General Instructions: Twelve problems are given; you should do any four. If you do more than four problems, you must choose which four should be graded, and only submit those four.

Each problem counts 20 points, and the solution should typically take less than 45 minutes.

Some of the problems spill onto two pages. Make sure you do all the parts of each problem you choose.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use a one page help sheet, a calculator, and with the proctor’s approval, a foreign language dictionary. No other materials may be used.

Some potentially useful information:

The atomic mass of hydrogen is 1.00794 amu.
The atomic mass of helium is 4.002602 amu.
1 amu is 1.66×10^{-27} kg.
c = 2.998 × 10^8 ms^{-1}.
G = 6.673 × 10^{-11} m^3 kg^{-1} s^{-2}.
The solar luminosity is 3.85×10^{26} W.
The mass of the Sun is 1.989×10^{30} kg.
The radius of the Sun is 7.00 × 10^{8} meters.
$h = 1.055 × 10^{-34}$ J
$e = 1.602 × 10^{-19}$ C
$k_B = 1.38 × 10^{-23}$ J/K
$mc^2 ≃ 0.5$ MeV
$hc ≃ 197$ MeV-fm.
A non-interacting Fermi gas has an occupation index $f$ related to the chemical potential $\mu$ and temperature $T$ by

$$f = \left[ \exp\left( \frac{E - \mu}{T} \right) + 1 \right]^{-1},$$

where $E = \sqrt{(mc^2)^2 + (pc)^2}$. In the limit of extreme degeneracy for a non-interacting fermionic gas, the number density is related to the Fermi momentum $p_F$ by

$$n = \frac{mc^2}{3\pi^2} \left( \frac{p_F}{\hbar} \right)^3.$$

a. (5 pts) Derive the above relation for the number density.

b. (5 pts) In the extremely non-relativistic and extremely relativistic limits, find the exponents $\gamma_{NR}$ and $\gamma_R$ in the relation $P \propto n^\gamma$. Estimate the approximate transition density between these two behaviors.

c. (4 pts) Assuming Newtonian gravity, establish the exponents $p_{NR}$ and $p_R$ in the mass-radius relation $M \propto R^p$ and sketch the mass-radius curve for white dwarfs, where the pressure is dominated by electrons. Put appropriate physical values for mass and radius on the axis.

d. (2 pts) In the case of ultra-dense matter, the pressure is dominated by degenerate non-relativistic neutrons. Estimate the density for which neutrons become relativistic, and show that this density is larger than the densities found in normal neutron stars.

e. (2 pts) At densities below the nuclear saturation density, $\rho_s = 3 \times 10^{14}$ g cm$^{-3}$, the neutrons are largely non-interacting, but above this density they are highly interacting and the equation of state behaves roughly like $P \propto n^2$. A neutron star could be described by this behavior. Sketch the mass-radius relation for a neutron star according to these relations.

f. (2 pts) In part e), you will find that in the limit of high densities that the neutron star mass is unbounded. In reality, neutron stars have a maximum mass. Why?

Solution:

a) For a non-interacting fermion gas,

$$n = \frac{g}{h^3} \int f d^3p$$
where \( f = \left[ \exp((E - \mu)/T) + 1 \right]^{-1} \). In the limit of extreme degeneracy, \( \mu/T \gg 1 \) and \( f = 1 \) for \( \mu < E_F = \sqrt{(mc^2)^2 + (p_Fc)^2} \) and \( f = 0 \) for \( \mu > E_F \).

\[
n = \frac{4\pi g}{h^3} \int_0^{p_F} p^2 dp = \frac{4\pi g}{3h^3} p_F^3.
\]

Setting \( \hbar = h/2\pi \) and \( g = 2 \) we find the indicated relation.

b) For a non-interacting fermion gas

\[
P = \frac{g}{3h^3} \int f_F \frac{\partial E}{\partial p} d^3p.
\]

We have \( \partial E/\partial p = p/E \). In the non-relativistic limit, \( E = mc^2 + p^2/[2(mc)^2] \) so that the integral is proportional to \( p_F^5 \). In the relativistic limit, \( E = pc \) and the integral is proportional to \( p_F^4 \). The number density is always proportional to \( p_F^3 \) and the indicated relations follow. The transition occurs when \( p_Fc \approx mc^2 \) or about

\[
n = \frac{g(mc^2)^4}{12\pi^2(hc)^3} \approx 10^{-9} \text{ fm}^{-3}
\]

or about \( 10^{6} \text{ g cm}^{-3} \).

c) For a Newtonian polytrope in which \( P \propto \rho^\gamma \) one finds from dimensional analysis of the equation of hydrostatic equilibrium

\[
\frac{dP}{dr} = -\frac{Gm\rho}{r^2}, \quad dm = 4\pi \rho r^2 dr
\]

that

\[
M \propto R^{(3\gamma-4)/(\gamma-2)}.
\]

In the NR case, \( M \propto R^{-3} \) and in the R case, \( M \propto R^0 \). The mass approaches a maximum value of about \( 1.4 M_\odot \) for small radii; but for radii larger than 1000 km, the mass asymptotically falls with the -3 power of the radius.

d) From the relation \( p_F = mc \) for relativity to dominate, and from \( n \propto p_F^3 \), it is clear that instead of \( 10^6 \text{ g cm}^{-3} \) for electrons we have a value \((m_n/m_e)^3 = (1837)^3 \approx 10^{10}\) times larger for neutrons, or \( 10^{16} \text{ g cm}^{-3}\).

e) At low densities, we have \( p = -3 \); for large densities we have \( p = \infty \) from part c. Thus, above nuclear density, the radius approaches a constant as mass increases. At nuclear density, the mass is about \( 1 \text{ M}_\odot \) using \( M \approx \rho_s R^3 \) and \( R \approx 12 \text{ km} \).

f) General relativity is important, and this results in a maximum mass.
Astronomy 2 (Interstellar Medium)

We consider a collisional excitation and emission of atoms which have three energy levels (1, 2, and 3). The excitation occurs only due to collision with background electrons, and neither absorption nor stimulated emission occur in this system. The transition between level 2 and 3 is rare and negligible. Use the following notation:

- $T_e$: electron temperature [K]
- $n_e$: electron density [cm$^{-3}$]
- $n_i$: density of atom at level $i$ [cm$^{-3}$]
- $A_{ij}$: Einstein A coefficient for transition from level $i$ to $j$ [s$^{-1}$]
- $C_{ij}$: Collisional excitation and de-excitation coefficients [s$^{-1} \cdot$ cm$^3$]
- $E_{ij}$: Energy gap between level $i$ and $j$ [erg]
- $\nu_{ij}$: Frequency corresponding to $E_{ij}$ [Hz$^{-1}$]
- $E_{ij}$: Number of states at level $i$

a. (4 pts) Describe the detailed balance equations between level 1 and 2, and between 1 and 3.

b. (4 pts) Describe the specific intensities of transition 2 to 1 ($I_{21}$) and 3 to 1 ($I_{31}$) using Einstein coefficients.

c. (4 pts) Derive the ratio $I_{21}/I_{31}$ using $A_{ij}$, $C_{ij}$, $\nu_{ij}$, and $n_e$.

d. (4 pts) Describe the relation between $C_{12}$ and $C_{21}$.

e. (4 pts) Show that the ratio $I_{21}/I_{31}$ does not depend on $C_{ij}$, in the limit of $n_e \rightarrow \infty$. Explain why this is the case even though frequent collisions are expected in this limit.

