General Instructions: Twelve problems are given; you should solve four problems. If you do more than four problems, you must choose which four should be graded, and only submit those four. You may do two problems from the same field only once, except for Astronomy, for which you can do up to four problems.

Each problem counts 20 points, and the solution should typically take less than 45 minutes.

Some of the problems cover multiple pages. Make sure you do all the parts of each problem you choose.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use a one page help sheet, a calculator, and with the proctor’s approval, a foreign language dictionary. No other materials may be used.

Some potentially useful information:

- The atomic mass of hydrogen is 1.00794 amu.
- The atomic mass of helium is 4.002602 amu.
- 1 amu is $1.66 \times 10^{-27}$ kg.
- $c = 2.998 \times 10^8$ ms$^{-1}$.
- $G = 6.673 \times 10^{-11}$ m$^3$kg$^{-1}$s$^{-2}$.
- The solar luminosity is $3.85 \times 10^{26}$ W.
- The mass of the Sun is $1.989 \times 10^{30}$ kg
- The radius of the Sun is $7.00 \times 10^8$ meters
- $\hbar = 1.055 \times 10^{-34}$ J
- $e = 1.602 \times 10^{-19}$ C
- $k_B = 1.38 \times 10^{-23}$ J/K
- $mc^2 \simeq 0.5$ MeV
- $\hbar c \simeq 197$ MeV-fm.
Astronomy 1

This problem concerns hydrogen burning at the surface of a neutron star. A neutron star that accretes gas from the envelope of a nearby companion star can fuse hydrogen near the surface through the hot CNO cycle. Here you will derive the properties of this surface layer. You can assume that the neutron star has a mass of $1.4 M_\odot = 2.8 \times 10^{33}$ g, a radius of 10 km, that the accreted material has approximately solar abundance (hydrogen mass fraction $X = 0.7$, helium mass fraction $Y = 0.3$) and that Newtonian gravity is correct (true here to within a factor of 2).

a. (5 pts) Show that the column density above any layer is proportional to the pressure in the layer. Assuming a typical pressure of $P = 1.0 \times 10^{22}$ dyne cm$^{-2}$ at the depth of the hydrogen-burning layer, estimate the column density of the gas.

b. (7 pts) Assuming that pressure is dominated by non–relativistic degenerate electrons ($P = 1.0 \times 10^{13} \rho^{5/3}$ dyne cm$^{-2}$), estimate the density of the gas at the depth of the hydrogen–burning layer. Below what temperature will the degenerate electron pressure assumption be valid?

c. (5 pts) Estimate the depth (in cm) of this layer.

d. (3 pts) Estimate the time it will take for photons to escape from that depth. Assume the temperature is high enough so that all electrons are unbound, and hence the opacity is just that of free electron, $\kappa_e \approx 0.2(1 + X)$.

SOLUTION:

a. Start by integrating the equation of hydrostatic equilibrium. In the surface layers of a neutron star gravity is constant, so it can be pulled out of the integral.

$$\frac{dP}{dr} = g \rho \Rightarrow P(r) = \int_r^R g \rho dr' = \frac{GM}{R^2} \int_r^R \rho dr'.$$

The integral in the rightmost equation is identically the column density $\Sigma(r)$. So, in the surface layers, the pressure is the column density above that layer times the surface gravity. The surface gravity of a neutron star ($R = 10$ km, $M = 1.4 M_\odot$) is $g = 1.9 \times 10^{14}$ cm/s. Then

$$\Sigma = \frac{P}{g} = \frac{1.0 \times 10^{22}}{1.9 \times 10^{14}} \text{ dyne cm}^{-2} = 5.3 \times 10^7 \text{ g cm}^{-2}.$$
b. The pressure from degenerate electrons is $P = 1.0 \times 10^{13} \rho^{5/3}$ dyne cm$^{-2}$, so
\[ \rho = \left( \frac{1.0 \times 10^{22}}{1.0 \times 10^{13}} \right)^{3/5} = 2.5 \times 10^5 \text{ g cm}^{-3}. \]  
(3)

The degeneracy boundary is when the ideal gas pressure is equal to the required pressure from hydrostatic equilibrium, so
\[ \frac{\rho k T}{\mu m_H} = 1.0 \times 10^{22} \text{ dyne cm}^{-2} \Rightarrow \]  
(4)

