General Instructions: Twelve problems are given; you should do any four. If you do more than four problems, you must choose which four should be graded, and only submit those four. Only two problems can be from the same subject area, except if astronomy is chosen in which case all four problems can be from astronomy.

Each problem counts 20 points, and the solution should typically take less than 45 minutes.

Some of the problems spill onto two pages. Make sure you do all the parts of each problem you choose.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use a one page help sheet, a calculator, and with the proctor’s approval, a foreign language dictionary. No other materials may be used.

Some potentially useful information:

- Proton mass: \( m_p = 1.67 \times 10^{-24} \text{gm} \)
- Electron mass: \( m_e = 9.11 \times 10^{-28} \text{gm} \)
- Hydrogen atomic mass is 1.00794 amu
- Helium atomic mass is 4.002602 amu
- 1 amu is \( 1.66 \times 10^{-27} \text{kg} \)
- 1 eV = \( 1.6 \times 10^{-19} \text{J} \)
- \( m_e c^2 \approx 0.5 \text{MeV} \)
- \( c = 2.998 \times 10^8 \text{ ms}^{-1} \)
- \( G = 6.673 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2} \)
- The solar luminosity is \( 3.85 \times 10^{26} \text{ W} \)
- The solar mass is \( 1.989 \times 10^{30} \text{kg} \)
- The solar radius is \( 7.00 \times 10^8 \text{ m} \)
- \( \hbar = 1.055 \times 10^{-34} \text{J} \)
- \( e = 1.602 \times 10^{-19} \text{C} \)
- \( k_B = 1.38 \times 10^{-23} \text{ J/K} \)
- \( \hbar c \approx 197 \text{ MeV-fm} \)
- Thomson cross section: \( \sigma_T = 6.65 \times 10^{-25} \text{ cm}^2 \)
- \( \sigma = 5.67 \times 10^{-5} \text{ erg cm}^{-2} \text{ K}^{-4} \)
Astronomy 1 (Stars)

There are limits to the masses of stars. In this problem you will estimate the highest and lowest mass stars that can exist on the hydrogen-burning main sequence.

A. The high mass limit: On the upper main sequence the luminosity \( L \) scales as roughly as the stellar mass \( M \) cubed \((L \propto M^3)\). For sufficiently high masses, radiation pressure will dominate over the gravitational forces holding the star together. The luminosity at which these two pressures are equal is called the Eddington luminosity \( (L_{Edd}) \).

a) (5 pts) Derive the Eddington luminosity in terms of the stellar mass and radius.
b) (3 pts) Scaling from the Sun, estimate the maximum stellar mass. State your answer in solar masses.
c) (2 pts) Speculate whether the maximum mass of stars may have changed, and why, as the universe has evolved.

B. The low mass limit: Stars are in hydrostatic equilibrium. This sets the central temperature. Note that on the lower main sequence mass \( M \) is proportional to radius \( R \), to a good approximation.

a) (3 pts) Show that the central pressure scales as \( M^2/R^4 \), and that the central temperature scales as \( M/R \).
b) (4 pts) In a sufficiently dense and cool gas, Fermi pressure due to degenerate electrons will contribute to the central pressure. Show that the Fermi pressure scales as \( \hbar^2 n_e^{\frac{5}{3}}/m_e \), where \( n_e \) and \( m_e \) are the number density and mass of the electrons, respectively. Hint: use the Heisenberg uncertainty principle to estimate the momentum of the electrons in terms of their density.
c) (1 pt) Why does a Fermi pressure-supported stellar core not support stable nuclear fusion?
d) (2 pts) The Brown Dwarf limit is the minimum mass that will support stable Hydrogen fusion. Estimate this mass. State all assumptions. In the Sun about 25% of the central pressure comes from degenerate electrons.

Solution:

A a) The pressure gradient outwards from radiation is given by \( dP/dr = L\rho\kappa/4\pi r^2 \), where \( L \) is the luminosity, \( \rho \) is the density, and \( \kappa \) is the opacity (units of \( \text{cm}^2/\text{gm} \)). This is balanced by the gravitational pressure gradient, \( dP/dr = -GM\rho/r^2 \). Solve for \( L \) to get \( L_{Edd} = 4\pi GcM/\kappa \). The opacity \( \kappa = \sigma_T/m_p \), where \( \sigma_T \) is the Thompson cross section of the electron and \( m_p \) is the mass of the proton. (The proton itself provides minimal cross section to photons. Conceptually, the radiation pressure acts on the electrons, gravity works on the protons, and electrostatic attraction is large enough to ensure they stay together.)

b) Plug in with \( L = L_\odot (M/M_\odot)^3 \). The maximum mass is about 150 \( M_\odot \).
c) Initially the universe was 90% H by number. Today it is less because of ongoing nucleosynthesis. There are therefore fewer nucleons per electron today. Hence the mean opacity is larger. Population III stars could have been much more massive.

B a) By hydrostatic equilibrium, \( \frac{dP}{dr} = -g \rho \), where \( \rho \) is the density. \( \frac{dP}{dR} \sim \frac{P_c}{R} \), where \( P_c \) is the central pressure and \( R \) is the stellar radius. \( g \sim GM/R^2; \rho \sim M/R^3 \). Therefore \( P_c \sim M^2/R^4 \).

\( P_c \sim \rho kT_c \), by the ideal gas law. Substitute \( P_c \sim M^2/R^4 \) and \( \rho \sim M/R^3 \) to find \( T_c \sim M/R \).

b) The electron pressure \( P_e = n_e p_e v_e \), where \( n_e \) is the electron number density, \( p_e \) is the momentum of the electrons, and \( v_e \) is the velocity of the electrons. The Heisenberg uncertainty principle can be stated as \( \Delta p_e \Delta x > \hbar \), where \( X \) is the position of the electron. Set \( \Delta x = n_e^{-1/3} \), the mean distance between electrons to get the mean momentum \( p_e \sim \hbar n_e^{1/3} \). Plug in to find \( P_e = \hbar n_e^{5/3}/m_e \).

c) When support is provided by degenerate electrons, there is no feedback on the temperature. If it falls, nuclear reaction rate decreases, and \( T \) decreases further, stopping the nuclear furnace.

d) The ratio of electron to total pressure, \( \frac{P_e}{P_c} \sim n_e^{5/3}/M^{-2} \), since \( M \sim R \). Also, \( n_e \sim M^{-2} \), so \( P_e/P_c \sim n_e^{2/3} \sim M^{-4/3} \). Arbitrarily assuming that \( P_g \) is unimportant when it provides less than 10% of the pressure support gives a brown dwarf mass limit of 0.078 \( M_\odot \).
Astronomy 2 (Galaxies, Interstellar Medium)

a) (2 pts) Name the galaxies other than the Milky Way that humans can see with the unaided eye if they know where to look.

b) (3 pts) The diversity of galaxy characteristics can be distilled down to 3 major Hubble types. What are these, and briefly describe their characteristics? What is the spectral type of our Galaxy?

c) (2 pts) When we admire the night sky with the unaided eye, the stars we see represent only a fraction of the baryonic matter in the universe. What are the major forms of the other baryonic matter and why don’t we see it?

d) (5 pts) Most of the matter in the universe is dark. Expressed as a fraction of the matter density required to close the universe, the so-called density parameter $\Omega$, the density parameter baryonic matter of the universe is about 0.04. The majority of matter in the universe is stuff that doesn’t absorb, emit, or scatter photons and is called dark matter.