Solution:

a.

$$n_2(A_{21} + C_{21}n_e) = n_1n_eC_{12} \quad (1)$$

$$n_3(A_{31} + C_{31}n_e) = n_1n_eC_{13} \quad (2)$$
b.

\[ I_{21} = \frac{h \nu_{21}}{4\pi} n_2 A_{21} \]  
\[ I_{31} = \frac{h \nu_{31}}{4\pi} n_3 A_{31} \] 

\( (3) \) \( (4) \)

c.

\[ \frac{I_{21}}{I_{31}} = \frac{h \nu_{21} n_2 A_{21}}{h \nu_{31} n_3 A_{31}} \]  
\[ = \frac{\nu_{21}}{\nu_{31}} \cdot \frac{A_{21}}{A_{31}} \cdot \frac{C_{12}}{C_{13}} \cdot \frac{A_{31} + C_{31} n_e}{A_{21} + C_{21} n_e} \] 

\( (5) \) \( (6) \)

d.

\[ C_{12} = \frac{g_2}{g_1} C_{21} \exp \left[ -\frac{E_{12}}{kT_e} \right] \] 

\( (7) \)

e. Using eq. (6) (7) and

\[ C_{13} = \frac{g_3}{g_1} C_{31} \exp \left[ -\frac{E_{13}}{kT_e} \right] \] 

\[ \frac{I_{21}}{I_{31}} \to \frac{\nu_{21}}{\nu_{31}} \cdot \frac{A_{21}}{A_{31}} \cdot \frac{g_2}{g_3} \exp \left( \frac{E_{32}}{kT_e} \right) \] 

\( (8) \) \( (9) \)

Frequent collisions bring the system into thermal equilibrium. Under the equilibrium, the population of each level is determined by the Boltzmann distribution, and does not depend on the underlying mechanism.

**Astronomy 3 (Observation/Experiment)**

**Radius of a transiting extrasolar planet (20 pts).** The apparent radius of an extrasolar planet transiting the disk of its host star is different from the planet’s photospheric radius. As photons from the star travel tangentially through the planet’s upper atmosphere, they can get partially or completely absorbed. The planet’s photospheric radius \( R \) is defined as the optical depth \( \tau = 2/3 \) surface in the atmosphere ("Side View" diagram). The apparent edge of a transiting planet is at a slightly larger distance \( \Delta R_{\text{ch}} \) from the projected planet center ("View Towards Star"). Here you will estimate the magnitude of this “transit radius effect.”

The optical depth \( \tau_{\text{ch}} \) along a chord followed by the stellar beam through the planet’s upper atmosphere is

\[ \tau_{\text{ch}} \sim \kappa \rho H \sqrt{\frac{2\pi R}{H}} \exp \left( -\frac{\Delta R_{\text{ch}}}{H} \right), \] 

\( (10) \)

where \( \kappa \) is the opacity, \( \rho \) is the mass density of the atmosphere, and

\[ H \approx \frac{kT}{\mu g m_p} \] 

\( (11) \)
is the atmospheric density scale height \(k\) is the Boltzmann constant, \(\mu\) is the mean molecular weight of the atmosphere, \(g\) is the planet’s surface gravity, and \(m_p\) is the proton mass). By definition, assuming an exponential atmosphere, \(\tau = \kappa \rho H = 2/3\).

![Diagram showing the atmospheric density scale height](image)

**View Towards Star**

**Side View**

**a. (8 pts) Derive an expression for the excess radius \(\Delta R_{ch}\) in terms of the planet’s radius \(R\) and the atmospheric scale height \(H\).**

**b. (10 pts) Estimate \(\Delta R_{ch}\) for the hot-Jupiter type transiting planet HD 209458b \((M = 0.64 M_{Jupiter}, R = 1.32 R_{Jupiter}, T \approx 2000 K)\). Assume 75% H\(_2\) and 25% He atmospheric mass composition for estimating \(\mu\). Use: \(M_{Jupiter} = 1.90 \times 10^{30}\) g, \(G = 6.67 \times 10^{-8}\) cm\(^3\) s\(^{-2}\) g\(^{-1}\), \(R_{Jupiter} = 7.10 \times 10^{9}\) cm, \(k/m_p = 8.26 \times 10^7\) erg K\(^{-1}\) g\(^{-1}\).**

**c. (2 pts) What is \(\Delta R_{ch}/R\) for HD 209458b? Can the transit radius effect explain the planet’s recorded \(\sim 10\%\) oversized appearance?**

**Solution:**

**a.** We make the same assumption for \(\tau_{ch}\) as for the optical depth level \(\tau\) of the photosphere.

\[
\tau_{ch} \sim \kappa \rho H \sqrt{\frac{2\pi R}{H}} \exp \left(-\frac{\Delta R_{ch}}{H}\right) = \frac{2}{3}. \tag{12}
\]

Since by definition also \(\tau = \kappa \rho H = 2/3\), then

\[
\sqrt{\frac{2\pi R}{H}} \exp \left(-\frac{\Delta R_{ch}}{H}\right) \sim 1 \quad \Rightarrow \quad \Delta R_{ch} \sim H \ln \sqrt{\frac{2\pi R}{H}}. \tag{13}
\]
b. We can estimate $\Delta R_{\text{ch}}$ from (13), but we first need an estimate for $H$ from (11); i.e., we need $\mu$ and $g$.