The accreted material is fully ionized, and its mean molecular weight $\mu$ is
\[ \frac{1}{\mu} = \frac{1}{\mu_I} + \frac{1}{\mu_e}, \]  
(5)

where for the ions $\mu_I = 4/(1+3X) = 1.3$ and for the electrons $\mu_e = 2/(1+X) = 1.2$. So $\mu = 0.6$. Hence,
\[ T = \frac{(1.0 \times 10^{22} \text{ dyne cm}^{-2}) \times 0.6 \times (1.67 \times 10^{-24} \text{ g})}{(2.5 \times 10^5 \text{ g cm}^{-3}) \times (1.38 \times 10^{-16} \text{ erg K}^{-1})} = 2.9 \times 10^8 \text{ K}. \]  
(6)

c. With $P = P_0 \rho^{5/3}$ the equation of hydrostatic equilibrium becomes
\[ \frac{d}{dr} (P_0 \rho^{5/3}) = P_0 \frac{5}{3} \rho^{2/3} \frac{d\rho}{dr} = g \rho \]  
(7)

Integrate to get $\rho(r)$:
\[ \rho^{-1/3} d\rho = \frac{3g}{5P_0} dr \Rightarrow \frac{3}{2} \rho^{2/3} = \frac{3g}{5P_0} r \Rightarrow \]  
(8)

\[ \rho = \left( \frac{2g}{5P_0} \right)^{3/2} r^{3/2}. \]  
(9)

Integrate this with respect to radius an set it equal to the surface mass (i.e., column) density:
\[ \Sigma(r) = \left( \frac{2g}{5P_0} \right)^{3/2} \frac{2}{5} r^{5/2} = 5.3 \times 10^7 \text{ g cm}^{-2}. \]  
(10)

Plug in $g = 1.9 \times 10^{14} \text{ cm s}^{-2}$ and $P_0 = 1.0 \times 10^{13}$ to get $r = 530 \text{ cm}$.

d. If the mean free path of a photon is denoted by $l$, the number of optical depths to the surface is
\[ N = \frac{r}{l} = \frac{\Sigma/\rho}{\kappa e_\rho} = \Sigma \kappa = 1.8 \times 10^7 \]  
(11)

The total distance traveled by an average escaping photon is the length of the distance to be traversed times the number of optical depths, so the photon travel distance is $Nr = 1.8 \times 10^7 \times 530 \text{ cm} = 9.6 \times 10^9 \text{ cm}$. Dividing by the speed of light gives the escape time, 0.3 seconds.

3
Astronomy 2

A reasonably close compact binary system will radiate enough energy in the form of gravitation radiation that its components will merge in significantly less than a Hubble time. Observations of binary systems in which one member is a pulsar allow precise timing of the orbital period and have served as experimental confirmation of the theory of General Relativity.

In 1964, P. C. Peters obtained an expression for the time–average rate of change in the semi–major axis of such a system by assuming the stars are point masses:

\[
\langle \frac{da}{dt} \rangle = -\frac{4}{5} \frac{G^3 m_1 m_2 (m_1 + m_2)}{c^5 a^3 (1 - e^2)^{7/2}} \left( 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right)
\]

where \(a\) is the semi–major axis, \(e\) is the orbital eccentricity, \(m_1\) and \(m_2\) are the masses of the objects, \(c\) is the speed of light and \(G\) is Newton’s gravitational constant.

a. (6 pts) Assuming circular orbits, and the result of Peters, obtain an expression for the semi–major axis as a function of time.

b. (8 pts) PSR J0737-3039 is the only known double pulsar. The masses of the neutron stars are 1.377 and 1.250 solar masses, and the period is 2.45 hours. Use the expression for the semi–major axis as a function of time to find the in–spiral (merge) time for this system.

c. (3 pts) The amplitude of the gravitational waveform is tiny, but experiments are underway to observe the waveform produced by compact object mergers. Why does the behavior of the frequency at the late stages of the in–spiral make analysis of any observed waveform difficult?

d. (3 pts) Why might Peters’s formula be inaccurate for the last few orbits before the merger?

