2\ \}$ $P^{\dot\alpha\alpha}Q^i{}_\alpha = 0$

$\{\ 3\ \}$ $\bar{Q}_i{}^{\dot\alpha}\bar{Q}_{j\dot\alpha} = 0$

$\{\ 4\ \}$ $Q^{j\alpha}\bar{Q}_i{}^{\dot\alpha} + 2\delta^j{}_i P^{\dot\gamma\alpha}M^{\dot\alpha}{}_{\dot\gamma} + i\delta^j{}_i P^{\dot\alpha\alpha}D - 2P^{\dot\alpha\alpha}R^j{}_i = 0$

$\{\ 5\ \}$ $-\bar{Q}_j{}^{\dot\alpha}Q^{i\alpha} + 2\delta^i{}_j P^{\dot\alpha\gamma}M_\gamma{}^\alpha - i\delta^i{}_j P^{\dot\alpha\alpha}D - 2P^{\dot\alpha\alpha}R^i{}_j = 0$

$\{\ 6\ \}$ $Q^{i\alpha}Q^j{}_\alpha = 0$

$\{\ 7\ \}$ $0 = 0$ (no new constraint)

$\{\ 8\ \}$ $\left\{\bar{Q}_i{}^{\dot\alpha}\left[\delta^k{}_j\left(M^{\dot\beta}{}_{\dot\gamma} + \frac{i}{2}\delta^{\dot\beta}{}_{\dot\gamma}(D-2i)\right) - \delta^{\dot\beta}{}_{\dot\gamma}R^k{}_j\right] + (i \leftrightarrow j)\right\} = 0$

$\{\ 9\ \}$ $$\left\{\bar{Q}_i{}^{\dot\alpha}\epsilon^{\alpha\gamma}\left[\delta^j{}_k\left(M_\gamma{}^\beta - \frac{i}{2}\delta_\gamma{}^\beta D\right) - \delta^{\dot\beta}{}_{\dot\gamma}R^j{}_k\right]\right.$$
$$\left. + \bar{Q}_k{}^{\dot\gamma}\epsilon^{\alpha\beta}\left[\delta^j{}_i\left(M^{\dot\alpha}{}_{\dot\gamma} + \frac{i}{2}\delta^{\dot\alpha}{}_{\dot\gamma}(D-2i)\right) - \delta^{\dot\alpha}{}_{\dot\gamma}R^j{}_i\right]\right\} = 0$$

$\{\ 10\ \}$ $Q^{j\alpha}\epsilon^{\dot\alpha\dot\gamma}\left[\delta^k{}_i\left(M^{\dot\beta}{}_{\dot\gamma} + \frac{i}{2}\delta^{\dot\beta}{}_{\dot\gamma}D\right) - \delta^{\dot\beta}{}_{\dot\gamma}R^k{}_i\right] - (j, \dot\alpha \leftrightarrow k, \dot\beta) = 0$

$\{\ 11\ \}$ $\left\{\bar{Q}_j{}^{\dot\alpha}\epsilon^{\alpha\gamma}\left[\delta^i{}_k\left(M_\gamma{}^\beta - \frac{i}{2}\delta_\gamma{}^\beta D\right) - \delta^{\dot\beta}{}_{\dot\gamma}R^i{}_k\right] - (j, \alpha \leftrightarrow k, \beta) = 0\right.$

$\{\ 12\ \}$ $$Q^{i\alpha}\epsilon^{\dot\alpha\dot\gamma}\left[\delta^k{}_j\left(M^{\dot\beta}{}_{\dot\gamma} + \frac{i}{2}\delta^{\dot\beta}{}_{\dot\gamma}D\right) - \delta^{\dot\beta}{}_{\dot\gamma}R^k{}_j\right]$$
$$+ Q^{k\gamma}\epsilon^{\dot\alpha\dot\beta}\left[\delta^i{}_j\left(M_\gamma{}^\alpha - \frac{i}{2}\delta_\gamma{}^\alpha(D-2i)\right) - \delta^\alpha{}_\gamma R^i{}_j\right] = 0$$

$\{\ 13\ \}$ $Q^{k\gamma}\left[\delta^i{}_j\left(M_\gamma{}^\alpha - \frac{i}{2}\delta_\gamma{}^\alpha(D-2i)\right) - \delta^\alpha{}_\gamma R^i{}_j\right] + (k \leftrightarrow i) = 0$

$\{\ 14\ \}$ $0 = 0$ (no new constraint)

{ 15 } $0 = 0$ (no new constraint)

{ 16 } $\epsilon^{\dot\alpha\dot\gamma}\left[\delta^{\ell}{}_{i}\left(M^{\dot\rho}{}_{\dot\alpha}+\frac{i}{2}\delta^{\dot\rho}{}_{\dot\alpha}D\right)-\delta^{\dot\rho}{}_{\dot\alpha}R^{\ell}{}_{i}\right]\left[\delta^{k}{}_{j}\left(M^{\dot\beta}{}_{\dot\gamma}+\frac{i}{2}\delta^{\dot\beta}{}_{\dot\gamma}(D-2i)\right)-\delta^{\dot\beta}{}_{\dot\gamma}R^{k}{}_{j}\right]$
$+(i\leftrightarrow j))=0$

{ 17 } $0 = 0$ (no new constraint)

{ 18 } $\left[\delta^{\ell}{}_{i}\left(M^{\dot\rho\dot\alpha}+\frac{i}{2}\epsilon^{\dot\rho\dot\alpha}D\right)-\epsilon^{\dot\rho\dot\alpha}R^{\ell}{}_{i}\right]\left[\delta^{k}{}_{j}\left(M^{\alpha\beta}-\frac{i}{2}\epsilon^{\alpha\beta}D\right)-\epsilon^{\alpha\beta}R^{k}l\ {}_{j}\right]$
$+\epsilon^{\alpha\beta}\left[\delta^{\ell}{}_{k}\left(M^{\dot\rho\dot\gamma}+\frac{i}{2}\epsilon^{\dot\gamma\dot\rho}D\right)-\epsilon^{\dot\gamma\dot\rho}R^{\ell}{}_{k}\right]\left[\delta^{j}{}_{i}\left(M^{\dot\alpha}{}_{\dot\gamma}+\frac{i}{2}\delta^{\dot\alpha}{}_{\dot\gamma}(D-2i)\right)-\delta^{\dot\alpha}{}_{\dot\gamma}R^{j}{}_{i}\right]$
$=0$

{ 19 } $0 = 0$ (no new constraint)

{ 20 } $\left[\delta^{\ell}{}_{i}\left(M^{\dot\rho\dot\alpha}+\frac{i}{2}\epsilon^{\dot\rho\dot\alpha}D\right)-\epsilon^{\dot\rho\dot\alpha}R^{\ell}{}_{i}\right]\left[\delta^{k}{}_{j}\left(M^{\alpha\beta}-\frac{i}{2}\epsilon^{\alpha\beta}D\right)-\epsilon^{\alpha\beta}R^{k}{}_{j}\right]$
$-(\ell\dot\alpha\leftrightarrow k,\dot\rho)=0$