The mean molecular weight $\mu$ of the atmosphere is $\mu = \rho/(nm_p)$, where $n$ is the volume density of particles. Given $X = 0.75$ H$_2$ mass fraction and $Y = 0.25$ He mass fraction, and $\approx 2m_p$ of mass per H$_2$ molecule and $\approx 4m_p$ of mass per He atom,

$$n = \left(\frac{X}{2} + \frac{Y}{4}\right) \frac{\rho}{m_p} = \frac{\rho}{\mu m_p} \Rightarrow \mu = \left(\frac{X}{2} + \frac{Y}{4}\right)^{-1} = 2.29 \quad (14)$$

The surface gravity on HD 209458b is

$$g = \frac{GM}{R^2} = 6.76 \times 10^{-8} \text{ cm}^3 \text{s}^{-2} \text{ g}^{-1} \frac{0.64 \times 1.90 \times 10^{30} \text{ g}}{(7.10 \times 10^9 \text{ cm})^2} = 1590 \text{ cm s}^{-2}. \quad (15)$$

Then, using (11),

$$H \approx 8.26 \times 10^7 \text{ erg K}^{-1} \text{ g}^{-1} \frac{2000 \text{ K}}{2.99 \times 1590 \text{ cm s}^{-2}} = 4.5 \times 10^7 \text{ cm}. \quad (16)$$

Finally, from (13)

$$\Delta R_{\text{ch}} \sim 4.5 \times 10^7 \text{ cm} \ln \sqrt{\frac{2\pi \times 1.32 \times 7.10 \times 10^9 \text{ cm}}{4.5 \times 10^7 \text{ cm}}} = 1.6 \times 10^8 \text{ cm} = 0.023R_{\text{Jupiter}}. \quad (17)$$

c. For HD 209458b

$$\frac{\Delta R_{\text{ch}}}{R} = \frac{0.023R_{\text{Jupiter}}}{1.32R_{\text{Jupiter}}} = 0.017. \quad (18)$$

This is insufficient to explain the $\sim 10\%$ oversized appearance of HD 209458b. Rather, the planet is likely puffed up because of heat trapped below an optically thick stratosphere.
Astronomy 4 (Cosmology)

In the 1940s, George Gamow suggested that a consequence of the Big Bang cosmological scenario is that nucleosynthesis could take place at the high densities and temperatures of the early Universe. Specifically, Gamow predicted the epoch of nucleosynthesis, the nucleon density at the epoch of nucleosynthesis, and the current temperature of the Universe by assuming a radiation-dominated cosmological model, i.e. a cosmological model for which the energy density of the Universe $\epsilon$ is given by the Stefan–Boltzman relation

$$\epsilon = aT^4,$$

where $a = 7.6 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$ is the radiation constant and $T$ is temperature. This work pre-dated the discovery of the cosmic microwave background radiation by almost 20 years.

a. (5 pts) Derive from the Friedman equation the relationship between temperature of the Universe $T$ and time $t$ for a flat, radiation-dominated cosmological model in terms of the radiation constant, the gravitational constant $G = 6.7 \times 10^{-8} \text{ cm}^3 \text{ g}^{-1} \text{ s}^{-2}$, and the speed of light $c = 3.0 \times 10^{10} \text{ cm s}^{-1}$.

b. (6 pts) Gamow recognized that nucleosynthesis would begin with the formation of deuterium via the reaction

$$p + n \rightarrow D + \gamma$$

at a temperature around $T = 10^9 \text{ K}$. Use the result of Part A to estimate the age of the Universe at the epoch of nucleosynthesis $t_{\text{nuc}}$.

c. (3 pts) Gamow further recognized that the necessary condition for significant nucleosynthesis is that the product of the reaction rate of the reaction of Equation (1) and $t_{\text{nuc}}$ is of order unity, i.e. that

$$n_{\text{nuc}}\langle \sigma v \rangle t_{\text{nuc}} \sim 1,$$

where $n_{\text{nuc}}$ is the nucleon density at the epoch of nucleosynthesis and $\langle \sigma v \rangle$ is the average product of the cross section of the reaction of Equation (1) and the nucleon speed. Taking $\langle \sigma v \rangle = 10^{-21} \text{ cm}^3 \text{ s}^{-1}$, estimate the nucleon density at $t_{\text{nuc}}$.

d. (3 pts) Use the current age of the Universe $t_0 = 13.7 \times 10^9 \text{ yr}$ and the result of Part A to estimate (1) the current temperature of the Universe $T_0$ and (2) the current nucleon density of the Universe $n_0$.

e. (5 pts) Of course, the Universe contains matter as well as radiation. Assuming that the current Universe is matter dominated (rather than radiation dominated), is the current temperature of the Universe higher or lower than the value obtained in Part D? Is the current nucleon density higher or lower than the value obtained in Part D? Justify your answer.
Solution:

a. Consider the continuity equation for radiation

\[ \epsilon R^4 = \epsilon_0 R_0^4 \]  \hspace{1cm} (21)

together with the Stefan–Boltzman relation

\[ \epsilon = a T^4 \]  \hspace{1cm} (22)

to derive the relationship between \( R \) and \( T \) as

\[ R = \frac{\epsilon_0 R_0}{a T} \]  \hspace{1cm} (23)

and

\[ dR = -\frac{\epsilon_0 R_0}{a T^2} dT. \]  \hspace{1cm} (24)

Integrate the Friedmann equation for a flat cosmological model

\[ \dot{R}^2 = \frac{8 \pi G \epsilon}{3 c^2} R^2 \]  \hspace{1cm} (25)

with Equations (4), (5), and (6) and the boundary condition \( R = 0 \) at \( t = 0 \) to get

\[ T = \left( \frac{3 c^2}{32 \pi G a} \right)^{1/4} t^{-1/2}. \]  \hspace{1cm} (26)

b. Solve Equation (8) for \( t \) to get

\[ t = \left( \frac{3 c^2}{32 \pi G a} \right)^{1/2} T^{-2}. \]  \hspace{1cm} (27)

Evaluation Equation (9) at \( T = 10^9 \) K to get

\[ t_{\text{nuc}} = 230 \text{ s}. \]  \hspace{1cm} (28)

c. Solve Equation (2) for \( n_{\text{nuc}} \) and evaluate at \( t_{\text{nuc}} = 230 \) s to get

\[ n_{\text{nuc}} = 8.7 \times 10^{16} \text{ cm}^{-3}. \]  \hspace{1cm} (29)
d. Evaluate Equation (8) at $t = 13.7 \times 10^9$ yr to get

$$T_0 = 23 \text{ K.} \quad (30)$$

Matter density scales as

$$n = n_0 \left( \frac{R_0}{R} \right)^3 = n_0 \left( \frac{T}{T_0} \right)^3. \quad (31)$$

Solve Equation (12) for $n_0$ and evaluate at $T = 10^9$ K and $T_0 = 23$ K to get

$$n_0 = 1.1 \times 10^{-6} \text{ cm}^{-3}. \quad (32)$$

e. Consider the continuity equation for matter

$$\rho R^3 = \rho_0 R_0^3. \quad (33)$$

Integrate Equation (7) with Equation (15) and the boundary condition $R = 0$ at $t = 0$ to get

$$R \propto t^{2/3} \quad (34)$$

in a matter-dominated Universe, compared with

$$R \propto t^{1/2} \quad (35)$$

in a radiation-dominated Universe. Since $T \propto R^{-1}$ and $n \propto R^{-3}$, the values of both $T_0$ and $n_0$ are lower than the values predicted in Part D if the current Universe is matter dominated rather than radiation dominated.