SOLUTION:

a. For \(e = 0\), Peters’s formula becomes

\[
\frac{da}{dt} = -\frac{4}{5} \frac{G^3 m_1 m_2 (m_1 + m_2)}{c^5 a^3}
\]

Taking

\[
\beta = \frac{4}{5} \frac{G^3 m_1 m_2 (m_1 + m_2)}{c^5 a^3}
\]

we may integrate

\[
\int_{a_0}^{a} a^\beta \, da' = -\beta \int_{t_0}^{t} dt'
\]
which with \( t_0 = 0 \) gives
\[
a^4 - a_0^4 = -4\beta t
\]
or
\[
a = (a_0^4 - 4\beta t)^{\frac{1}{4}}
\]

b. Any choice of units should work. Here we choose cgs units.

\[
m_1 = (1.337 M_{\odot}) (1.9989 \times 10^{33} \text{ g}/M_{\odot}) = 2.6592 \times 10^{33} \text{ g}
\]
\[
m_2 = (1.250 M_{\odot}) (1.9989 \times 10^{33} \text{ g}/M_{\odot}) = 2.4861 \times 10^{33} \text{ g}
\]
\[
P_0 = (2.45 \text{ h}) (60 \text{ min/h}) (60 \text{ s/min}) = 8820 \text{ s}
\]

Apply Kepler’s 3rd law
\[
P_0 = \frac{4\pi^2}{G (m_1 + m_2)} a_0^3
\]

Solving for \( a_0 \),
\[
a_0 = \left[ \frac{G (m_1 + m_2)}{4\pi^2} P_0^2 \right]^{\frac{1}{3}}
\]

Plugging in the numbers
\[
a_0 = \left[ \frac{1 \times 10^{-8} \text{ cm}^3/\text{g s}^2}{4\pi^2} \left( 5.1453 \times 10^{33} \text{ g} \right) (8820 \text{ s}) \right]^{\frac{1}{3}}
\]
\[
= 8.7793 \times 10^{10} \text{ cm}
\]

To find the inspiral time, put this in the expression for \( a(t) \). At \( t = t_{\text{inspiral}} \) we have,
\[
0 = (a_0^4 - 4\beta t_{\text{inspiral}})^{\frac{1}{4}}
\]

Re-arranging,
\[
t_{\text{inspiral}} = \frac{a_0^4}{4\beta}
\]

In these units, \( \beta = 3.8372 \times 10^{26} \text{ cm}^4/\text{s} \). Evaluating, we find
\[
t_{\text{inspiral}} = 3.8705 \times 10^{16} \text{ s} \approx 123 \text{ Myr}
\]

Note that including the eccentricity gives a shorter inspiral time. The accepted result is about 85 Myr.
c. In Peters’s expression, $\frac{da}{dt} \propto \frac{1}{a^3}$. As $a$ becomes small, $\frac{da}{dt}$ becomes large so the rate of inspiral greatly increases as the components complete the last tens of orbits. From Kepler, $P \propto a^{3/2}$, so $f \propto a^{-3/2}$. Thus the frequency rapidly increases or “chirps” at the end of the inspiral, making standard Fourier analysis difficult.

Note that the amplitude of the waves greatly increases as the stars merge together, which makes the signal easier to detect. And, depending on the detector, as the frequency increases the detector may get more sensitive. Credit given for answers correctly addressing these issues.

d. Toward the end of the inspiral, tidal effects become important and the assumption of point masses is not valid.
Astronomy 3

Strange worlds: An intrepid explorer, you find yourself on a strange Earth–like planet around a foreign star. You begin to make some basic observations.

a. (3 pts) The rotation period of the planet is much shorter than its year. Over the course of the planet’s year, you make note of the highest position the star reaches in the sky each day. You see on one day the star gets as high as $10^\circ$ from the zenith, while on a different day, its highest point is $90^\circ$ from the zenith. These are the extremes over the course of the year. By convention, we will take your location to be in the Northern hemisphere. What is your latitude and what is the planet’s polar axial tilt?

b. (3 pts) You (an alien) meet a friendly local resident (who remarkably speaks English!). The resident tells you that if you were to walk 1000 km due south from your position, you would find that on the summer solstice the star is directly overhead. What is the radius of the planet?

c. (4 pts) With a spectrograph, you determine that the star is an early M star. If the length of the year is 31 Earth days, then what is the semi–major axis of the planet’s orbit?

d. (5 pts) Neglecting any greenhouse effect, but otherwise taking the appearance of the planet to be Earth–like, and using your knowledge of stars, what average temperature would you expect the planet to experience?

e. (5 pts) Mysteriously, the planet has a small ring system. Folklore says that it only recently formed when an asteriod wandered too close to the planet. Neglecting the internal tensile strength of the asteriod and rotational effects, at what distance from the planet do you expect a typical asteriod to break apart. State any assumptions you need to make.