{ 21 } $\left[\delta^{\ell}{}_{i}\left(M^{\dot\rho\dot\alpha}+\frac{i}{2}\epsilon^{\dot\rho\dot\alpha}D\right)-\epsilon^{\dot\rho\dot\alpha}R^{\ell}{}_{i}\right]\left[\delta^{k}{}_{j}\left(M^{\alpha\beta}-\frac{i}{2}\epsilon^{\alpha\beta}D\right)-\epsilon^{\alpha\beta}R^{k}{}_{j}\right]$
$-(i,\alpha\leftrightarrow j,\beta)=0$

{ 22 } $0 = 0$ (no new constraint)

{ 23 } $\left[\delta^{i}{}_{\ell}\left(M^{\alpha\rho}-\frac{i}{2}\epsilon^{\alpha\rho}D\right)-\epsilon^{\alpha\rho}R^{i}{}_{\ell}\right]\left[\delta^{k}{}_{j}\left(M^{\dot\beta\dot\alpha}+\frac{i}{2}\epsilon^{\dot\alpha\dot\beta}D\right)-\epsilon^{\dot\alpha\dot\beta}R^{k}{}_{j}\right]$
$+\epsilon^{\dot\alpha\dot\beta}\left[\delta^{k}{}_{\ell}\left(M^{\gamma\rho}+\frac{i}{2}\epsilon^{\gamma\rho}D\right)-\epsilon^{\gamma\rho}R^{k}{}_{\ell}\right]\left[\delta^{i}{}_{j}\left(M_{\gamma}{}^{\alpha}-\frac{i}{2}\delta_{\gamma}{}^{\alpha}(D-2i)\right)-\delta_{\gamma}{}^{\alpha}R^{i}{}_{j}\right]$
$=0$

{ 24 } $\left[\delta^{i}{}_{\ell}\left(M_{\alpha}{}^{\rho}-\frac{i}{2}\delta_{\alpha}{}^{\rho}D\right)-\delta_{\alpha}{}^{\rho}R^{i}{}_{\ell}\right]\left[\delta^{j}{}_{k}\left(M^{\alpha\beta}+\frac{i}{2}\epsilon^{\alpha\beta}(D-2i)\right)-\epsilon^{\alpha\beta}R^{j}{}_{k}\right]=0$

{ 25 } $0 = 0$ (no new constraint)

## B.3 Appendix: Closure of shortening

In this appendix, we will use equation (3.4) to prove the closure of shortening conditions induced by setting $g_i{}^{\alpha}=0$. We let $i$ and $\alpha$ be a fixed value, and the remaining indices are arbitrary. The nontrivial (nonvanishing) commutation relations are:

1. $0=\{g_i{}^{\alpha},g_{\beta}{}^{j}\}=\delta^{\alpha}_{\beta}g_i{}^{j}+\delta^{j}_{i}g_{\beta}{}^{\alpha}$. This implies $g_i{}^{j}$ and $g_{\beta}{}^{\alpha}$ should also vanish.

2. Since $g_i{}^j = 0$, then we have $0 = [g_i{}^j, g_i{}^\alpha] = \delta_i^j g_i{}^\alpha$. $g_i{}^\alpha = 0$ is our starting point, therefore, this gives no new constraint.

3. $0 = [g_\beta{}^k, g_i{}^j] = \delta_i^k g_\beta{}^j$. This doesn't give a new condition.

4. $0 = [g_i{}^\alpha, g_\beta{}^\alpha] = \delta^\alpha_\beta g_i{}^\alpha$. This is again the starting point.

5. $0 = [g_\beta{}^\alpha, g_\gamma{}^j] = \delta^\alpha_\gamma g_\beta{}^j$. No new condition.

Therefore, the superfield vanishes under $g_i{}^\alpha$, $g_i{}^{\dot\beta}$, $g_\beta{}^j$, $g_\beta{}^{\dot\beta}$, $g_i{}^j$, and $g_\beta{}^\alpha$. It won't imply the vanishing of any other $g_i{}^\beta$ or $g_{\dot\alpha}{}^i$. The algebra of shortening condition $g_{\dot\alpha}{}^i = 0$ is the complex conjugate of the above ones.

The consequence of this can be easily realized diagrammatically. We first write down the generator matrix with superconformal and special conformal generators vanishing:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \alpha & 1 & 2 & \cdots & \mathcal{N} & \dot\alpha \end{array} \\
\begin{array}{c} \beta \\ 1 \\ 2 \\ \vdots \\ \mathcal{N} \\ \dot\beta \end{array} &
\begin{pmatrix}
g_\beta{}^\alpha & 0 & 0 & 0 & 0 & 0 \\
g_1{}^\alpha & g_1{}^1 & g_1{}^2 & \cdots & g_1{}^{\mathcal{N}} & 0 \\
g_2{}^\alpha & g_2{}^1 & g_2{}^2 & \cdots & g_2{}^{\mathcal{N}} & 0 \\
\vdots & & & \ddots & & \vdots \\
g_{\mathcal{N}}^\alpha & g_{\mathcal{N}}^1 & g_{\mathcal{N}}^2 & \cdots & g_1{}^{\mathcal{N}} & 0 \\
g_{\dot\beta}{}^\alpha & g_{\dot\beta}{}^1 & g_{\dot\beta}{}^2 & \cdots & g_{\dot\beta}{}^{\mathcal{N}} & g_{\dot\beta}{}^{\dot\alpha}
\end{pmatrix}
\end{array}.
$$

If we choose $g_i{}^\alpha = 0$, then the whole row with such an element should completely vanish (also $g_\beta{}^\alpha$):

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \alpha & 1 & 2 & \cdots & \mathcal{N} & \dot\alpha \end{array} \\
\begin{array}{c} \beta \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ \mathcal{N} \\ \dot\beta \end{array} &
\begin{pmatrix}
\boxed{0} & 0 & 0 & 0 & 0 & 0 \\
g_1{}^\alpha & g_1{}^1 & g_1{}^2 & \cdots & g_1{}^{\mathcal{N}} & 0 \\
g_2{}^\alpha & g_2{}^1 & g_2{}^2 & \cdots & g_2{}^{\mathcal{N}} & 0 \\
\vdots & & & \ddots & & \vdots \\
\boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & 0 \\
\vdots & & & \ddots & & \vdots \\
g_{\mathcal{N}}^\alpha & g_{\mathcal{N}}^1 & g_{\mathcal{N}}^2 & \cdots & g_1{}^{\mathcal{N}} & 0 \\
g_{\dot\beta}{}^\alpha & g_{\dot\beta}{}^1 & g_{\dot\beta}{}^2 & \cdots & g_{\dot\beta}{}^{\mathcal{N}} & g_{\dot\beta}{}^{\dot\alpha}
\end{pmatrix}
\end{array}.
$$

For $g_{\dot{\alpha}}{}^{i} = 0$, instead of row, it is the column with $g_{\dot{\alpha}}{}^{i}$ that vanishes (together with $g_{\dot{\alpha}}{}^{\dot{\beta}}$).