Evidence for very large quantities of dark matter was derived by observations of clusters of galaxies in the 1930’s. Describe clearly the observations and the interpretation that led to this conclusion.

Problem continues on next page
e) (8 pts) The above figure shows the observed deviation, in magnitudes, of distant type Ia supernovae and gamma ray bursters (GRBs) from those expected in an empty Einstein-deSitter universe (green horizontal line). At high redshift there seems to be a systematic deviation from this cosmological model, in that the standard candles are too bright for their redshifts.

i Explain why type Ia supernovae and GRBs are expected to be “standard candles”.

ii Supernovae have spectra with very broad emission lines; GRBs have power-law spectra. How are their redshifts measured?

iii The standard interpretation of the deviation in the figure is that there is a non-zero cosmological constant. What does this mean for the expansion of the universe?

Solution:

a) Galaxies visible to the naked eye are M31 (the Andromeda Galaxy), and the Large and Small Clouds of Magellan.

b) The 3 major Hubble types are Spiral, Elliptical, and Irregular. Spirals are have a disk and a well-defined rotation axis. They include the barred spirals (SB) and the spheroidal (S0) galaxies. The arms consist of population I (young and metal-rich) stars and dust; the disk population consists of an older and redder population of stars. The bulge and halo are population II. Ellipticals are elliptical or spherical in shape. There is no preferred rotation axis. There is little dust present, and little evidence of recent star formation. Irregulars show little or no overt structure, and often show recent star formation.

The Milky Way is thought to be a barred spiral - perhaps an SBC or SBd.

c) “Dark” baryonic matter is mostly gas and dust (about 10% of the baryonic mass in the Milky Way). The dust is cool and radiates only in the IR and sub-mm, where the naked eye is not sensitive. Some hot gas is visible, but most gas is also cool. The amount of mass tied up in planets, brown dwarfs, or black holes, none of which radiate significantly in the optical, is negligible.

On large scales about 50% phase of the intergalactic medium. This is gas that never cooled and condensed into galaxies.

d) In the 1930’s Fritz Zwicky measured the velocity dispersion of galaxies in clusters. In a bound, relaxed system virial equilibrium holds (2K=-U). Zwicky measured the velocities, and assumed a mass-to-light ratio typical of the Milky Way, to estimate the kinetic energy. The gravitational potential gives the total mass of the cluster. The two can be reconciled only if the mass-to-light is an order of magnitude larger than accounted for from stars.

e) i. Type Ia supernovae are expected to be standard candles because all involve the implosion of a 1.4 M_⊙ core; GRBs are also expected to be standard candles based on similar arguments. In actuality, they are “standardizable” candles.

ii. In this case redshifts are determined not from the spectra of the SN or GRB, but from superposed absorption lines. This absorption lines are from gas entrained in the Hubble flow, and formally provide only a lower limit to the redshift, but in practice they arise in the interstellar medium of the host galaxy of the SN or GRB.
iii. The SN in the plot are brighter than expected for their redshifts, so are closer to us than expected in a uniform Hubble flow. Therefore, the universe must have been expanding more slowly with time in the past. The evidence is that the expansion of the universe is accelerating with time. This is expected in the case of a cosmological constant, which exerts a constant pressure with time, while the deceleration due to the gravity of the mass in the universe decreases as the universe expands.
Astronomy 3 (Observation/Experiment)

**Temperature of an extrasolar planet (20 pts).** A hot-Jupiter type extrasolar planet is found in an orbit that periodically places it between us and the planet’s host star. As shown during Dr. Jayawardhana’s colloquium this fall, the total light from the star + planet system exhibits a characteristic periodic variation, as shown in the diagram. The deeper dip in the light curve is due to the transit of the planet in front of the star; the shallower dip is due the eclipse of the planet by the star. You observe the system with the Spitzer Space Telescope over the full period of the orbiting planet, and find that:

- at wavelength $\lambda_1 = 4.5$ $\mu$m the depth of the transit is $\Delta F_t(\lambda_1) = 2.30\% \pm 0.02\%$ and of the eclipse is $\Delta F_e(\lambda_1) = 0.07\% \pm 0.02\%$;
- at wavelength $\lambda_2 = 8.0$ $\mu$m the depth of the transit is $\Delta F_t(\lambda_2) = 2.34\% \pm 0.04\%$ and of the eclipse is $\Delta F_e(\lambda_2) = 0.23\% \pm 0.04\%$.

For the following, assume that the host star is an exact solar analog, with radius $R_* = 1R_\odot$ ($6.96 \times 10^{10}$ cm) and effective temperature $T_{\text{eff},*} = 5777$ K, that the star and the planet radiate like blackbodies, and that the stellar and planetary disks have uniform apparent surface brightness (i.e., ignore limb darkening). Ignore atmospheric edge effects that may increase the apparent radius of the transiting planet.

![Diagram of planet transiting star with light curve](image)

a) (3 pts) What is the radius of the planet, based on the two transit depth measurements? Express your result in terms of $R_\odot$ and include your estimate of the uncertainty.

b) (10 pts) What is the brightness temperature of the planet at each observing wavelength? Are the two temperatures above mutually consistent?

c) (7 pts) You obtain a third measurement with Spitzer of the eclipse only, at a wavelength of $\lambda_3 = 24$ $\mu$m. You find that its depth is $\Delta F_e(\lambda_3) = 0.26\% \pm 0.05\%$. Is the assumption that the planet radiates like a single-temperature blackbody still adequate? Justify your answer in terms of the reduced $\chi^2$ of the blackbody model.
Solution:

a) The depth $\Delta F_t$ of the photometric transit is

$$\Delta F_t(\lambda) = \frac{\pi R_p^2}{\pi R_s^2 F_{\lambda,s} + F_{\lambda,p}} = \frac{R_p^2}{R_s^2} (1 - \Delta F_e(\lambda)) \Rightarrow$$

$$\frac{\Delta F_t(\lambda)}{1 - \Delta F_e(\lambda)} = \frac{R_p^2}{R_s^2} \Rightarrow$$

$$R_p = R_s \left( \frac{\Delta F_t(\lambda)}{1 - \Delta F_e(\lambda)} \right)^{1/2} \approx R_s \sqrt{\Delta F_t(\lambda)},$$

where $F_{\lambda}$ denotes flux density [erg s$^{-1}$ cm$^{-2}$ micron$^{-1}$] and subscripts $p$ denote properties of the planet. The approximation above is acceptable because $\Delta F_e(\lambda) \ll \frac{\sigma_{\Delta F_t(\lambda)}}{\Delta F_t(\lambda)} < 1$. Note that $R_p$ is dependent on $\lambda$.