SOLUTION:

a. The maximum altitude occurs on the summer solstice and the minimum occurs on the winter solstice.

From the figure above, we have:

$$z_{\text{summer}} = l - \theta = 10^\circ$$

$$z_{\text{winter}} = l + \theta = 90^\circ$$

where $z$ is the angle from the zenith that the Sun makes on the summer and winter solstices, respectively. Solving, we have a latitude of $l = 50^\circ$ and axial tilt of $\theta = 40^\circ$. 
b. The observer to the south is at the equivalent of the Tropic of Cancer on the solstice, which is a latitude of $40^\circ$ on this planet. This means that $10^\circ$ of latitude corresponds to 1000 km, giving a circumference of 36,000 km and a radius of 5,700 km.

c. An early M star has a mass of $\sim 0.5 \, M_\odot$. Using (Newton’s version of) Kepler’s third law:

$$\frac{4\pi^2 a^3}{G} = M_\star P^2$$

we find $a = 2.3 \times 10^{12}$ cm.

d. In thermal equilibrium, the planet will radiate the same power it absorbs from the star. The absorbed power is

$$P_{\text{abs}} = \pi R_p^2 (1 - a_B) \frac{L_\star}{4\pi d^2}$$

where $a_B$ is the albedo. We take $a_B = 0.3$, as Earth. The luminosity of an early M star is $\sim 0.1 \, L_\odot$. The radiated power is as a blackbody,

$$P_{\text{emit}} = 4\pi R_p^2 \sigma T^4$$

giving the equilibrium temperature as

$$T = \left[ \frac{L_\star (1 - a_B)}{16\pi \sigma d^2} \right]^{1/4} = 300 \, \text{K}$$
e. A small asteroid will be broken apart if the tidal force across it is greater than its gravitational acceleration. This is the Roche limit. The gravitational acceleration at the surface is

\[ a_g = \frac{GM_a}{R_a^2}. \tag{18} \]

The tidal acceleration at the surface is:

\[ a_{\text{tidal}} = \frac{2GM_p R_a}{r^3}, \tag{19} \]

where \( R_a \) is the radius of the asteroid and \( r \) is the distance from the planet. Solving for the distance, we have

\[ r = R_s \left( \frac{2M_p}{M_a} \right)^{1/3} = R_p \left( \frac{2\rho_p}{\rho_a} \right)^{1/3}. \tag{20} \]

Taking the average density of the planet to be (like Earth), \( \rho_p = 5 \text{ g/cc} \) and the average density of the asteroid to be \( \rho_a = 2 \text{ g/cc} \), we have \( r = 9,700 \text{ km} \).
Astronomy 4

In the Galactic disk, there is the continuous recycling of material between stars and the gas. Stars form from the gas and return some fraction of the mass back into the gas through supernovae. The next generation of stars forms from the gas again. Let’s calculate the evolution of gas and stellar masses and metallicity in the galaxy through the recycling.

For simplicity, assume that the Galactic disk is a one-zone, closed box. Just before the \((i+1)\)-th generation of stars form, the gas and stellar masses in the box are \(M_g(i)\) and \(M_s(i)\). The total mass in the box is \(M_{\text{tot}}\). The mass fraction of metal in the gas is \(Z_g(i)\). The star formation efficiency is \(E\), and the mass fraction of the material that is ejected from stars back into the gas is \(\alpha\). Assume that the ejecta are all metals, are released before the next generation of stars forms, and are mixed instantaneously in the box.

For example, the recycling starts from the pristine gas without metal at the formation of the Galaxy. Therefore, \(M_g(0) = M_{\text{tot}}, M_s(i) = 0,\) and \(Z_g(0) = 0\).

a. (3 pts) Write the mass of gas used to form the first generation of stars, the mass ejected back to the gas as metal from supernovae and the mass of the stars after the ejection.

b. (4 pts) Derive the relation between the gas mass of \(i\)-th generation \("M_g(i)"\) and that of \((i + 1)\)-th generation \("M_g(i + 1)"\).

c. (3 pts) Derive \(M_g(i)\) using \(M_{\text{tot}}, E, \alpha,\) and \(i\).

d. (4 pts) Derive the relation between the metal mass in \(i\)-th generation \("Z_g(i)M_g(i)"\) and \((i + 1)\)-th generation \("Z_g(i + 1)M_g(i + 1)"\). Note that all the supernovae ejecta are metal.