## B.4 Appendix: Proof of equation (3.13)

In this section, we are going to prove the identity:

$$[g^n, \mathcal{O}\} = \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \tilde{g}^{n-i} \mathrm{ad}_{\tilde{g}}{}^{i} \mathcal{O}, \tag{B.1}$$

where

$$\begin{cases} \binom{m}{n} \equiv \dfrac{(m+n)!}{m!n!} \\ \mathrm{ad}_x y \equiv [x, y] \\ \tilde{g}^{n-i} \mathrm{ad}_{\tilde{g}}{}^{i} = g_{(\mathcal{A}_1}{}^{(\mathcal{B}_1} \cdots g_{\mathcal{A}_{n-i}}{}^{\mathcal{B}_{n-i}} \left[ g_{\mathcal{A}_{n-i+1}}{}^{\mathcal{B}_{n-i+1}} \cdots \left[ g_{\mathcal{A}_{n-1}}{}^{\mathcal{B}_{n-1}}, \left[ g_{\mathcal{A}_n]}{}^{\mathcal{B}_n]} \mathcal{O} \right\} \right\} \right\} \end{cases}$$

This identity can be proven by using mathematical induction. Before starting this, it is useful to derive the equation:

$$\begin{aligned} \mathrm{ad}_{g^n} \mathcal{O} = {} & \left[ g_{(\mathcal{A}_1}{}^{(\mathcal{B}_1} \cdots g_{\mathcal{A}_n]}{}^{\mathcal{B}_n]}, \mathcal{O} \right\} \\ = {} & g_{(\mathcal{A}_1}{}^{(\mathcal{B}_1} \left[ g_{\mathcal{A}_2}{}^{\mathcal{B}_2} \cdots g_{\mathcal{A}_n]}{}^{\mathcal{B}_n]}, \mathcal{O} \right\} \\ & + (-1)^{\kappa(\sum_{i=2}^{n} (\mathcal{A}_i + \mathcal{B}_i)) \kappa(\mathcal{O})} \left[ g_{(\mathcal{A}_1}{}^{(\mathcal{B}_1}, \mathcal{O} \right\} g_{\mathcal{A}_2}{}^{\mathcal{B}_2} \cdots g_{\mathcal{A}_n]}{}^{\mathcal{B}_n]} \\ = {} & g_{(\mathcal{A}_1}{}^{(\mathcal{B}_1} \left[ g_{\mathcal{A}_2}{}^{\mathcal{B}_2} \cdots g_{\mathcal{A}_n]}{}^{\mathcal{B}_n]}, \mathcal{O} \right\} \\ & + (-1)^{\kappa(\sum_{i=2}^{n} (\mathcal{A}_i + \mathcal{B}_i)) \kappa(\mathcal{A}_1 + \mathcal{B}_1)} g_{(\mathcal{A}_2}{}^{(\mathcal{B}_2} \cdots g_{\mathcal{A}_n}{}^{\mathcal{B}_n} \left[ g_{(\mathcal{A}_1]}{}^{(\mathcal{B}_1]}, \mathcal{O} \right\} \\ & + (-1)^{\kappa(\sum_{i=2}^{n} (\mathcal{A}_i + \mathcal{B}_i)) \kappa(\mathcal{O})} \left[ \left[ g_{(\mathcal{A}_1}{}^{(\mathcal{B}_1}, \mathcal{O} \right\}, g_{\mathcal{A}_2}{}^{\mathcal{B}_2} \cdots g_{\mathcal{A}_n]}{}^{\mathcal{B}_n]} \right\} \\ = {} & g_{(\mathcal{A}_1}{}^{(\mathcal{B}_1} \left[ g_{\mathcal{A}_2}{}^{\mathcal{B}_2} \cdots g_{\mathcal{A}_n]}{}^{\mathcal{B}_n]}, \mathcal{O} \right\} + g_{(\mathcal{A}_1}{}^{(\mathcal{B}_1} \cdots g_{\mathcal{A}_{n-1}}{}^{\mathcal{B}_{n-1}} \left[ g_{(\mathcal{A}_n]}{}^{(\mathcal{B}_n]}, \mathcal{O} \right\} \\ & - \left[ {}_{(\mathcal{A}_1}{}^{(\mathcal{B}_1} \cdots g_{\mathcal{A}_{n-1}}{}^{\mathcal{B}_{n-1}}, \left[ g_{\mathcal{A}_n]}{}^{\mathcal{B}_n]}, \mathcal{O} \right\} \right\} \\ = {} & \tilde{g}\ \mathrm{ad}_{\tilde{g}^{n-1}} \mathcal{O} + \tilde{g}^{n-1} \mathrm{ad}_{\tilde{g}} \mathcal{O} - \mathrm{ad}_{\tilde{g}^{n-1}} \left( \mathrm{ad}_{\tilde{g}} \mathcal{O} \right). \end{aligned} \tag{B.2}$$

Now we start the proof:

- For $n = 2$, equation (B.1) is obviously true since it is nothing but equation (3.9). (This can also be seen by taking $n = 2$ in equation (B.2).)

- Assume equation (B.1) is true for $n = k$. Then we can check if $n = k+1$ is also true by direct calculation:

$$
\begin{aligned}
\mathrm{ad}_{g^{k+1}}\mathcal{O} = &\quad \tilde{g}^k \mathrm{ad}_{\tilde{g}}\mathcal{O} + \tilde{g}\ \mathrm{ad}_{\tilde{g}^k}\mathcal{O} - \mathrm{ad}_{\tilde{g}^k}\left(\mathrm{ad}_{\tilde{g}}\mathcal{O}\right) \\
= &\quad \tilde{g}^k \mathrm{ad}_{\tilde{g}}\mathcal{O} + \tilde{g}\left[\sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i}\tilde{g}^{k-i}\mathrm{ad}_{\tilde{g}}{}^i\mathcal{O}\right] \\
&\quad - \left[\sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i}\tilde{g}^{k-i}\mathrm{ad}_{\tilde{g}}{}^i\left(\mathrm{ad}_{\tilde{g}}\mathcal{O}\right)\right] \\
= &\quad \binom{k}{0}\tilde{g}^k \mathrm{ad}_{\tilde{g}}\mathcal{O} + \binom{k}{1}\tilde{g}^k \mathrm{ad}_{\tilde{g}}\mathcal{O} \\
&\quad + \left[\sum_{i=2}^{k}(-1)^{i-1}\binom{k}{i}\tilde{g}^{k-i+1}\mathrm{ad}_{\tilde{g}}{}^i\mathcal{O}\right] \\
&\quad + \left[\sum_{i=2}^{k+1}(-1)^{i-1}\binom{k}{i-1}\tilde{g}^{k-i+1}\mathrm{ad}_{\tilde{g}}{}^i\mathcal{O}\right] \\
= &\quad \binom{k+1}{1}\tilde{g}^k \mathrm{ad}_{\tilde{g}}\mathcal{O} + \left[\sum_{i=2}^{k+1}(-1)^{i-1}\binom{k+1}{i}\tilde{g}^{k-i+1}\mathrm{ad}_{\tilde{g}}{}^i\mathcal{O}\right] \\
= &\quad \left[\sum_{i=1}^{k+1}(-1)^{i-1}\binom{k+1}{i}\tilde{g}^{(k+1)-i}\mathrm{ad}_{\tilde{g}}{}^i\mathcal{O}\right]
\end{aligned}
$$