**AMO 1**

Consider the hyperfine structure of an alkali atom in an external magnetic field $B_0$, which is described by

$$V_{HFS} \approx A \mathbf{I} \cdot \mathbf{J} - \mathbf{\mu}_J \cdot \mathbf{B}_0 \quad (36)$$

where $\mathbf{I}$ is the nuclear spin, $\mathbf{J}$ the electronic angular momentum, and $\mathbf{\mu}_J = -g_J \mu_B \mathbf{J}$ the electronic magnetic moment (we neglect the coupling of the much smaller nuclear magnetic moment to the external field). The hyperfine structure of the ground level $n^2S_{1/2}$ in the field is illustrated in the figure.

a. (7 pts) Use the information in the figure to deduce the value of $I$, and calculate the energy shifts at zero field (in units of A).
b. (8 pts) Show explicitly that in the weak-field limit, the three lower levels have the same (uniform) separation between them as the five upper ones. Furthermore show that in the strong-field limit, the lower four levels have the same uniform separation between them as the four upper ones (hint: you can use the vector model for this).

c. (5 pts) For both limits, label all the levels with the appropriate quantum numbers.

Solution:

a. \(n^2S_{1/2}\) means \(L = 0, J = 1/2\). Coupling to total angular momentum \(F = J + I\), with \(|F| = \sqrt{F(F+1)}\) and \(F = I \pm 1/2\). 5-fold degeneracy of upper HF level at zero field means \(F = 2\), and therefore \(I = 3/2\).

Energy splitting at zero field from \(I \cdot J = \frac{1}{2} [F(F + 1) - I(I + 1) - J(J + 1)]\); one obtains \(V_{HFS} = 3A/4\) for \(F = 2\), and \(V_{HFS} = -5A/4\) for \(F = 1\).

b. Weak field limit \(|\vec{\mu}_J \cdot \vec{B}_0| \ll AI \cdot J\). In this case \(I\) and \(J\) rotate rapidly around \(F\), which precesses around \(B_0\). Interaction with external field is \(\propto J \cdot B_0 = \left[\frac{(J \cdot F)}{(F+1)}\right] \cdot B_0\) where \(J \cdot F = I \cdot J + |I|^2 = \frac{1}{2} (F(F + 1) + J(J + 1) - I(I + 1))\). This means \(\Delta V_{HFS} = \mu_B g_F m_F B_0\), with \(g_F = g_J \frac{F(F+1)+J(J+1)-I(I+1)}{2F(F+1)} = g_J/4(F = 2), -g_J/4(F = 1)\), with \(g_J = 2\). As a result, all levels are equally spaced. Good quantum numbers: \(F, m_F\).

Strong field limit \(|\vec{\mu}_J \cdot \vec{B}_0| \gg AI \cdot J\). In this case \(J\) precesses around \(B_0\). \(I\) does not separately precess around \(B_0\) b/c of small nuclear magnetic moment. However \(I\) precesses around \(J\), and hence around \(B_0\). This means \(m_I, m_J\) are good quantum numbers. Interaction energy \(V_{HFS} = g_J \mu_B m_J B_0 + A m_J m_J\).

Upper manifold: \(m_J = 1/2\), lower manifold: \(m_J = -1/2\), in both manifolds have spacing \(A m_J\) between levels.

[The much longer non-vector model solution of 2. still needs to be typed up].
c. (From bottom to top:) Weak field: $F = 1(m_F = 1, 0, -1), F = 2(m_F = -2, -1, 0, 1, 2)$. Strong field: $m_J = -1/2(m_I = 3/2, 1/2, -1/2, -3/2); m_J = 1/2(m_I = -3/2, -1/2, 1/2, 3/2).

AMO 2

Equilibrium and sound in a Bose-Einstein condensate.

An atomic Bose-Einstein condensate (BEC) with wave function $\Psi$ in a potential $V$ fulfills the Schrödinger equation

$$
-i\hbar \frac{d\Psi}{dt} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V + g|\Psi|^2 \right) \Psi,
$$

where $g = 4\pi\hbar^2 a/m$ and $|\Psi|^2 = n(\vec{r})$ is the atomic density. In the following, consider a BEC containing $N_0 = 5 \times 10^5 \ 87\text{Rb}$ atoms ($a = 5 \text{nm}$) in a spherically symmetric harmonic trapping potential $V = \frac{1}{2}m(\omega r)^2$ with trap frequency $\omega/(2\pi) = 50 \text{Hz}$.

a. (5 pts) What is the origin of the mean field term $g|\Psi|^2$? What are the conditions for which one can describe a cloud of atoms by a macroscopically populated wave function $\Psi$?

b. (5 pts) Assuming that all atoms are in the ground state of the trap, give an estimate for the ratio between kinetic energy and the interaction energy?

c. (5 pts) Neglecting the kinetic energy, show that the wave function $\Psi = \psi(\vec{r})e^{-i\mu t/\hbar}$ has an inverted parabolic density $n(\vec{r})$ with maximum $n(0) = \mu/g$ and radial extent $R = \sqrt{2\mu/(m\omega^2)}$. Using $N_0 = \int n(\vec{r})d^3r$, derive an expression for the chemical potential $\mu$ and calculate it in units of $k_B \times nK$.

d. (5 pts) Now consider the case $V = 0$ for which $|\psi|^2 = \mu/g$ is constant. Show that a small perturbation $\delta\Psi = (ue^{iqx-\omega t}) + v^*e^{-iqx-\omega t})e^{-i\mu t/\hbar}$ to the wave function, $\Psi + \delta\Psi (|\delta\Psi| \ll |\Psi|)$, results in

$$
-i\hbar \frac{d\delta\Psi}{dt} = \frac{\hbar^2 q^2}{2m} \delta\Psi + 2g|\psi|^2 \delta\Psi + g\psi^2 \delta\Psi^*.
$$

By equating terms with the same time dependence, this leads to two coupled equations for the amplitudes $u$ and $v$. Derive the dispersion relation $\omega(q)$, and show that for small momenta $q$ the excitation is a sound wave with velocity $v_s = \sqrt{\mu/m}$. How large is $v_s$ (with $\mu$ as in (3))? How would you experimentally excite such sound waves?