The uncertainty in $R_p$ is

$$\sigma_{R_p} = \sigma_{\Delta F_t(\lambda)} \frac{\partial R_p}{\partial (\Delta F_t(\lambda))} = \sigma_{\Delta F_t(\lambda)} R_s \frac{1}{2} (\Delta F_t(\lambda))^{-1/2} = R_s \frac{\sigma_{\Delta F_t(\lambda)}}{2 \sqrt{\Delta F_t(\lambda)}}$$

The two transit depth measurements at $\lambda_1 = 4.5$ $\mu$m and $\lambda_2 = 8.0$ $\mu$m give $R_p(\lambda_1) = 0.1517 \pm 0.0007 R_s$ and $R_p(\lambda_2) = 0.1530 \pm 0.0013 R_s$. The weighted mean of the planet radius is

$$\langle R_p \rangle = \sum w_i R_p(\lambda_i),$$

where

$$w_i = \frac{1}{\sum 1/\sigma_i^2}.$$

Hence,

$$\langle R_p \rangle = \frac{R_p(\lambda_1)/\sigma_{R_p(\lambda_1)}^2 + R_p(\lambda_2)/\sigma_{R_p(\lambda_2)}^2}{1/\sigma_{R_p(\lambda_1)}^2 + 1/\sigma_{R_p(\lambda_2)}^2} = 0.1520 \pm 0.0006 R_s,$$

where the uncertainty of the weighted mean is obtained as

$$\sigma_{\langle R_p \rangle} = \sqrt{\sum w_i^2 \sigma_i^2} = \sqrt{\sum \frac{1/\sigma_i^2}{(\sum 1/\sigma_i^2)^2}} = \sqrt{\sum \frac{1/\sigma_i^2}{\sum 1/\sigma_i^2}}.$$

b) The depth of the eclipse is

$$\Delta F_e(\lambda) = \frac{\pi R_p^2 F_{\lambda,p}}{\pi R_s^2 F_{\lambda,s} + \pi R_p^2 F_{\lambda,p}} \approx \frac{R_p^2 F_{\lambda,p}}{R_s^2 F_{\lambda,s}}.$$

Assuming that both the star and the planet radiate like blackbodies, where the blackbody radiation law in terms of wavelength is

$$I(\lambda, T) = \frac{2h c^2}{\lambda^5} \left( \exp \left( \frac{hc}{\lambda kT} \right) - 1 \right)^{-1},$$

(10)
c) For \( \lambda \) the squares of the errors, which is

\[
\frac{F_{\lambda_p}}{F_{\lambda,*}} = \left( \frac{\exp \left( \frac{hc}{\lambda k T_p} \right) - 1}{\exp \left( \frac{hc}{\lambda k T_*} \right) - 1} \right)^{-1} = \frac{\exp \left( \frac{hc}{\lambda k T_*} \right) - 1}{\exp \left( \frac{hc}{\lambda k T_p} \right) - 1}.
\]  

(11)

Setting

\[
B(\lambda) \equiv \frac{hc}{\lambda k}
\]

(12)

\[
C_*(\lambda) \equiv \exp \left( \frac{hc}{\lambda k T_*} \right) - 1 = \exp \left( \frac{B(\lambda)}{T_*} \right) - 1,
\]

(13)

and substituting into Eq. (9), we obtain

\[
\exp \left( \frac{B(\lambda)}{T_p} \right) = 1 + \frac{R_p^2 C_*(\lambda)}{R_*^2 \Delta F_e(\lambda)} \Rightarrow \quad T_p = \frac{B(\lambda)}{\ln \left( 1 + \frac{R_p^2 C_*(\lambda)}{R_*^2 \Delta F_e(\lambda)} \right)}.
\]

(15)

(16)

Also,

\[
\sigma_{T_p}^2 = \sigma_{R_p}^2 \left( \frac{\partial T_p}{\partial R_p} \right)^2 + \sigma_{\Delta F_e}^2 \left( \frac{\partial T_p}{\partial \Delta F_e} \right)^2, \quad \text{where}
\]

\[
\frac{\partial T_p}{\partial R_p} = -\frac{B(\lambda)}{\ln \left( 1 + \frac{R_p^2 C_*(\lambda)}{R_*^2 \Delta F_e(\lambda)} \right)^2} \left( 1 + \frac{R_p^2 C_*(\lambda)}{R_*^2 \Delta F_e(\lambda)} \right)^{-1} \frac{R_p^2 C_*(\lambda)}{R_*^2 \Delta F_e(\lambda)}
\]

\[
\frac{\partial T_p}{\partial \Delta F_e} = \frac{B(\lambda)}{\ln \left( 1 + \frac{R_p^2 C_*(\lambda)}{R_*^2 \Delta F_e(\lambda)} \right)^2} \left( 1 + \frac{R_p^2 C_*(\lambda)}{R_*^2 \Delta F_e(\lambda)} \right)^{-1} \frac{R_p^2 C_*(\lambda)}{R_*^2 \Delta F_e(\lambda)}.
\]

(17)

(18)

(19)

For \( \lambda_1 = 4.5 \mu m, B(\lambda_1) = 3200 \text{ K}, \) and \( C_*(\lambda_1) = 0.739, \) so from Eqs. (16–19) \( T_p(\lambda_1) = 990 \pm 80 \text{ K}. \) Likewise, for \( \lambda_2 = 8.0 \mu m, T_p(\lambda_2) = 1170 \pm 100 \text{ K}. \)

Both brightness temperatures are within \( \approx 1 \) standard deviation of the weighted mean, 1060 K, and so are mutually consistent.

c) For \( \lambda_3 = 24 \mu m, \) we find \( T_p(\lambda_3) = 880 \pm 120 \text{ K}. \) The weighted average, \( \langle T_p \rangle = 1020 \text{ K}, \) of the three brightness temperature determinations will give the greatest possible consistency between the measurements and the assumption that the planet radiates like a single-temperature blackbody. We can construct the goodness of fit statistic \( \chi^2_{\text{red}} \), weighted by the squares of the errors, which is \( \chi^2 \)-distributed:

\[
\chi^2_{\text{red}} = \frac{1}{\nu} \sum_{i=1}^{N=3} \frac{(T_p(\lambda_i) - \langle T_p \rangle)^2}{\sigma_{T_p(\lambda_i)}^2} = 1.7,
\]

(20)

where \( \nu = N - 1 = 2 \) is the number of degrees of freedom. The somewhat high value of \( \chi^2_{\text{red}} \) betrays that a single-temperature blackbody may not be a good approximation for the planet’s radiation at all three wavelengths.
Astronomy 4 (Cosmology)

There are two general types of supernovae, and for convenience we refer to them here as gravitational collapse (GC) supernovae and thermonuclear (T) supernovae.

a) (2 pts) Describe the type of star from which of each type of supernova evolves.

b) (8 pts) Estimate on physical grounds the total energy released in each type of supernova. Show your arguments and work.

c) (8 pts) In GC supernovae, nearly all the released energy appears in the form of neutrinos. However, in contrast to neutrinos emitted from the Sun which freely stream to infinity, the supernova neutrinos are trapped within for a finite time. Estimate the duration of this ”trapped neutrino” stage on theoretical grounds, assuming the neutrino-matter cross section is about \( \sigma = 10^{-44} (E_\nu / \text{MeV})^2 \) cm\(^{-2} \), where the energy of a neutrino is \( E_\nu \). Has this prediction ever been observationally confirmed?

d) (2 pts) Explain how type T supernovae, in contrast to type GC supernovae, are significant sources for exploring cosmology.