  where we have used equation (B.2) and $\binom{k+1}{i} = \binom{k}{i-1} + \binom{k}{i}$. Hence, equation (B.1) is also true for $n = k + 1$.

- By mathematical induction, equation (B.1) is true for every integer $n \geq 2$.

## B.5 Appendix: Full set of $g^3$-constraints.

This appendix is the list of all possible $g^3$-constraints. This set can be induced by the highest scale dimension constraint: $g_{(\dot\alpha}{}^{(\alpha} g_{\dot\beta}{}^{\beta} g_{i]}{}^{j]} = 0$. Since negative scale dimension constraints always kill the superfield by construction ($s\phi = 0$, $\bar{s}\phi = 0$, or $k\phi = 0$), we list only the constraints with non-negative scale dimension in the table below.

| | $\alpha\beta i$ | $\alpha\beta\dot\alpha$ | $\alpha i j$ | $\alpha i \dot\alpha$ | $ijk$ |
|---|---|---|---|---|---|
| $\dot\rho\dot\sigma\ell$ | $g_{(\dot\rho}{}^{(\alpha} g_{\dot\sigma}{}^{\beta} g_{\ell]}{}^{i]}$ | $g_{(\dot\rho}{}^{(\alpha} g_{\dot\sigma}{}^{\beta} g_{\ell]}{}^{\dot\alpha]}$ | $g_{(\dot\rho}{}^{(\alpha} g_{\dot\sigma}{}^{i} g_{\ell]}{}^{j]}$ | $g_{(\dot\rho}{}^{(\alpha} g_{\dot\sigma}{}^{i} g_{\ell]}{}^{\dot\alpha]}$ | $g_{(\dot\rho}{}^{(i} g_{\dot\sigma}{}^{j} g_{\ell]}{}^{k]}$ |
| $\dot\rho\dot\sigma\rho$ | $g_{(\dot\rho}{}^{(\alpha} g_{\dot\sigma}{}^{\beta} g_{\rho]}{}^{i]}$ | $g_{(\dot\rho}{}^{(\alpha} g_{\dot\sigma}{}^{\beta} g_{\rho]}{}^{\dot\alpha]}$ | $g_{(\dot\rho}{}^{(\alpha} g_{\dot\sigma}{}^{i} g_{\rho]}{}^{j]}$ | $g_{(\dot\rho}{}^{(\alpha} g_{\dot\sigma}{}^{i} g_{\rho]}{}^{\dot\alpha]}$ | $g_{(\dot\rho}{}^{(i} g_{\dot\sigma}{}^{j} g_{\rho]}{}^{k]}$ |
| $\dot\rho\ell m$ | $g_{(\dot\rho}{}^{(\alpha} g_{\ell}{}^{\beta} g_{m]}{}^{i]}$ | $g_{(\dot\rho}{}^{(\alpha} g_{\ell}{}^{\beta} g_{m]}{}^{\dot\alpha]}$ | $g_{(\dot\rho}{}^{(\alpha} g_{\ell}{}^{i} g_{m]}{}^{j]}$ | $g_{(\dot\rho}{}^{(\alpha} g_{\ell}{}^{i} g_{m]}{}^{\dot\alpha]}$ | $g_{(\dot\rho}{}^{(i} g_{\ell}{}^{j} g_{m]}{}^{k]}$ |
| $\dot\rho\ell\rho$ | $g_{(\dot\rho}{}^{(\alpha} g_{\ell}{}^{\beta} g_{\rho]}{}^{i]}$ | $g_{(\dot\rho}{}^{(\alpha} g_{\ell}{}^{\beta} g_{\rho]}{}^{\dot\alpha]}$ | $g_{(\dot\rho}{}^{(\alpha} g_{\ell}{}^{i} g_{\rho]}{}^{j]}$ | $g_{(\dot\rho}{}^{(\alpha} g_{\ell}{}^{i} g_{\rho]}{}^{\dot\alpha]}$ | $g_{(\dot\rho}{}^{(i} g_{\ell}{}^{j} g_{\rho]}{}^{k]}$ |
| $\ell mn$ | $g_{(\ell}{}^{(\alpha} g_{m}{}^{\beta} g_{n]}{}^{i]}$ | $g_{(\ell}{}^{(\alpha} g_{m}{}^{\beta} g_{n]}{}^{\dot\alpha]}$ | $g_{(\ell}{}^{(\alpha} g_{m}{}^{i} g_{n]}{}^{j]}$ | $g_{(\ell}{}^{(\alpha} g_{m}{}^{i} g_{n]}{}^{\dot\alpha]}$ | $g_{(\ell}{}^{(i} g_{m}{}^{j} g_{n]}{}^{k]}$ |
| $\dot\rho\rho\sigma$ | $g_{(\dot\rho}{}^{(\alpha} g_{\rho}{}^{\beta} g_{\sigma]}{}^{i]}$ | $g_{(\dot\rho}{}^{(\alpha} g_{\rho}{}^{\beta} g_{\sigma]}{}^{\dot\alpha]}$ | $g_{(\dot\rho}{}^{(\alpha} g_{\rho}{}^{i} g_{\sigma]}{}^{j]}$ | 0 | 0 |
| $\ell m\rho$ | $g_{(\ell}{}^{(\alpha} g_{m}{}^{\beta} g_{\rho]}{}^{i]}$ | $g_{\ell}{}^{(\alpha} g_{m}{}^{\beta} g_{\rho]}{}^{\dot\alpha]}$ | $g_{\ell}{}^{(\alpha} g_{m}{}^{i} g_{\rho]}{}^{j]}$ | 0 | 0 |
| $\ell\rho\sigma$ | $g_{(\ell}{}^{(\alpha} g_{\rho}{}^{\beta} g_{\sigma]}{}^{i]}$ | 0 | 0 | 0 | 0 |