Solution:
a. The term $g|\Psi|^2$ originates from $s$-wave collisions between atoms. Mean-field description is valid as long as the product $na^3 \ll 1$ (weakly interacting), and the number of atoms $N_0 \gg 1$.

b. Kinetic energy $K \sim N_0\hbar\omega$, interaction energy $U \sim N_0gn$ with $n \approx N_0 l_{ho}^{-3} = N_0 \left(\frac{m\omega}{\hbar}\right)^{3/2}$. Using the expression for $g$ yields $U/K \sim 4\pi N_0 a/l_{ho}$. With $l_{ho} = 1.2 \mu m$, obtain $U/K \sim 10^{14}$.

c. Expressions for $n(r)$ and $R$ follow directly from eq. (1). Performing straightforward integration yields

$$
\mu = \left[\frac{15}{8\pi} N_0 g \left(\frac{m\omega}{2}\right)^{3/2}\right]^{2/5} = \frac{15^{2/5}}{2} \left(\frac{N_0 a}{l_{ho}}\right)^{2/5} \hbar \omega = k_B \times 68 \text{nK}.
$$

d. Eq. (2) follows after substituting $\Psi + \delta \Psi$ into eq. (1), cancelling out unperturbed terms for $\Psi$, and neglecting terms of order $\delta \Psi^2$. With prescription, and using $g|\psi|^2 = \mu$ and $\psi^2 = |\psi|^2 e^{-2i\mu t/\hbar}$, then obtain coupled equations

$$
0 = (E - \hbar \omega) u + \mu v \quad (39)
$$

$$
0 = \mu u + (E + \hbar \omega) v \quad (40)
$$

where $E = (\hbar q)^2/2m + \mu$. Solving this yields $\omega(q) = \frac{1}{\hbar} \sqrt{\left(\frac{\hbar q}{2m}\right)^2 + 2\mu}$. For $q \to 0$ have linear dispersion $\omega(q) \approx q\sqrt{\mu/m}$, with speed of sound $v_s = \omega/q = \sqrt{\mu/m} = 2.5 \text{ mm/s}$. Excite sound by locally perturbing the density of the condensate, e.g. with focused laser beam.

Condensed Matter 1

A two-dimensional electron gas can be described by the Hamiltonian

$$
H = \frac{p_x^2 + p_y^2}{2m^*} + \frac{\alpha_R}{\hbar} (p_y \sigma_x - p_x \sigma_y).
$$

Here $m^*$ is the effective mass of an electron moving in a 2D layer, $\sigma_{x,y}$ are Pauli matrices acting on electron’s spin and $\alpha_R$ is some constant known as the coupling constant of Rashba’s spin-orbit interaction. This coupling constant is proportional to the fine structure constant and can be considered small in this problem. The Rashba term breaks the 2D parity of the system $\sigma \to -\sigma$ and $p \to -p$, and its origin can be traced back to the fact that for the 2D gas confined to the surface of the 3D semiconductor the 3D parity is broken explicitly by the average confining electric field $E$ perpendicular to the 2D layer.

a. (5 pts) For $\alpha_R = 0$ the ground state of the 2D electron gas is the completely filled Fermi sphere with Fermi momentum $p_F = \sqrt{2m^*\mu}$, where $\mu$ is the chemical
potential of the gas. Express the chemical potential $\mu$ and the total energy of the ground state per particle $E/N$ in terms of the 2D concentration $n$ of the electron gas.

b. (5 pts) Diagonalize the Hamiltonian for $\alpha_R \neq 0$. What is the ground state for a model with $\alpha_R \neq 0$? Sketch the “Fermi sphere picture” for this ground state.

c. (5 pts) Show that the relations derived in a. do not change to the first order in $\alpha_R \neq 0$.

d. (5 pts) Estimate the energy difference between the “spin-up” and “spin-down” states close to the Fermi surface (i.e., spin splitting of the energy $\frac{p^2}{2m^*}$) if the 2D concentration of electron gas is $n = 2 \times 10^{12} \text{cm}^{-2}$ and $\alpha_R = 1.5 \times 10^{-11} \text{eV} \cdot \text{m}$.

Solution:

a. The concentration of electrons is

$$n_{\uparrow, \downarrow} = \frac{\pi p^2_F}{(2\pi \hbar)^2} = \frac{m^*}{2\pi \hbar^2} \mu.$$  \hfill (41)

The total concentration is

$$n = 2 \frac{\pi p^2_F}{(2\pi \hbar)^2} = \frac{m^*}{\pi \hbar^2} \mu$$  \hfill (42)

and

$$\mu = \frac{\hbar^2}{m^*} \pi n.$$  \hfill (43)

Similarly for the energy per unit area we have

$$\frac{E}{A} = 2 \int \frac{d^2p}{(2\pi \hbar)^2} \frac{p^2}{2m^*} = 2 \frac{p_F^4}{16\pi \hbar^2 m^*} = \frac{2(2m^* \mu)^2}{16\pi \hbar^2 m^*} = \frac{\pi \hbar^2}{2m^*} n^2$$  \hfill (44)

or per particle

$$\frac{E}{N} = \frac{\pi \hbar^2}{2m^*} n.$$  \hfill (45)

b. The Hamiltonian can be rewritten as

$$H = \frac{p^2}{2m^*} - \frac{\alpha_R}{\hbar} ( \sigma_p = -p_y + \sigma_y p_x ) = \frac{p^2}{2m^*} - \frac{\alpha_R}{\hbar} p \cdot \hat{\sigma},$$

where we used polar coordinates in momentum plane $(p_x, p_y) \rightarrow (p, \theta)$ and a unit vector $\hat{\theta} = (-\sin \theta, \cos \theta)$.

The eigenvalues of this Hamiltonian are obviously $\frac{p^2}{2m^*} \pm \frac{2\alpha_R}{\hbar} p$ for spin along and opposite to $\hat{\theta}$ ($\langle \sigma \rangle = \pm \hat{\theta}$) respectively.
The Fermi surface is split into two Fermi surfaces for \( \langle \sigma \rangle = \pm \hat{\theta} \) with Fermi momenta \( p_\pm \) defined respectively by

\[
\frac{p_\pm^2}{2m^*} + \frac{\alpha_R}{\hbar} p_\pm = \mu. \tag{46}
\]

This is the picture of the ground state in this case:

c. Solving (46) in linear order in \( \alpha_R \) we have

\[
p_\pm^2 = 2m^*\mu \pm \frac{\alpha_R}{\hbar} 2m^* p_\pm \approx 2m^*\mu \pm \frac{\alpha_R}{\hbar} 2m^* \sqrt{2m^*\mu} \tag{47}
\]

We see that

\[
n = n_+ + n_- = \frac{p_+^2 + p_-^2}{4\pi\hbar^2} = \frac{m^*}{\pi\hbar^2} \mu + O(\alpha_R^2) \tag{48}
\]