Solution:

a) The progenitor of a T supernova is a white dwarf in a close binary system: either the white dwarf accretes matter from a companion and grows to its Chandrasekhar limit, or it merges with a companion white dwarf due to orbital energy loss from gravitational radiation. The progenitor of a GC supernova is a star initially more massive than about 8 \( M_\odot \).

b) T supernovae: energy due to thermonuclear fusion of approximately 1.4 \( M_\odot \) of carbon into iron. The difference in binding energies is about 1 MeV/nucleon. The energy released is

\[
1.4 \, M_\odot \times 2 \times 10^{33} \text{(g/M}_\odot\text{)} \times 6 \times 10^{23} \text{(nucleons/g)} \times 1.6 \times 10^{-6} \text{ erg/nucleon} = 2.8 \times 10^{51} \text{ erg}
\]

GC supernovae: energy due to gravitational binding energy of a neutron star. Assuming a constant density, this is \((3/5)GM^2/R\). Taking the neutron star mass to be 1.4 \( M_\odot \) with a radius of 14 km one finds \( 2 \times 10^{53} \) erg.

c) The neutrinos diffuse out of the interior. The diffusion timescale is \( 3R^2/(\lambda c) \approx 12 \) s where the neutrino mean free path is \( \lambda = 1/(n\sigma) \) with \( n \) the baryon number density and \( \sigma \) is the neutrino-matter cross section.

The neutrino cross section is approximately \( \sigma = 10^{-44} (E_\nu / \text{MeV})^2 \) cm\(^{-2} \). The interior neutrino energy \( E_\nu \) can be estimated by assuming they are degenerate fermions, then their average energy is of the order of their chemical potential, which is \( h c (6\pi^2 n Y_\nu)^{1/3} \) where \( Y_\nu = 0.1 \) is the number of neutrinos per baryon. The protoneutron star central density is about the nuclear saturation density \( n = 0.16 \) fm\(^{-3} \) or \( \rho = 2.7 \times 10^{14} \) g cm\(^{-3} \). The neutrino energy in the interior of the protoneutron star core is therefore about 200 MeV, although the average energy of the escaping neutrinos is about 30 MeV.

The protoneutron star radius is \( R \approx (3M/(4\pi\rho))^{1/3} \approx 14 \) km.

Timescale of order 10 s was observed from SN 1987A.

d) Since the energy source for T supernovae stems from an exploding white dwarf near the Chandrasekhar mass, the energies of different events are quite similar. The “standard candle” nature of the sources makes them ideal distance indicators.
The Michelson interferometer at the right is illuminated by a laser system that emits a wide beam of coherent light of wavelength $\lambda$ so that the two horizontal rays emerging from it are all parallel. The front surface of the beam splitter (BS) facing the laser (solid line) reflects 50% of the incident light and the back surface of the BS (dashed line) is anti-reflection coated. The path length $d$ between its reflecting surface and the reflecting surface of each mirror is the same (including traversal of the beam splitter for the horizontal path). The optical path through the beam splitter material causes an additional phase shift of $\phi$. The device can be used to make a pattern of nearly straight-line fringes on the screen at the bottom.

a) (4 pts) Calculate the difference in phase at the screen between rays that travel along the two paths marked A (central rays). Show the phase difference for rays travelling along the path B is the same.

b) (3 pts) With the phase difference the same along all paths, there can be no fringes - the screen can be either fully illuminated, fully dark, or partially (uniformly) illuminated depending on the choice of $\phi$. Suppose $\phi$ is chosen so that it’s fully dark. What happens to the photons that are incident from the laser system? Explain your answer.

c) (5 pts) What needs to be done to change the light field on the screen from uniform illumination as described above to a pattern of straight-line fringes? Draw a careful diagram of how the fringe spacing $x$ is determined, and what can be done to change it. Draw a wave diagram (NOT a ray diagram) to illustrate your point.

d) (3 pts) The pattern can be shifted by one fringe (i.e. from dark to bright to dark) by moving the top mirror. Which way should it be moved and how far?

e) (5 pts) If the upper mirror is moved vertically at constant velocity $v$ and the screen replaced by a photodetector smaller than 1/2 fringe, what would be the time-dependence of the photocurrent? How would it change if the laser system emitted two closely spaced frequencies? How would the time-dependent signal be related to the laser spectrum? What would “closely spaced” mean? Repeat the above questions for a detector larger than one fringe.
Solution:

a) From the laser to the beamsplitter’s reflecting surface, and from the beamsplitter’s reflecting surface to the screen, light travels the same path so there is no net phase difference. Let the phase shift from the beamsplitter’s reflecting surface to the upper mirror be \( \Phi_1 = \frac{2\pi d}{\lambda} \), and an equal amount coming back so the total phase accumulation is \( 2\Phi_1 \). For the beam that travels horizontally there is a phase accumulation of \( \Phi_1 + \phi \) from the beamsplitter’s reflecting surface to the mirror, then an equal accumulation coming back to the the reflecting surface, so the total phase accumulation is \( 2(\Phi_1 + \phi) \). However, the reflection for this beam is from a medium of higher index of refraction than the medium outside of the beamsplitter, so there is an addition phase shift of \( \pi \) making the total \( 2(\Phi_1 + \phi) + \pi \). The phase difference over the two paths is thus \( 2\phi + \pi \). Straightforward geometry for rays along path B shows that the phase shifts are the same as along paths A. (This is true even if the beam splitter is tilted away from the 45° angle as drawn here, and the mirrors tilted to retroreflect the incident beams.)

b) The light returns to the laser system because transmission through the beam splitter of the horizontal beam suffers no extra \( \pi \) phase shift there either, and reflection of the vertical beam is from a medium of higher index, so there’s no phase shift. Thus the beam going to the screen and the beam returning to the laser have a \( \pi \) phase difference for any path length differences to the mirrors.