| | $\alpha\dot\alpha\dot\beta$ | $\dot\alpha ij$ | $\dot\alpha\dot\beta i$ |
|---|---|---|---|
| $\dot\rho\dot\sigma\ell$ | $g_{(\dot\rho}{}^{(\alpha} g_{\dot\sigma}{}^{\dot\alpha} g_{\ell]}{}^{\dot\beta]}$ | $g_{(\dot\rho}{}^{(\dot\alpha} g_{\dot\sigma}{}^{i} g_{\ell]}{}^{j]}$ | $g_{(\dot\rho}{}^{(\dot\alpha} g_{\dot\sigma}{}^{\dot\beta} g_{\ell]}{}^{i]}$ |
| $\dot\rho\dot\sigma\rho$ | $g_{(\dot\rho}{}^{(\alpha} g_{\dot\sigma}{}^{\dot\alpha} g_{\rho]}{}^{\dot\beta]}$ | $g_{(\dot\rho}{}^{(\dot\alpha} g_{\dot\sigma}{}^{i} g_{\rho]}{}^{j]}$ | 0 |
| $\dot\rho\ell m$ | $g_{(\dot\rho}{}^{(\alpha} g_{\ell}{}^{\dot\alpha} g_{m]}{}^{\dot\beta]}$ | $g_{(\dot\rho}{}^{(\dot\alpha} g_{\ell}{}^{i} g_{m]}{}^{j]}$ | 0 |
| $\dot\rho\ell\rho$ | 0 | 0 | 0 |
| $\ell mn$ | 0 | 0 | 0 |
| $\dot\rho\rho\sigma$ | 0 | 0 | 0 |
| $\ell m\rho$ | 0 | 0 | 0 |
| $\ell\rho\sigma$ | 0 | 0 | 0 |

0 means it is negative scale dimension constraint, therefore no additional constraints.