Similarly the total energy

\[
\frac{E}{A} = \frac{p_+^4 + p_-^4}{16\pi\hbar^2 m^*} \approx \frac{2(2m^*\mu)^2}{16\pi\hbar^2 m^*} = \frac{\pi\hbar^2}{2m^*} n^2 \tag{49}
\]

and

\[
\frac{E}{N} = \frac{\pi\hbar^2}{2m^*} n + O(\alpha_R^2). \tag{50}
\]

d. From (47) we have

\[
\delta\epsilon = \frac{p_+^2 - p_-^2}{2m^*} \approx 2 \frac{\alpha_R}{\hbar} \sqrt{2m^*\mu} \approx 2\alpha_R \sqrt{2\pi n} \approx 11 \text{ meV} \tag{51}
\]

The term breaks the symmetry with respect to independent rotations of spin and space. The Hamiltonian is still invariant with respect to simultaneous rotations of spin and orbital parts of the wave function. The symmetry breaking originates from relativistic corrections (spin-orbit interaction). Namely, the relativistic correction to the Schrödinger equation coming from the Dirac equation has a spin-orbit term

\[-\frac{e\hbar}{4m^*c^2} \sigma \cdot \mathbf{E} \times \mathbf{p}, \]

where \( \mathbf{E} \) is an external electric field. We notice that this term is parity invariant (\( \sigma \rightarrow \sigma \) and \( p \rightarrow -p \), \( E \rightarrow -E \)). The Rashba term breaks 2D parity symmetry \( \sigma \rightarrow \sigma \) and \( p \rightarrow -p \). To have an effective theory of 2D electron gas with inversion symmetry broken terms one should have the 3D parity broken explicitly. This is what happens when the 2D gas is confined to the surface of the
3D semiconductor. (One can also think that the average confining $E$ perpendicular to the 2D layer is present). For $\mathbf{E} = E\hat{z}$ the term $-\frac{e\hbar}{4m^*c^2}\mathbf{\sigma} \cdot \mathbf{E} \times \mathbf{p}$ becomes of the Rashba type.

Condensed Matter 2

Most of us learned in freshman physics that a metal film of small thickness $t$ placed in a magnetic field $B$ normal to its surface (see figure above) and carrying a uniform current $I$ along the film (in the $x$ direction) will develop a voltage $V_H$, transverse to the current flow, which is called the Hall voltage, $V_H = R_H IBE$, where $R_H = (1/(ne))$ and $n$ is the sheet density of electrons, i.e. the number of electrons in a 1m×1m square of the film. Now consider a sample of the same shape that is an intrinsic (no doping or impurities) semiconductor instead of a metal film. To make things easy, we will take the sample to be isotropic and include the band structure effects through the effective masses of the electrons and holes giving mobilities of $\mu_e$ and $\mu_h$ respectively. Assuming the temperature is high enough that there is an appreciable density $n$ of electrons thermally excited to the conduction band, derive a formula for $R_H(n, \mu_e, \mu_h)$ that will give $V_H$ to first order in $B$.

Solution:

An electron or hole in crossed electric and magnetic fields has a drift velocity $\mathbf{v}$, due to the forces of the fields and scattering, given by

$$ \mathbf{v} = \mu (\mathbf{E} + \mathbf{v} \times \mathbf{B}) $$

where $\mu = e\tau/m^*$ and is taken to be positive for holes and negative for electrons. Writing out the components of this gives, e.g. for electrons

$$ v_x = \mu_e (E_x + Bv_y) $$

and

$$ v_y = \mu_e (E_y - Bv_x) $$

where the magnetic field is $\mathbf{B} = B\hat{z}$. Solving for $v_x$ and $v_y$ gives, to first order in $B$,

$$ v_x = \mu_e (E_x + \mu_e B E_y) $$
\[ v_y = \mu_e (E_y - \mu_e B E_x) \]

In contrast to the single carrier case of a simple metal, the boundary condition is now \( j_y = ne (v_{yh} - v_{ye}) = 0 \), so \( v_{yh} = v_{ye} \), taking the density of holes and electrons to be equal since the semiconductor is undoped. This gives for the Hall field

\[ E_y = \frac{\mu_h^2 - \mu_e^2}{\mu_h - \mu_e} B E_x \]

since \( j_x = ne (\mu_h - \mu_e) E_x \) so, writing \( E_y \) in terms of \( B \) and \( j_x \) gives

\[ E_y = \frac{\mu_h + \mu_e}{(\mu_h - \mu_e) ne} B j_x \]

or

\[ V_y \equiv V_H = R_H B I \]

where

\[ R_H = \frac{\mu_h + \mu_e}{(\mu_h - \mu_e) ne} = \frac{1}{ne} \frac{|\mu_h| - |\mu_e|}{|\mu_h| + |\mu_e|} \]

**Nuclear 1**

The goal of the RHIC experiments is to create and to study the properties of QCD plasmas. Throughout this problem set the Boltzmann constant \( k_B = 1 \), but keep all factors of \( \hbar \) and \( c \) explicit.

a. (4 pts) Treat hot QCD plasma as a massless ideal gas of gluons at temperature \( T \). Determine the number of gluons per unit volume, counting spin and color as internal degrees of freedom. You do not need to evaluate the final integrals, but provide an estimate of the final result in terms of the temperature and fundamental constants \( \hbar \) and \( c \).

b. (4 pts) One scattering process in weakly coupled plasma is elastic gluon scattering \( gg \leftrightarrow gg \). Draw the Feynman diagrams describing this process. Use dimensional reasoning to give an estimate for the typical \( gg \leftrightarrow gg \) cross section in plasma in terms of the temperature and the strong coupling constant. Assume that all momenta are of order the (temperature)/\( c \).

c. (3 pts) Using parts (1) and (2), estimate the gluon the mean free path \( \ell_{mfp} \), and the typical time between collisions \( \tau_c \) at a given temperature.