tilt one mirror by \( \theta \) so its beam reflected beam is deviated by \( 2\theta \) and the wavefronts arrive at the screen at an angle \( 2\theta \) to one another as shown at the right (\( k_1 \) and \( k_2 \) are the wave vectors at angle \( 2\theta \), the sets of parallel lines represent the wavefronts). Arrange the drawing so that the bisector is vertical as shown, and then do the geometry on the little diamonds. The fringes are located at the places where there are vertical rows of dots so they’re separated by \( x = \frac{\lambda}{2 \sin \theta} \).

d) Moving it either way by \( \frac{\lambda}{2} \) will change the path length difference by \( \lambda \) and hence shift one full fringe.

e) It will be sinusoidal with frequency \( 2v/\lambda \). If there were two frequencies of light corresponding to two wavelengths, there would be a superposition of them and the total signal would resemble a carrier at their average and a beat at half their difference. It’s the Fourier transform of the laser spectrum. “Closely spaced” in this case means that their difference is small compared to their average. If the detector were large, several of the fringes would flow across it and the total time dependence would vanish. Thus there would be only a constant voltage (current) from the detector.
AMO 2

Consider a two-level atom in an excited state which can decay via spontaneous emission to the ground state. The transition frequency is $\omega_0$.

a) (4 pts) Spontaneous emission is sometimes described as leading to natural broadening. How does this natural broadening arise? How does the spontaneous emission rate depend on the dipole moment of the transition? Justify your answer.

b) (8 pts) Suppose that the excited atom is located in a resonant cavity with frequency $\omega_0$ and volume $V$. Calculate the energetic shift $\Delta E$ due to coupling of the exited atom to the vacuum field in the cavity. What is the corresponding Rabi frequency, and how does the ground state population change as a function of time?

c) (8 pts) Spontaneous emission is not an inherent property of the atom, but of the atom-vacuum system. Why is this the case? Describe an experimental setup with which spontaneous emission can be suppressed.

Solution:

a) Broadening arises from energy-time uncertainty $\delta E \sim \hbar/\tau$ associated with the decay lifetime $\tau$. The spontaneous emission rate ($1/\tau$) is proportional to the square of the electric dipole moment $\mu$. Justification: interaction Hamiltonian $H = -\mu \cdot \vec{E}$, which means that the transition amplitude $\propto \mu$, and therefore the transition rate $\propto \mu^2$.

b) The coupling is between the states $|e, 0\rangle$ (excited atom, empty cavity) and $|g, 1\rangle$ (ground-state atom, photon in cavity). The field energy density with a single photon in the cavity is $\hbar \omega_0/V$, which is equal to $\frac{1}{2} \epsilon_0 E_0^2$, with electric-field amplitude $E_0$. Solving for $E_0$ yields $E_0 = \sqrt{2\hbar \omega_0/\epsilon_0 V}$. The resulting total energy shift is $\Delta E = \mu_{eg} \sqrt{2 \hbar \omega_0/\epsilon_0 V}$, where $\mu_{eg}$ is the dipole matrix element. The (vacuum) Rabi frequency is $\Omega = \Delta E/\hbar = \sqrt{2 \omega_0 \mu_{eg}^2/\epsilon_0 \hbar V}$. The resonantly coupled two-state system performs coherent oscillations between the two states, with $P_g = \cos^2(\frac{1}{2} \Omega t)$.

c) Emission requires a mode into which a photon can be emitted. In free space, the atom can radiate into a continuum of field modes that are resonantly coupled to the atom (since it is a continuum of modes, the process is irreversible). However, in a cavity of finite size there may be no allowed mode at the transition frequency of the atom, and therefore spontaneous emission can be suppressed. An apparatus could consist of a high-finesse Fabry Perot cavity that is tuned such that the atomic transition frequency is between allowed modes of the cavity (see Physics Today, January 1989, p.4).
One of the most basic electron transport phenomena in nano-scale solid-state structures is the “quantization of conductance”. It refers to the electric conductance of a narrow ballistic channel (see the Figure) between the two reservoirs of electrons in equilibrium at temperature $T$ and different chemical potentials $\mu_1, \mu_2$. To understand this phenomenon, it is sufficient to model the channel as a one-dimensional (1D) gas of non-interacting electrons, with the chemical potentials of electrons moving forward and backward equal to the respective chemical potentials of the reservoirs. Neglect electron spin.

![Figure: Ballistic channel between two reservoirs](image)

**a)** (3 pts) If the dispersion relation of the 1D electrons is $\epsilon(k)$, where $k$ is the electron wavevector, what is their propagation velocity $v(k)$?

**b)** (6 pts) For the 1D electron gas with dispersion $\epsilon(k)$, calculate the density $n$ of the single-particle states per unit energy and unit length, separately for the forward- and backward-moving electrons. If the temperature of electrons in the gas is $T$, what are the occupation probabilities of these states?

**c)** (8 pts) Combining the results or parts (a) and (b) calculate the average electric current $J$ that flows along the channel as a function of the voltage $V$ across it, defined as $V = (\mu_1 - \mu_2)/e$. Assume that $T$ and $eV$ are small on the scale of both chemical potentials $\mu_{1,2}$: $eV, T \ll \mu_{1,2}$. Show that the current satisfies the Ohm’s law, $J = \sigma V$, and find expression for the conductance $\sigma$.

**d)** (3 pts) Calculate the corresponding resistance $1/\sigma$ in Ohms, giving at least 3 significant digits.

**Solution:**

**a)** As usual for all quasiparticles in solid-state systems, the propagation velocity is $v(k) = (1/\hbar)\frac{d\epsilon(k)}{dk}$.

**b)** For the 1D gas, the number of states $dN$ per unit length in an interval $dk$ of wavevectors is $dN = dk/(2\pi)$. The interval $dk$ correspond to the energy interval $d\epsilon = |d\epsilon(k)/dk|dk$. Therefore, the density of the single-particle states per unit length and unit energy is

$$n = dN/d\epsilon = 1/[2\pi|d\epsilon(k)/dk|] = 1/(2\pi\hbar v)$$
both for the forward- and backward-moving electrons. The occupation probabilities of these states are \( f(\epsilon, \mu_1) \) for the forward- and \( f(\epsilon, \mu_2) \) for the backward-moving electrons, where \( f(\epsilon, \mu) \) is the equilibrium Fermi distribution function with the chemical potential \( \mu \).

c) The average electric current \( J \) is obtained from the difference between the flux of the forward- and backward-moving electrons:

\[
J = \int_0^\infty d\epsilon \text{env}[f(\epsilon, \mu_1) - f(\epsilon, \mu_2)] = \frac{e}{2\pi \hbar} \int_0^\infty d\epsilon [f(\epsilon, \mu_1) - f(\epsilon, \mu_2)].
\]

For \( eV, T \ll \mu_1, 2 \), the last integral reduces simply to \((\mu_1 - \mu_2)\). This gives, finally:

\[
J = \sigma V, \quad \sigma = \frac{e^2}{2\pi \hbar} = \frac{e^2}{\hbar}.
\]

d) The quantum of resistance \( R_Q = 1/\sigma = h/e^2 \). Substituting values of \( e \) and \( \hbar \) in SI units, we obtain its magnitude in Ohms:

\[
R_Q = 6.62607 \cdot 10^{-34}/(1.6018 \cdot 10^{-19})^2 \simeq 25813 \Omega.
\]
Bragg’s Law for X-ray diffraction is $n \lambda = 2d \sin \theta$

a) (4 pts) Draw a diagram to illustrate the meaning of the variables $d$ and $\theta$. Use your diagram to deduce that the Bragg condition is satisfied when the detector is at $2\theta$ with respect to the incoming beam when the planes from which diffraction occur are oriented at an angle of $\theta$ with respect to the incoming beam.