## B.6 C++ Code For Calculating the $D^5$ Constraint

Calculating equation (3.18) would be almost impossible. Just by expanding out terms in $D_{(\dot\rho}{}^{(\alpha}D_{\dot\sigma}{}^{\beta}D_i{}^{j}D_k{}^{l}D_{m]}{}^{n]}$ gives 14400 terms. They can, of course, be further simplified by symmetries. However, the number would still be huge. To find equation (3.18), we use computer to calculate it. The following is the C++ code used to calculate the term with out put in TEX form.

```
#include<stdio.h>
#include<iostream>
#include<string>
using namespace std;

int all[2000][2][5], num[2000], count, N = 5, result[2000][2][5], rnum[2000],
total = 0;

int print(int arr[]){
  int i, j, k;
  string u[] = {"\\alpha","\\beta","j","l","n"};
  string d[] = {"\\dot{\\rho}","\\dot{\\sigma}","i","k","m"};
  string t[2 * N];

  j = 0;
  k = 0;

  for(i = 0; i < N; i++){

    if (arr[2 * i + 1] == 1){
      t[2 * i + 1] = u[j];
      j++;
    }else{
      t[2 * i + 1] = u[k + 2];
      k++;
    }
  }

  j = 0;
```

```
  k = 0;

  for(i = 0; i < N; i++){

    if (arr[2 * i] == 1){
      t[2 * i] = d[j];
      j++;
    }else{
      t[2 * i] = d[k + 2];
      k++;
    }
  }

  for(i = 0; i < N; i++){
    if(arr[2*i] == 2){
      printf("\\delta_{%s}^{~%s}", t[2*i].c_str(), t[2*i+1].c_str());
    }else{
      printf("g_{%s}^{~%s}", t[2*i].c_str(), t[2*i+1].c_str());
    }
  }


  return 0;
}

bool compare(int i, int j){
  bool r;
  int k, l;
  r = true;
  for(k = 0; k < 2; k++){
    for(l = 0; l < N; l++){
      if(result[i][k][l] != all[j][k][l]){
        r = false;
      }
    }
  }
  return r;
}

void equal(int l, int i){
```

```
  int j, k;

  for(j = 0; j < 2; j++){
    for(k = 0; k < N; k++){
      result[l][j][k] = all[i][j][k];
    }
  }
}

void rearrange(int i){
  int j, k, l, temp[2][5];
  bool move;
// =============== Rearrange =================

// =================== P =====================
  for(move = true; move == true; ){
    move = false;
    for(j = 1; j < N; j++){
       if((all[i][0][j] == 1 && all[i][1][j] == 1) && !(all[i][0][j - 1] == 1 &&
all[i][1][j - 1] == 1)){
        if(all[i][0][j - 1] == 1 || all[i][1][j - 1] == 1){
          num[i] = - num[i];
        }

        for(k = 0; k < 2; k++){
          temp[k][j - 1] = all[i][k][j - 1];
          all[i][k][j - 1] = all[i][k][j];
          all[i][k][j] = temp[k][j - 1];
        }
        move = true;
      }
    }
  }
// =========================================

// =================== q ======================
  for(move = true; move == true; ){
    move = false;
    for(j = 1; j < N; j++){
```

```
  if((all[i][0][j] == 0 && all[i][1][j] == 1) && !(all[i][0][j - 1] == 1 &&
all[i][1][j - 1] == 1) && !(all[i][0][j - 1] == 0 && all[i][1][j - 1] == 1)){
    if(all[i][1][j - 1] == all[i][0][j]  && all[i][0][j] != 1){
      for(k = 0; k < N; k++){
        for(l = 0; l < 2; l++){
          if(k != j && k != j - 1){
            all[count][l][k] = all[i][l][k];
          }else if(k == j - 1){
            all[count][0][k] = all[i][0][k];
            all[count][1][k] = all[i][1][k + 1];
          }else{
            all[count][l][k] = 2;
          }
        }
      }
      num[count] = num[i];
      for(k = 0; k < N - 1; k++){
        if(all[count][0][k] == 2){
          all[count][0][k] = all[count][0][k + 1];
          all[count][0][k + 1] = 2;
          all[count][1][k] = all[count][1][k + 1];
          all[count][1][k + 1] = 2;
        }
      }
      count++;
    }

    if(all[i][0][j - 1] == all[i][1][j]  && all[i][1][j] != 1){
      for(k = 0; k < N; k++){
        for(l = 0; l < 2; l++){
          if(k != j && k != j - 1){
            all[count][l][k] = all[i][l][k];
          }else if(k == j - 1){
            all[count][0][k] = all[i][0][k + 1];
            all[count][1][k] = all[i][1][k];
          }else{
            all[count][l][k] = 2;
          }
```

```
          }
        }
        num[count] = - num[i];
        for(k = 0; k < N - 1; k++){
          if(all[count][0][k] == 2){
            all[count][0][k] = all[count][0][k + 1];
            all[count][0][k + 1] = 2;
            all[count][1][k] = all[count][1][k + 1];
            all[count][1][k + 1] = 2;
          }
        }
        count++;
      }

      if(all[i][0][j - 1] == 1){
        num[i] = - num[i];
      }

      for(k = 0; k < 2; k++){
        temp[k][j - 1] = all[i][k][j - 1];
        all[i][k][j - 1] = all[i][k][j];
        all[i][k][j] = temp[k][j - 1];
      }
      move = true;
    }
  }
}
// ==========================================

// ================== qb ====================
  for(move = true; move == true; ){
    move = false;
    for(j = 1; j < N; j++){
      if((all[i][0][j] == 1 && all[i][1][j] == 0) && !(all[i][0][j - 1] == 1 &&
all[i][1][j - 1] == 1) && !(all[i][0][j - 1] == 0 && all[i][1][j - 1] == 1) &&
!(all[i][0][j - 1] == 1 && all[i][1][j - 1] == 0)){
        if(all[i][1][j - 1] == all[i][0][j]  && all[i][0][j] != 1){
          for(k = 0; k < N; k++){
```

```
    for(l = 0; l < 2; l++){
      if(k != j && k != j - 1){
        all[count][l][k] = all[i][l][k];
      }else if(k == j - 1){
        all[count][0][k] = all[i][0][k];
        all[count][1][k] = all[i][1][k + 1];
      }else{
        all[count][l][k] = 2;
      }
    }
  }
  num[count] = num[i];
  for(k = 0; k < N - 1; k++){
    if(all[count][0][k] == 2){
      all[count][0][k] = all[count][0][k + 1];
      all[count][0][k + 1] = 2;
      all[count][1][k] = all[count][1][k + 1];
      all[count][1][k + 1] = 2;
    }
  }
  count++;
}
if(all[i][0][j - 1] == all[i][1][j]  && all[i][1][j] != 1){
  for(k = 0; k < N; k++){
    for(l = 0; l < 2; l++){
      if(k != j && k != j - 1){
        all[count][l][k] = all[i][l][k];
      }else if(k == j - 1){
        all[count][0][k] = all[i][0][k + 1];
        all[count][1][k] = all[i][1][k];
      }else{
        all[count][l][k] = 2;
      }
    }
  }
  num[count] = - num[i];
  for(k = 0; k < N - 1; k++){
    if(all[count][0][k] == 2){
```

```
            all[count][0][k] = all[count][0][k + 1];
            all[count][0][k + 1] = 2;
            all[count][1][k] = all[count][1][k + 1];
            all[count][1][k + 1] = 2;
          }
        }
        count++;
      }

      for(k = 0; k < 2; k++){
        temp[k][j - 1] = all[i][k][j - 1];
        all[i][k][j - 1] = all[i][k][j];
        all[i][k][j] = temp[k][j - 1];
      }
      move = true;
    }
  }
 }
// ========================================
}

void work(void){
 int i, j;
 bool sign;

 for(i = 0; i < count; i++){
   sign = false;
   for(j = 0; j < total + 1; j++){
     if(compare(j,i)){
       rnum[j] = rnum[j] + num[i];
       sign = true;
     }
   }
   if(sign == false){
     rnum[total] = num[i];
     equal(total,i);
     total++;
   }
```

```
  }
}

void output(void){
  int arr[10];
  int i, j, width;

  width = 0;
  printf("& ");
  for(i = 0; i < total; i++){
    if(rnum[i] > 0 && rnum[i] != 1){
      printf(" + %i", rnum[i]);
      for(j = 0; j < N; j++){
        arr[2 * j] = result[i][0][j];
        arr[2 * j + 1] = result[i][1][j];
      }
      print(arr);
      width++;
    }else if(rnum[i] == 1){
      printf(" + ");
      for(j = 0; j < N; j++){
        arr[2 * j] = result[i][0][j];
        arr[2 * j + 1] = result[i][1][j];
      }
      print(arr);
      width++;
    }else if(rnum[i] < 0 && rnum[i] != -1){
      printf(" - %i", - rnum[i]);
      for(j = 0; j < N; j++){
        arr[2 * j] = result[i][0][j];
        arr[2 * j + 1] = result[i][1][j];
      }
      print(arr);
      width++;
    }else if(rnum[i] == -1){
      printf(" - ");
      for(j = 0; j < N; j++){
        arr[2 * j] = result[i][0][j];
```

```
        arr[2 * j + 1] = result[i][1][j];
      }
      print(arr);
      width++;
    }
    if(width == 3){
      width = 0;
      printf("\\\\ \n &");
    }
  }
}

int main(void){
  int i, j, k, l, u[2], d[2];
  std::fill_n(num, 2000, 1);
  std::fill_n(rnum, 2000, 0);

  for(i = 0; i < 2000; i++){
    for(j = 0; j < N; j++){
      all[i][0][j] = 0;
      all[i][1][j] = 0;
    }
  }

  count = 0;

  for(i = 0; i < N - 1; i++){
    for(k = i + 1; k <= N - 1; k++){
      for(j = 0; j < N - 1; j++){
        for(l = j + 1; l <= N - 1; l++){
          u[0] = j;
          u[1] = l;
          d[0] = i;
          d[1] = k;
          if( (((u[0] >= d[1]) || (d[0] > u[1])) || (u[0] >= d[0] && d[1] > u[1]))
|| (d[1]<=u[1] && d[0] > u[0]) ){
            num[count] = -1;
          }else{
            num[count] = 1;
```

```
          }
          all[count][0][d[0]] = 1;
          all[count][0][d[1]] = 1;
          all[count][1][u[0]] = 1;
          all[count][1][u[1]] = 1;
          count++;
        }
      }
    }
  }

  for(i = 0; i < count; i++){
    rearrange(i);
  }

  work();

  printf("Total = %i terms\nAfter rearragement = %i\n\n",count , total);

  output();

  return 0;
}
```

# Appendix C

# F-theory

## C.1 Notations

Instead of explaining our notations all over chapter 4, some common notations are defined in this section so that the readers don't have to hunt for them.

i) $f(1) \equiv f(\sigma_1)$, where $\sigma$ is worldvolume coordinates, $f(1-2) \equiv f(\sigma_1 - \sigma_2)$, $f((1)+(2)) \equiv f(1) + f(2)$, and, similarly, $f((1)-(2)) \equiv f(1) - f(2)$.

ii) Worldvolume vector indices are denoted as $q$, $r$, $\cdots$; spacetime spinor indices are $\alpha$, $\beta$, $\cdots$; superspace indices (which include $\{D, P, \Omega\}$) are $M$, $N$, $O$, $\cdots$; covariantized superspace indices are $A$, $B$, $C$, $\cdots$ (also include $\{D, P, \Omega\}$); group coordinate indices are denoted as $I$, $J$, $K$, $\cdots$; the covariantized index for $H$ group is $S$, and the full set of covariantized superspace indices, including all "$A$" indices, $S$, and $\Sigma$, are $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\cdots$.