Now consider two streams of the gluon gas separated by a small transition layer of width \( \Delta y \) (see below). The upper stream moves at a slow speed \( \Delta v^x \ll c \), and the
lower stream is at rest. The force per area pulling the lower stream forward (or the upper stream backward) is determined by the shear viscosity and the velocity gradient

\[ \frac{F_x}{A} = \eta \frac{\Delta v_x}{\Delta y}. \]

\( \Delta y \) is vastly larger than the mean free path. Thus, the force between the two streams arises as the excess momentum carried by the gas at the top stream diffuses to the bottom.

d. (3 pts) A typical gluon near the top has slightly more \( x \) momentum than a typical gluon near the bottom. Estimate this difference.

e. (3 pts) Estimate the time it takes for this momentum to diffuse across the transition layer in terms of \( \ell_{\text{mfp}} \) and \( \tau_c \).

f. (3 pts) Estimate the momentum transferred per unit time from the upper stream to the lower stream, and use this result to estimate the shear viscosity \( \eta \) in terms of the temperature and strong coupling constant. \( \eta/n \) where \( n \) is the density of

Solutions:

a. Gluons, two spins, eight colors \( g_s = 16 \).

\[
\begin{align*}
n &= g_s \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{e^{p/T} - 1} \\
 &= g_s T^3 \int \frac{d^3x}{(2\pi)^3} \frac{1}{e^x - 1} \\
 &= g_s \left( \frac{T}{\hbar c} \right)^3 \frac{\zeta(3)}{\pi^3} \\
 &\approx g_s \left( \frac{T}{\hbar c} \right)^3 0.12
\end{align*}
\]
b. Dimensional analysis + counting $\alpha_s$.

$$
\sigma \sim \alpha_s^2 \left( \frac{hc}{T} \right)^2
$$

c. Then

$$
\ell_{\text{mfp}} \sim \frac{1}{n\sigma} \sim \frac{hc}{\alpha_s^2 T}
$$

The gluons move with speed of light:

$$
\tau_c \sim \frac{\ell_{\text{mfp}}}{c} \sim \frac{\hbar}{\alpha_s^2 T}
$$

d. The temperature is blue shifted

$$
T_{\text{top}} \sim T \sqrt{\frac{1 + u/c}{1 - u/c}} \approx T + Tu^x/c
$$

Thus

$$
\Delta p \simeq \Delta E/c \simeq \frac{Tu^x}{c^2}
$$

e. The process is diffusive. Over a time $\tau$ there are $\tau/\tau_c$ hops of distance $\ell_{\text{mfp}}$, which add in square

$$
(\Delta y)^2 = (\ell_{\text{mfp}})^2 \frac{\tau}{\tau_c}
$$

Thus

$$
\tau = \frac{(\Delta y)^2}{c^2 \tau_c}
$$

f. Putting together the pieces. The momentum transfer over time $\tau$ is the total number of gluons $\times$ the moment transfer per $\tau$ per gluon

$$
\frac{\Delta P^x}{\tau} = nA \Delta y \frac{\Delta p}{\tau} = nA \Delta y \frac{Tu^x}{c^2} \frac{c^2 \tau_c}{(\Delta y)^2} = nT \tau_c A \frac{\Delta u^x}{\Delta y}
$$

So the force per area

$$
\frac{F^x}{A} \simeq \frac{\Delta P^x}{A\tau} \simeq nT \tau_c \frac{\Delta u^x}{\Delta y}
$$

Thus the shear viscosity is of order

$$
\eta \sim \left( \frac{T}{hc} \right)^3 \frac{\hbar}{\alpha_s^2}
$$
Nuclear 2

In a recent colloquium Steve Peggs discussed the concept and the status of Thorium energy amplifiers. In this nuclear reactor type neutrons from a spallation source bombard a $^{232}$Th target. (See figures below when answering questions.)

a. (5 pts) Sketch the "fuel cycle", i.e. the decay chain that eventually leads to the nucleus X that actually fissions in this reactor? The sequence of reactions are neutron capture, beta-decay, beta-decay.

b. (5pts) Estimate the energy gained per fission process assuming fission to equal mass daughters.

A fundamental advantage of Thorium amplifiers is that they can be operated with a non critical fuel mass and thus are safe from meltdowns of the reactor core.

a. (5 pts) How is the critical mass defined and what factors determine it?

b. (5 pts) Explain why Thorium amplifiers work even below the critical mass and how a meltdown of the core can be prevented.
Solutions:

a. \( n + {^{232}}\text{Th} \rightarrow {^{233}}\text{Th} (\beta \text{ decay}) \rightarrow {^{233}}\text{Pa} (\beta \text{ decay}) \rightarrow {^{233}}\text{U} \) (Aside: \( {^{233}}\text{U} \) has a large thermal neutron fission cross section similar to \( {^{235}}\text{U} \))

b. Read off graph of binding energy. Binding energy from \( 233 \rightarrow 116 \) changes by \( \sim 1 \text{ MeV/A} \). That gives about 230 MeV per fission. Done more accurately the estimate is 10-15% smaller.

c. The critical mass is the amount of fuel it takes to sustain a chain reaction. There are two basic components that contribute: number of neutrons produced per fission and number of neutrons lost through surface before they interact. For \( {^{235}}\text{U} \) the critical mass is \( \sim 52 \text{ kg} \) or a sphere of 17 cm diameter (don’t need to know this). In practice needs to be enriched from 0.7% in natural Uranium to at least 2-3% to be usable as fuel in a water moderated reactor. Thus much larger amounts of fuel are needed.

d. In conventional fission reactors the fission process itself will provide the neutrons for further induced fission reactions. In order to maintain a chain reaction one needs to provide a sufficient mass of fuel, i.e. the critical mass. In the case of \( {^{235}}\text{U} \) one needs to enrich the \( {^{235}}\text{U} \) if water is used as a neutron moderator. The Thorium amplifier provides neutrons from spallation source. As soon as the neutron flux is stopped the chain reaction will end.
High Energy 1

Charged π’s can decay via the weak interaction.

a. (2 pts) List the possible two body decay modes of the charged pion, π−, and denote what is the dominant decay mode.

b. (9 pts) Calculate the integrated two body Lorentz invariant phase space for a pion at rest with mass \( m_\pi \), decaying into a massless anti-neutrino and a negatively charged lepton with mass \( m_L \).

c. (3 pts) Based on your calculation in part b, one charged lepton decay mode has a much larger phase space available than the other(s). Explain why the structure of the weak interaction causes the charged lepton mode with the smallest available phase space to have a decay rate that is approximately \( \sim 10^4 \) times larger than the other(s).

d. (6 pts) Consider now the \( \pi^0 \) meson. What is its main decay mode? How can we deduce from its decay that there are 3 colors? Draw the relevant Feynman diagram. Estimate the ratio of the \( \pi^- \) and \( \pi^0 \) lifetime by using dimensional arguments and taking the ratio of the corresponding coupling constants (use that \( Gm^2_p \sim 10^{-5} \)).