Consider the above x-ray diffraction pattern which shows measured intensity (on a logarithmic scale) vs the detector angle $2\theta$ in degrees. The scan is obtained by a coupled motion of the sample and detector, such that at any point in the scan the angle of incidence of the x-ray beam is $\theta$, and the detector angle is $2\theta$ with respect to the incoming beam. The sample measured is an artificially layered superlattice composed of a finite number of bi-layers stacked on top of each other. Several periodicities are visible in the x-ray diffraction pattern, and, to a first approximation, the diffraction pattern can be considered to be a superposition of the individual diffraction patterns generated from each periodicity. The highest intensity peak (at $22.75^\circ$) is due to the single crystal substrate on which the film is grown. The next highest intensity peak (at $22.27^\circ$) reflects the average out of plane lattice parameter of the film. Two longer periodicities are also visible. The first can be seen in the intense peaks marked with arrows (one of which is also the peak that reflects the average out of plane lattice parameter of the film), and is due to the superlattice periodicity. The second, longest, periodicity, gives rise to the lower intensity and finer period oscillations, which reflect the total thickness of the sample.

b) (4 pts) If the substrate is known to be SrTiO$_3$, which is cubic with lattice parameter 3.905 Å, what is the wavelength of the x-rays used for this diffraction experiment? (Note: You know that the peak you are looking corresponds to $n = 1$.)

c) (2 pts) What is the energy of the x-rays (in eV)? ($h = 6.626 \times 10^{-34} \text{m}^2\text{kg}\text{s}^{-1}, c = 3 \times 10^8 \text{ms}^{-1}, e = 1.6 \times 10^{-19} \text{C}$)

Problem continues on next page
d) (4 pts) What is the average out of plane lattice parameter of the film? (As for part B, this peak corresponds to \( n = 1 \).) Why is the corresponding peak broader than the peak in part B?

e) (4 pts) Using the angular positions of the superlattice-related peaks, calculate the superlattice periodicity. Here you don’t know what \( n \) is, but you should be able to eliminate it.

f) (2 pts) How many bi-layers make up the film? Explain briefly.

Solution:

a) Draw a diagram to illustrate the meaning of the variables \( d \) and \( \theta \). Use your diagram to deduce that the Bragg condition is satisfied when the detector is at \( 2\theta \) with respect to the incoming beam when the planes from which diffraction occur are oriented at an angle of \( \theta \) with respect to the incoming beam.

b) If the substrate is known to be SrTiO\(_3\), which is cubic with lattice parameter 3.905 Å, what is the wavelength of the x-rays used for this diffraction experiment. (Note: You know that the peak you are looking corresponds to \( n=1 \)).

\[
\lambda = 2 \times 3.905 \text{Å} \times \sin\left(\frac{22.75^\circ}{2}\right) = 1.5404 \text{Å}
\]

c) What is the energy of the x-rays (in eV)? (\( h = 6.626 \times 10^{-34} \text{m}^2\text{kg}\text{s}^{-1}, c = 3 \times 10^8 \text{ms}^{-1}, e = 1.6 \times 10^{-19} \text{C} \))

\[
E = \frac{hc}{\lambda} = \frac{6.626 \times 10^{-34} \times 3 \times 10^8}{1.5404 \times 10^{-10} \times 1.6 \times 10^{-19} \text{eV}} = 8.1 \text{keV}
\]

d) What is the average out of plane lattice parameter of the film? (As for part B, this peak corresponds to \( n=1 \)).

\[
d = \frac{\lambda}{2\sin\theta} = \frac{1.5404}{2 \sin 22.27^\circ/2} = 3.99 \text{Å}
\]

e) Using the angular positions of two superlattice peaks calculate the superlattice periodicity. Here you don’t know what \( n \) is, but you should be able to eliminate it.

If you consider the two successive peaks to be the \( n \)th and \( n + 1 \)th peak, then:
\((n + 1)\lambda = 2d \sin \theta_{n+1}\)
\((n)\lambda = 2d \sin \theta_n\)
\(\lambda = 2d(\sin \theta_{n+1} - \sin \theta_n)\)
\(d = \frac{\lambda}{2(\sin \theta_{n+1} - \sin \theta_n)}\)
\(d = \frac{1.504^\circ}{2(\sin(22.27^\circ/2) - \sin(18.55^\circ/2))} = 24.1\text{Å}\)

(This question can also can be done with the higher superlattice peak).

f) How many bi-layers make up the film?

The periodicity of the film thickness oscillations is 10 times that of the superlattice oscillations, therefore there are 10 bi-layers making up the film.

The same result can be obtained from taking the difference between two of the oscillation peaks, e.g.:
\(d_{\text{film}} = \frac{1.504^\circ}{2(\sin(22.65^\circ/2) - \sin(21.3^\circ/2))} = 250\text{Å} \approx 10 \text{ times the bi-layer thickness.}\)
Nuclear 1

The $J/\psi$ particle has a mass of 3.097 GeV/c$^2$ and a width of 63 keV. A 10 GeV/c $J/\psi$ is created in a symmetric proton-proton collisions at $\sqrt{s} = 100$ GeV and it subsequently decays according to

$$J/\psi \rightarrow e^+ + e^-.$$  

a) (3 pts) Find the mean distance traveled by the $J/\psi$ in the laboratory before decaying.

b) (4 pts) For a symmetric decay, i.e. a decay where the magnitudes of the momenta of the electron and the positron are equal in the laboratory frame, find the energy of the decay electron in the laboratory.

c) (3 pts) Find the laboratory angle of the electron with respect to the direction of the $J/\psi$.

d) (7 pts) A compact cylindrical calorimeter made up of lead tungstate crystals (PbWO$_4$) (dimension: $2.3 \times 2.3 \times 10$ cm$^3$, Molièr radius 2.19 cm) is to be designed to for a future detector at RHIC. Would the calorimeter be able to clearly separate the electron and positron hits if the inner radius of the calorimeter is planned at 40 cm? Justify your answer quantitatively. (The face of the calorimeter crystals that face the collision point are $2.3 \times 2.3$ cm$^2$.)

e) (3 pts) If the same detector is used to measure $J/\psi$’s created in $Au - Au$ collisions at similar energies, would any of the answers to above questions change?