iii) $\mathring{\rhd}_M(\sigma)$ = worldvolume current: e.g. $\mathring{\rhd}_D(\sigma) = \mathring{\rhd}_\alpha$, $\mathring{\rhd}_\Omega(\sigma) = \mathring{\rhd}^{\alpha r}$.

iv) $\eta_{MNr}$ is the generalized constant metric, $f_{MN}{}^O$ is the structure constants.

v) $\alpha^I(\sigma)$ is the coordinates of $H$ group (a function of the worldvolume).

vi) $e_S{}^I(\sigma)$ is the vielbein that converts functional derivatives $\left(\dfrac{\delta}{\delta\alpha^I(\sigma)}\right)$ into symmetry generators $(\rhd_S(\sigma))$.

vii) $\partial^r = \dfrac{\partial}{\partial \sigma_r}$, a worldvolume coordinate derivative. Sometimes we have to specify which coordinate we act on: Then we add an additional index $\partial_1^r = \dfrac{\partial}{\partial \sigma_{1r}}$.

viii) $\triangleright_A(\sigma)$ = covariantized worldvolume current.

ix) $\triangleright_{\mathcal{A}}(\sigma)$ = the full set of covariant worldvolume currents.

x) ${g_A}^M(\sigma)$ is a worldvolume field and is an element of $H$ group.

xi) Parenthesis [ ) in $f_{[m|n|o)}$ is the graded (anti)symmetrization, i.e. sum of index permutation (with a minus sign if not interchanging two spinor indices) in the parenthesis but not the ones in between the two vertical lines, | |.

## C.2 Relating $f$'s and $\eta$'s Using Jacobi

The following are the complete list of all the relations between $f$'s and $\eta$'s. The "zero modes" means no derivative on delta function ones ($\delta^2$) and the "oscillating modes" means the ones that have a derivative on a delta function ($\delta\partial\delta$). Equations (C.1 $\sim$ C.11) show that $f$'s are invariant under group $H$, and equation (C.12) gives nothing new.

**Zero modes:**

$$0 = f_{S_1S_2}{}^{S'} f_{S'S_3}{}^{S_4} + f_{S_2S_3}{}^{S'} f_{S'S_1}{}^{S_4} + f_{S_1S_3}{}^{S'} f_{S_2S'}{}^{S_4}, \tag{C.1}$$

$$0 = f_{S_1S_2}{}^{S'} f_{S'D_1}{}^{D_2} + f_{S_2D_1}{}^{D'} f_{D'S_1}{}^{D_2} + f_{S_1D_1}{}^{D'} f_{S_2D'}{}^{D_2}, \tag{C.2}$$

$$0 = f_{S_1S_2}{}^{S'} f_{S'P_1}{}^{P_2} + f_{S_2P_1}{}^{P'} f_{P'S_1}{}^{P_2} + f_{S_1P_1}{}^{P'} f_{S_2P'}{}^{P_2}, \tag{C.3}$$

$$0 = f_{S_1S_2}{}^{S'} f_{S'\Omega_1}{}^{\Omega_2} + f_{S_2\Omega_1}{}^{\Omega'} f_{\Omega'S_1}{}^{\Omega_2} + f_{S_1\Omega_1}{}^{\Omega'} f_{S_2\Omega'}{}^{\Omega_2}, \tag{C.4}$$

$$0 = f_{S_1S_2}{}^{S'} f_{S'\Sigma_1}{}^{\Sigma_2} + f_{S_2\Sigma_1}{}^{\Sigma'} f_{\Sigma'S_1}{}^{\Sigma_2} + f_{S_1\Sigma_1}{}^{\Sigma'} f_{S_2\Sigma'}{}^{\Sigma_2}, \tag{C.5}$$

$$0 = f_{SD_1}{}^{D'} f_{D'D_2}{}^{P} + f_{D_1D_2}{}^{P'} f_{P'S}{}^{P} + f_{SD_2}{}^{D'} f_{D_1D'}{}^{P}, \tag{C.6}$$

$$0 = f_{SD}{}^{D'} f_{D'P}{}^{\Omega} + f_{DP}{}^{\Omega'} f_{\Omega'S}{}^{\Omega} + f_{SP}{}^{P'} f_{DP'}{}^{\Omega}, \tag{C.7}$$

$$0 = f_{SD}{}^{D'} f_{D'\Omega}{}^{\Sigma} + f_{D\Omega}{}^{\Sigma'} f_{\Sigma'S}{}^{\Sigma} + f_{S\Omega}{}^{\Omega'} f_{D\Omega'}{}^{\Sigma}, \tag{C.8}$$

$$0 = f_{SP}{}^{P'} f_{P'D}{}^{\Omega} + f_{PD}{}^{\Omega'} f_{\Omega'S}{}^{\Omega} + f_{SD}{}^{D'} f_{PD'}{}^{\Omega}, \tag{C.9}$$

$$0 = f_{SP_1}{}^{P'} f_{P'P_2}{}^{\Sigma} + f_{P_1P_2}{}^{\Sigma'} f_{\Sigma'S}{}^{\Sigma} + f_{SP_2}{}^{P'} f_{P_1P'}{}^{\Sigma}, \tag{C.10}$$

$$0 = f_{S\Omega}{}^{\Omega'} f_{\Omega'D}{}^{\Sigma} + f_{\Omega D}{}^{\Sigma'} f_{\Sigma'S}{}^{\Sigma} + f_{SD}{}^{D'} f_{\Omega D'}{}^{\Sigma}, \tag{C.11}$$

$$0 = f_{D_1D_2}{}^{P'} f_{P'D_3}{}^{\Omega} + f_{D_2D_3}{}^{P'} f_{P'D_1}{}^{\Omega} + f_{D_3D_1}{}^{P'} f_{P'D_2}{}^{\Omega}, \tag{C.12}$$

$$0 = f_{D_1D_2}{}^{P'} f_{P'P}{}^{\Sigma} - f_{D_2P}{}^{\Omega'} f_{\Omega'D_1}{}^{\Sigma} + f_{PD_1}{}^{\Omega'} f_{\Omega'D_2}{}^{\Sigma}. \tag{C.13}$$