Solution:

a. To conserve lepton number, charge and the fact that the pion is the lightest charged hadronic state means the only possible 2 body decays of the pion are given by:

\[
\pi^- \rightarrow \mu^- \bar{\nu}_\mu \\
\pi^- \rightarrow e^- \bar{\nu}_e
\]

These are the dominant decay modes. There is no mode for the \( \tau \) since it is heavier than the pion.

b. The phase space integral is:

\[
\int d\Omega_2 = \int \frac{d^3p_l}{(2\pi)^32E_l} \frac{d^3p_\nu}{(2\pi)^32E_\nu} (2\pi)^4 \delta(-4)(p_\nu - p_\mu - p_l)
\]

\[
= \int \frac{dp_l p_l^2 d\Omega}{(2\pi)^32E_l2E_\nu} (2\pi) \delta(m_\pi - E_\nu - E_l)
\]

\[
= \int d\Omega \frac{p_l^2}{16\pi^2 E_l E_\nu} \left( \frac{p_l}{E_l} - \frac{p_\nu}{E_\nu} \right)^{-1}
\]

\[
= \int d\Omega - \frac{|p_l|}{16\pi^2 m_\pi} = \frac{1}{8\pi} \left( 1 - \frac{m_L^2}{m_\pi^2} \right)
\]
c. The pion has spin 0, while the $\mu$ and $\bar{\nu}$ are spin $1/2$. In the rest frame of the pion the anti-neutrino and muon are back to back, and their spin also has to be opposite one another to conserve angular momentum. However, the weak interactions are chiral and produce LH particles and RH antiparticles in the massless limit. Therefore in the massless limit the amplitude for this process would vanish. To generate this process via the weak interactions the amplitude must be proportional to the mass of the lepton. (in this case the square of the mass), and $m_\mu^2/m_e^2 \sim 10^4$.

d. 

\[ \begin{array}{c}
\pi^0 \\
\hline
u \text{ or } d \text{ quark} \\
\gamma \\
\gamma
\end{array} \]

The main decay mode is $\pi^0 \to \gamma + \gamma$. If one only takes into account the contribution of one up quark and one down quark in the loop, the decay rate comes out a factor 3 too small. The decay rate of $\pi^-$ is proportional to $G^2m_{\pi^-}^2$ and that of $\pi^0$ is proportional to $\alpha^2m_{\pi^0}$. The ratio is $\tau(\pi^0)/\tau(\pi^-) = [Gm_{\pi}^2/\alpha]^2 = [10^{-5}(m_\pi/m_p)^2/\alpha]^2 = 2.7 \cdot 10^{-8}$. The experimental values are $\tau(\pi^0) = 0.8 \cdot 10^{-16}$ s and $\tau(\pi^-) = 2.6 \cdot 10^{-8}$ s, and yield for the ratio $3.0 \cdot 10^{-9}$, which is not too far off, given the crude nature of the estimate.
A simple theory of the weak and electromagnetic forces describes the interaction of fermions with gauge bosons forming a weak isospin triplet (the SU(2) components) and with another boson forming a weak hypercharge singlet (the U(1) component). This theory gives rise to a Lagrangian with terms like

\[ g_W \left( \frac{1}{\sqrt{2}} j^\pm j_\mu^\pm W_\mu^+ + \frac{1}{\sqrt{2}} j^- j^- W^0 \right) + \frac{g'Y}{2} j_\mu^Y B^0_\mu \]

in which \( g_W \) and \( g' \) are the isospin and hypercharge couplings, \( W^\pm \) and \( W^0 \) are the weak isospin gauge fields and \( B^0 \) is the hypercharge field. The \( j \) terms involve fermions and Dirac matrices, and the details of these are not needed for this problem. Nature, however, does not correspond to this simple model.

a. (7 pts) Recast this Lagrangian in terms of the physical gauge fields \( W^\pm, Z \) and the photon \( A \) and the weak mixing angle \( \theta_W \). What are the coupling constants of the gauge fields \( Z \) and \( A \) to the electron and neutrino? (Write the physical couplings in terms of \( g_W, g' \) and \( \theta_W \).)

b. (7 pts) What measurements are used to fix the three parameters in the result of part (a). Briefly describe how the measurements are carried out. Specifically mention what is detected and how is that related to the model.

c. (6 pts) Another consequence of this mixing is a relation between the masses of the \( Z^0 \) and \( W^\pm \) bosons. What is the relation? Explain how the masses are measured.

Solution:

a. Set \( Z^\mu = W^\mu_0 \cos \theta_W - B^\mu \sin \theta_W \) and \( A^\mu = W^\mu_0 \sin \theta_W + B^\mu \cos \theta_W \) and substitute this into the original expression above. Group terms to identify the pure \( Z \) and pure \( A \) terms. So, at this level the \( Z \) term becomes \((g_W \cos \theta_W j^0_\mu - \frac{g'}{2 \sin \theta_W} j^Y_\mu) Z^\mu \) and the \( A \) term becomes \((g_W \sin \theta_W j^0_\mu + \frac{g'}{2 \cos \theta_W} j^Y_\mu) A^\mu \). The \( W^\pm \) terms are unchanged.

The \( A \) field is the photon. Defining the purely EM Lagrangian as \( e j^em A^0_\mu \) and the current relation \( j^em = j^0 + j^Y/2 \) and substituting this into the above gives the photon coupling to fermions as \( e = g_w \sin \theta_W = g' \cos \theta_W \). Correspondingly, this gives the \( Z \) current coupling to fermions as \( g_Z (j^3_\mu - \sin^2 \theta_W j^em_\mu) \) with \( g_Z = e/(\cos \theta_W \sin \theta_W) \).
b. Accept any set of three measurements which give \( g_W, g_e, g_Z \) and/or \( \theta_W \). Prefer modern experiments over older (e.g. \( W \) decay width over beta decay). For masses discuss either direct production in \( e^+e^- \) for which either (e.g. dimuon) rates vs. \( \sqrt{s} \) are measured to determine the total width or the peak cross section or beta decay or DIS. Talk about what the final state actually contains.

c. The additional relation is \( M_W = M_Z \cos \theta_W \). The \( Z \) mass is measured either by reconstructing the mass in dilepton decays \( Z \rightarrow ee \) or \( Z \rightarrow \mu\mu \) from the measured lepton momenta or by measuring the frequency of such decays as a function of the center of mass energy in \( e^+e^- \) collisions. The \( W \) mass is measured either by reconstructing the transverse mass \( M_T = 2\sqrt{p_T^e p_T^\nu (1 - \cos \delta)} \) in \( W \rightarrow e\nu \) decay (or in \( W \rightarrow \mu\nu \) decay) or by a measurement of the \( WW \) production rate as a function of center of mass energy in \( e^+e^- \) collisions. In the transverse mass expression, \( p_T^e \) is the electron momentum component perpendicular to the beam, \( p_T^\nu \) is the missing momentum component perpendicular to the beam and \( \delta \) is the opening angle between the two. The missing momentum is calculated as the negative of the vector sum of all measured momenta. (The neutrino cannot be directly detected.)