Solution:

a) The total width $\Gamma$ of the $J/\psi$ decay is 63 keV, so its proper lifetime is:

$$\tau_0 = \frac{h/2\pi}{\Gamma} = \frac{6.58 \times 10^{-16}}{63 \times 10^3} = 1.045 \times 10^{-20} \text{ s}$$

The laboratory lifetime is then $\tau = \tau_0 \gamma$ where $\gamma$ is the Lorentz factor. Hence the mean distance traveled by the $J/\psi$ in the laboratory before decaying is:

$$l = \tau \beta c = \tau_0 \gamma \beta c = \frac{\tau_0 p c}{m} = 1.045 \times 10^{-20} \times \frac{10}{3.097} \times 3 \times 10^8 = 1.012 \times 10^{-11} \text{ m}$$

b) For a symmetric decay, conservation of energy and momentum gives:

$$E_{J/\psi} = 2E_e$$  \quad and  \quad $$p_{J/\psi} = 2p_e \cos \theta$$

where $\theta$ is the angle the electron makes with the direction of the $J/\psi$ particle. Thus:

$$E_e = \frac{1}{2} E_{J/\psi} = \frac{1}{2} \sqrt{p_{J/\psi}^2 + m_{J/\psi}^2} = \frac{1}{2} \sqrt{(10)^2 + (3.097)^2} = 5.23 \text{ GeV}$$
c) To find the angle between the electron and the $J/\psi$ direction:

\[
\left( \frac{E_J}{2} \right)^2 - \left( \frac{p_{J/\psi}}{2 \cos \theta} \right)^2 = E_e^2 - p_e^2 = m_e^2
\]

Solve for $\cos \theta$ and $\theta$

\[
\cos \theta = \frac{p_{J/\psi}}{\sqrt{p_{J/\psi}^2 + m_{J/\psi}^2 - 4m_e^2}} = \frac{10}{\sqrt{10^2 + 3.007^2 - 4(0.511 \times 10^{-3})^2}} = 0.9552
\]

\[
\theta_{J/\psi-e} = 17.2^\circ
\]

d) Draw a simple diagram of the generated electron-positron pair and projection on to a flat calorimeter surface. The surface is made up of lead-tungstate crystals with 2.3 cm$^2$ units/granularity. Geometry tells you that to distinguish between the two electrons, they should hit two crystals separated by at least one crystal size (2.3 cm). If $R$ is the distance between the beam pipe and the location of the front face of the calorimeter:

\[
R \sin \theta = 1.15 \text{ cm} \implies R = \frac{1.15}{\sin(17.2^\circ)} = 3.95 \text{ cm}
\]

This means if you build the calorimeter at any distance beyond 3.95 cm, to first order you would not have a difficulty distinguishing the two (positron and electron) hits from each other without a concern for shower merging. The distance of 40 cm is (generously) sufficient to achieve this. Recall that the Molier radius of 2.19 cm contains 97% of the total shower.

e) There is no difference between $J/\psi$’s created in $Au - Au$ collisions and those created in $p - p$ collisions. Nothing regarding the above considerations changes.
Nuclear 2

The FAIR facility in Germany will collide an anti-proton beam with the lab energy of up to 15 GeV into fixed nuclear targets. Consider the scattering of an anti-proton with the lab energy of $E = 15$ GeV off a lead nucleus $^{208}$Pb. The following information is useful:

- Anti-protons have a short mean free path in nuclear matter because the $\bar{p}$ annihilation cross section on a single nucleon is large, $\sigma_{\text{abs}} \simeq 40$ mb (where 10 mb = 1 fm$^2 = 10^{-30}$ m$^2$). Again this is the absorption cross section of a $\bar{p}$ on the individual protons and neutrons.

- The elastic anti-proton scattering amplitude on the whole nucleus (without spin) can be parameterized with partial waves:

$$ f(\theta, \phi) = \frac{1}{2\hbar k} \sum_\ell (2\ell + 1)(S_\ell - 1)P_\ell(\cos \theta), $$

where $p = \hbar k$ is the momentum of the anti-proton, and $S_\ell$ is the S-matrix element for the $\ell$-th partial wave. $S_\ell$ is parameterized as

$$ S_\ell(p) = \eta_\ell(p) e^{i2\delta_\ell(p)}, $$

where $\delta_\ell$ is the phase shift, and $\eta_\ell$ is the inelasticity factor, which changes from zero to one depending on whether the process is totally elastic or totally inelastic in the $\ell$-th partial wave.

- The Legendre Polynomials satisfy the orthogonality relation:

$$ \int_{-1}^{1} dx P_\ell(x)P_\ell'(x) = \frac{2}{2\ell + 1} \delta_{\ell\ell'}, $$

Answer the following questions:

a) (2 pts) Estimate the size of a lead nucleus.

b) (2 pts) Estimate the absorption length of an anti-proton in a nucleus or nuclear matter.

c) (4 pts) Give the formula for the total elastic scattering cross section in terms of the partial waves, phase shifts and inelasticity factors.

d) (4 pts) Estimate the typical partial wave contributing to the scattering amplitude for the anti-proton $^{208}$Pb collisions. Estimate the maximum partial wave which is almost completely absorbed in anti-proton Pb scattering.

e) (5 pts) Estimate the total elastic and total inelastic cross sections. Be sure to state which one is which.

f) (3 pts) Qualitatively sketch the differential elastic cross section as a function of scattering angle for anti-proton on $^{208}$Pb at this energy. Estimate the characteristic scattering angle for these elastic events.
Solution:

a) \( R_{p_b} \simeq 6 \text{ fm} \).

b) Estimating
\[
\ell_{mfp} = \frac{1}{n\sigma} = \frac{1}{1/6 \text{ fm}^{-3} 4 \text{ fm}^2} \simeq 1.5 \text{ fm}
\]


c) The cross section is:
\[
\sigma = \int d\Omega \, |f|^2 = \frac{2\pi}{k^2} \sum_{\ell} \frac{2\ell + 1}{2} |S_{\ell} - 1|^2 \tag{21}
\]
\[
= \frac{2\pi}{k^2} \sum_{\ell} \frac{2\ell + 1}{2} |\eta e^{2i\delta_{\ell}} - 1|^2. \tag{22}
\]
\[
= \frac{2\pi}{k^2} \sum_{\ell} \frac{2\ell + 1}{2} |\eta e^{2i\delta_{\ell}} - 1|^2. \tag{23}
\]


d) When the impact parameter \( b = 6 \text{ fm} \) then the angular momentum in units of \( \hbar \) is
\[
\ell_* \sim \frac{pR}{\hbar} = \frac{15 \text{ GeV} \times 6 \text{ fm}}{197 \text{ MeV fm}} \tag{24}
\]
\[
= 457 \tag{25}
\]

e) Given the large number of partial waves we can apply semi-classical reasoning The inelastic cross section is clearly
\[
\sigma_{\text{inel}} = \pi R^2
\]
If the you know the result you can simply state
\[
\sigma_{\text{el}} = \pi R^2
\]
Otherwise one can the sum over \( \ell \) with an integral
\[
\sigma_{\text{el}} = \frac{2\pi}{p^2} \int_0^{\frac{\pi R}{p}} \ell \tag{26}
\]
\[
= \frac{2\pi}{k^2} \frac{1}{2} \left( \frac{pR}{\hbar} \right)^2 \tag{27}
\]
\[
= \pi R^2 \tag{28}
\]

f) The angle at which the differential cross section for the scattering off a black disk has a diffractive minimum is \( \theta_d \simeq 1/(kR_A) \simeq 2 \times 10^{-3} \text{ rad} \).
a) (4 pts) Write down the Lagrangian for a free complex scalar field $\phi$ of mass $m$, coupled to the electromagnetic potential $A_{\mu}$ by the minimal coupling prescription. Denote by $e$ the electric charge of $\phi$. Write down the gauge transformations and check that your Lagrangian is gauge invariant.