**Oscillating modes:**

$$0 = f_{S_1S_2}{}^{S'} \eta_{S'\Sigma a} + f_{S_1\Sigma}{}^{\Sigma'} \eta_{S_2\Sigma' a} + f_{S_1a}{}^{b} \eta_{S_2\Sigma b}, \tag{C.14}$$

$$0 = f_{SD}{}^{D'} \eta_{D'\Omega a} + f_{S\Omega}{}^{\Omega'} \eta_{D\Omega' a} + f_{Sa}{}^{b} \eta_{D\Omega b}, \tag{C.15}$$

$$0 = f_{SP_1}{}^{P'} \eta_{P'P_2a} + f_{SP_2}{}^{P'} \eta_{P_1P'a} + f_{Sa}{}^{b} \eta_{P_1P_2b}, \tag{C.16}$$

$$0 = f_{DS}{}^{D'} \eta_{D'\Omega a} + f_{D\Omega}{}^{\Sigma'} \eta_{S\Sigma' a} - \frac{1}{2} f_{Sa}{}^{b} \eta_{D\Omega b}, \tag{C.17}$$

$$0 = f_{D_1D_2}{}^{P'} \eta_{P'Pa} + f_{D_1P}{}^{\Omega'} \eta_{D_2\Omega' a}, \tag{C.18}$$

$$0 = f_{P_1S}{}^{P'} \eta_{P'P_2a} + f_{P_1P_2}{}^{\Sigma'} \eta_{S\Sigma' a} - \frac{1}{2} f_{Sa}{}^{b} \eta_{P_1P_2b}, \tag{C.19}$$

$$0 = f_{PD_1}{}^{\Omega'} \eta_{\Omega'D_2a} + f_{PD_2}{}^{\Omega'} \eta_{D_1\Omega' a}, \tag{C.20}$$

$$0 = f_{\Omega S}{}^{\Omega'} \eta_{\Omega'Da} + f_{\Omega D}{}^{\Sigma'} \eta_{S\Sigma' a} - \frac{1}{2} f_{Sa}{}^{b} \eta_{\Omega Db}, \tag{C.21}$$

$$0 = f_{\Sigma S_1}{}^{\Sigma'} \eta_{\Sigma'S_2a} + f_{\Sigma S_2}{}^{\Sigma'} \eta_{S_1\Sigma' a} - f_{S_1a}{}^{b} \eta_{S_2\Sigma b} - f_{S_2a}{}^{b} \eta_{S_1\Sigma b}. \tag{C.22}$$

## C.3 5-Brane Commutation Relations

This appendix shows all the nonvanishing commutation relations for the 5-brane.

1. $\left\{\triangleright_{D_1}(1), \triangleright_{D_2}(2)\right\} = \left\{\triangleright_{\alpha_1}(1), \triangleright_{\alpha_2}(2)\right\}$
   $= i\left(\gamma^{mn}\right)_{\alpha_1\alpha_2} \mathring{\triangleright}_{mn}\delta(1-2) = if_{D_1D_2}{}^{P}\triangleright_P\delta(1-2).$

2. $\left[\triangleright_D(1), \triangleright_P(2)\right] = \left[\triangleright_\alpha(1), \triangleright_{cd}(2)\right]$
   $= i\left(\gamma^{ef}\right)_{\alpha\beta}\epsilon_{efcda}\triangleright^{\beta a}\delta(1-2) = if_{D_1P}{}^{\Omega}\triangleright_\Omega\delta(1-2).$

3. $\left[\triangleright_{P_1}(1), \triangleright_{P_2}(2)\right] = \left[\triangleright_{c_1d_1}(1), \triangleright_{c_2d_2}(2)\right]$
   $= i\left(-\mathring{\eta}_{[c_1|[e}\epsilon_{f]|d_1]c_2d_2a} + \mathring{\eta}_{[c_2|[e}\epsilon_{f]|d_2]c_1d_1a}\right)\triangleright^{efa}\delta(1-2)$
   $+i\epsilon_{c_1d_1c_2d_2a}\triangleright^{a}((1)-(2)\delta(1-2)$
   $= if_{P_1P_2}{}^{\Sigma}\triangleright_\Sigma\delta(1-2) + i\eta_{P_1P_2a}\triangleright^{a}((1)-(2))\delta(1-2).$

4. $\left\{\triangleright_D(1), \triangleright_\Omega(2)\right\} = \left\{\triangleright_\alpha(1), \triangleright^{\beta b}(2)\right\}$
   $= \frac{i}{2}\left(\gamma_{ef}\right)_\alpha{}^{\beta}\triangleright^{efb}\delta(1-2) + i\delta_\alpha{}^{\beta}\delta_a^b\triangleright^{a}((1)-(2))\delta(1-2)$
   $= if_{D\Omega}{}^{\Sigma}\triangleright_\Sigma\delta(1-2) + i\eta_{D\Omega a}\triangleright^{a}((1)-(2))\delta(1-2).$

5. $\left[\triangleright_S(1), \triangleright_D(2)\right] = \left[\triangleright_{ef}(1), \triangleright_\alpha(2)\right]$
   $= \frac{i}{4}\left(\gamma_{ef}\right)_\alpha{}^{\rho}\triangleright_\rho\delta(1-2) = if_{SD}{}^{D'}\triangleright_{D'}\delta(1-2).$

6. $\left[\triangleright_S(1), \triangleright_P(2)\right] = \left[\triangleright_{ef}(1), \triangleright_{cd}(2)\right]$
   $= -i\mathring{\eta}_{[c|[e}\delta_{f]|d]}^{c'd'}\triangleright_{c'd'}\delta(1-2) = if_{SP}{}^{P'}\triangleright_{P'}\delta(1-2).$

7. $\left[\triangleright_S(1), \triangleright_\Omega(2)\right] = \left[\triangleright_{ef}(1), \triangleright^{\beta b}(2)\right]$
   $= i\left[-\frac{1}{4}\left(\gamma_{ef}\right)_\rho{}^{\beta}\delta_a^b + \delta_\rho{}^{\beta}\delta_{[e}^b\mathring{\eta}_{f]a}\right]\triangleright^{\rho a}\delta(1-2) = if_{S\Omega}{}^{\Omega'}\triangleright_{\Omega'}\delta(1-2).$

8. $\Big[\rhd_S(1), \rhd_S(2)\Big] = \Big[\rhd_{e_1 f_1}(1), \rhd_{e_2 f_2}(2)\Big]$

$= -i\mathring{\eta}_{[e_2|[e_1}\delta^{e'f'}_{f_1]|f_2]}\rhd_{e'f'}\delta(1-2) = if_{SS}{}^{S'}\rhd_{S'}\delta(1-2).$

9. $\Big[\rhd_S(1), \rhd_\Sigma(2)\Big] = \Big[\rhd_{ef}(1), \rhd^{gha}(2)\Big]$

$= i\mathring{\eta}_{[g'|[e}\delta^{gh}_{f]|h']}\delta^a_b\rhd^{g'h'b}\delta(1-2) + i\mathring{\eta}_{b'[e}\delta^a_{f]}\rhd^{ghb'}\delta(1-2) + 2i\delta^{gh}_{ef}\rhd^a(2)\delta(1-2)$

$= if_{S\Sigma}{}^{\Sigma'}\rhd_{\Sigma'}\delta(1-2) + 2i\eta_{S\Sigma a}\rhd^a(2)\delta(1-2).$