b) (6 pts) Draw all the tree-level Feynman diagrams contributing to Compton scattering, that is, the scattering a photon $\gamma$ off a $\phi$ particle,

$$\gamma + \phi \longrightarrow \gamma + \phi.$$ 

Label the diagrams by the momenta and polarizations of the incoming and outgoing particles, as follows: $p_{\mu}$ and $p'_{\mu}$ are the momenta of the incoming and outgoing $\phi$ particle; $k_{\mu}$ and $k'_{\mu}$ the momenta of the incoming and outgoing photon; $\epsilon_{\mu}$ and $\epsilon'_{\mu}$ the polarizations of the incoming and outgoing photon.

c) (6 pts) Write down the tree-level scattering amplitude by summing the contributions of the Feynman diagrams.

d) (4 pts) Check that your expression for the amplitude is invariant under

$$\epsilon_{\mu} \rightarrow \epsilon_{\mu} + \alpha k_{\mu},$$

where $\alpha$ is any constant. Explain why this must be the case.

**Solution:**

a) The Lagrangian density is

$$\mathcal{L} = -(\partial^\mu + ieA^\mu)\phi^* (\partial_\mu - ieA_\mu)\phi - m^2\phi\phi^*.$$ (29)

The infinitesimal gauge transformations are

$$\delta \phi(x) = i e \lambda(x) \phi(x), \quad \delta A_\mu = \partial_\mu \lambda(x).$$ (30)

Then

$$\delta[(\partial_\mu - i e A_\mu)\phi] = i e \lambda(x)(\partial_\mu - i e A_\mu)\phi = \frac{\partial}{\partial \lambda}(\partial_\mu - i e A_\mu)\phi - i e \lambda(x)\frac{\partial}{\partial \lambda}(\partial_\mu - i e A_\mu)\phi^*,$$ (31)

$$\delta[(\partial_\mu + i e A_\mu)\phi^*] = -i e \lambda(x)(\partial_\mu + i e A_\mu)\phi^*,$$ (32)

which imply invariance of the kinetic term. The mass term is immediately seen to be separately invariant.

b)
c) The total tree-level amplitude is

\[ A = A_I + A_{II} + A_{III}, \]  

(33)
where

\[ A_I = e^2 \frac{(2p + k) \cdot \epsilon (2p' + k') \cdot \epsilon'}{(k + p)^2 + m^2} = 2e^2 \frac{(p \cdot \epsilon)(p' \cdot \epsilon')}{k \cdot p} \]  \hspace{1cm} (34)

\[ A_{II} = e^2 \frac{(2p - k') \cdot \epsilon' (2p' - k) \cdot \epsilon}{(-k' + p)^2 + m^2} = -2e^2 \frac{(p' \cdot \epsilon')(p' \cdot \epsilon)}{k' \cdot p} \]  \hspace{1cm} (35)

\[ A_{III} = -2e^2 \epsilon \cdot \epsilon'. \]  \hspace{1cm} (36)

In simplifying the expressions we have used \( p^2 = -m^2 \) and \( k \cdot \epsilon = k' \cdot \epsilon' = 0 \).

d) Substituting \( \epsilon \to k \) in the total amplitude,

\[ 2e^2 \left[ \frac{(p \cdot k)(p' \cdot \epsilon')}{k \cdot p} - \frac{(p \cdot \epsilon')(p' \cdot k)}{k' \cdot p} - k \cdot \epsilon' \right] = 2e^2 \left[ p' \cdot \epsilon' - p \cdot \epsilon' - k \cdot \epsilon' \right] = 0, \]  \hspace{1cm} (37)

since \( p' - p - k = -k' \) and \( k' \cdot \epsilon' = 0 \).

This invariance of the amplitude is a consequence of gauge invariance.
High Energy 2

In colloquium this fall, we heard that Dark Matter (DM) makes up 23% of the universe, and that determining the nature of DM is active research.

a) (4 pts) One source of evidence for DM comes from rotation curves of galaxies, especially spiral galaxies. (1) Sketch the rotation curve expected in the absence of DM. (2) Sketch an observed rotation curve. For both cases, indicate the dependence on radial distance from the galactic center.

b) (4 pts) Indicate with “yes” or “no” which of the following interactions DM can undergo:

i. Strong
ii. Electromagnetic
iii. Weak
iv. Gravitational

c) (2 pts) List two possible candidates for DM

d) (6 pts) Assuming the velocity of our solar system about the galactic center is 200 km/s, the density of DM is 0.3 GeV/cm$^3$ and the DM on fermion scattering cross-section has the form $\alpha^4/M^2$ with $\alpha$ the appropriate coupling constant, estimate the scattering rate for a 100 GeV DM particle.

e) (4 pts) Describe an existing or planned experiment or observatory which could be used to detect DM candidates. What is the fundamental process being detected in this experiment?

Solution:

a) Assume the luminous region has a constant density. We know that $m v^2/r = G m M/r^2$. If $M$ is the mass inside the radius $r$, then inside the luminous region $v \propto r$. In the absence of DM outside the luminous region, $M$ is constant (equal to the mass of the luminous region) and $v \propto 1/\sqrt{r}$. Observed rotation curves are essentially constant outside the luminous region.

b) strong, no; EM, no; Weak, yes; gravity, yes

c) primordial black holes, SUSY LSP, WIMP (an example is SUSY LSP), axions, sterile neutrinos, MACHOS, Kaluza-Klein gravitons

d) The average scattering rate is $\sigma \times v \times n$ in which $\sigma$ is the cross section $\sigma = (\hbar c)^2 \alpha^4/M^2$, $v$ the velocity, $n$ the number density. Assuming a weak-scale cross section, $\alpha \approx G_F M_W^2 \approx 0.1$, converting the given energy density to a number density, $n = \rho/M$ and remembering to convert units ($\hbar c = 200$ MeV×fm), the scattering rate is $O(10^{-31})$ scatters/sec.

e) choose from CDMS/XENON style (bolometric scattering from nuclei) or LHC (appearance in decay chains visible as MET). Prefer CDMS/XENON because of direct link to Dark Matter.