e. (3 pts) Derive the metallicity \(Z_g(i)\) [use only \(E, \alpha,\) and \(i\) in the solution].

f. (3 pts) We can assume that the star formation efficiency is roughly \(E = 0.05\) and the fraction of mass that is released to the gas is about \(\alpha = 0.15\). How many cycles of the recycling are necessary to form the Sun’s metallicity of 0.02? Each cycle takes about \(10^8\) years (which is roughly the rotation timescale of the Galaxy). The age of the Galaxy is roughly \(10^{10}\) years, and the age of the Sun is about \(5 \times 10^9\) years. Is your calculation consistent with these ages? If not, how can we change the model?
 SOLUTION

a. The amount of mass "$M_{\text{tot}}E$" becomes the 1st generation of stars, "$M_{\text{tot}}E\alpha$" of which is ejected back to the gas due to supernovae. Therefore, after one cycle, the mass of "$M_{\text{tot}}E - M_{\text{tot}}E\alpha$" is in stars.

b. 

$$M_g(i + 1) = M_g(i) [1 - E(1 - \alpha)] \quad (21)$$

c. 

$$M_g(i) = M_{\text{tot}} [1 - E(1 - \alpha)]^i \quad (22)$$

(23)

d. 

$$Z_g(i + 1)M_g(i + 1) = Z_g(i)M_g(i) - Z_g(i)M_g(i)E + M_g(i)E\alpha \quad (24)$$

e. From the equation above,

$$Z_g(i + 1) - 1 = \frac{1 - E}{1 - E(1 - \alpha)}(Z_g(i) - 1), \quad (25)$$

so

$$Z_g(i) - 1 = \left(\frac{1 - E}{1 - E(1 - \alpha)}\right)^i (Z_g(0) - 1). \quad (26)$$

Therefore,

$$Z_g(i) = 1 - \left(\frac{1 - E}{1 - E(1 - \alpha)}\right)^i \quad (27)$$

f. 2-3 generations (more precisely 2.547 generations). So, the Sun must have born in $2 - 3 \times 10^8$ years after the formation of the Galaxy based on this model. This is not consistent with the Sun’s age. We need to consider an infall of the zero-metal gas from outside of the Galaxy (the Galaxy is not a closed one-box. ANY REASONABLE ANSWER THAT EXPLAINS GETS POINTS).
Focusing of a laser beam: Consider focusing a HeNe laser beam using a 10 cm focal length lens.

a. (10 pts) If the laser beam can be described by a Gaussian transverse profile with a waist of 1 cm, what is the size of the beam in the focus (i.e. \( \sim \) a focal length away from the lens)? Recall that the matrices describing the action of a lens with focal length \( f \) and propagation over a distance \( L \) are given by

\[
\begin{pmatrix}
1 & 0 \\
-1/f & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & L \\
0 & 1
\end{pmatrix}
\]

and the equation for the \( q \) parameter is given by

\[
\frac{1}{q(z)} = \frac{1}{R(z)} + \frac{i\lambda}{\pi w^2(z)}
\]

b. (10 pts) Imagine that you wanted to achieve a roughly uniform intensity distribution in the focal plane of the lens ("flat top") - i.e. to produce a beam whose transverse intensity distribution, \( I(r) \) (where \( r \) is the distance from the optical axis) approximates \( I(r) = I_0 \) for \( r < R_0 \) and \( I(r) = 0 \) for \( r > R_0 \).

How would you have to modify the spatial intensity and phase of the beam just before or at the lens in order to achieve this? What kind of device would allow you to do this? You might find the following relationship useful:

\[
\mathcal{F}\left\{ \text{Circ}(\sqrt{x^2 + y^2}) \right\} = \frac{J_1(2\pi\sqrt{u^2 + v^2})}{\sqrt{u^2 + v^2}}
\]

Here \( \mathcal{F} \) denotes the Fourier transform, \( J_1 \) is the first-order Bessel function of the first kind, and \((u, v)\) are conjugate variables to \((x, y)\).

SOLUTION:

\[
q_f = \frac{Aq_i + B}{Cq_i + D}; \quad \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
1 & L \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
-1/f & 1
\end{pmatrix} = \begin{pmatrix}
0 & f \\
-1/f & 1
\end{pmatrix}; \quad L = f
\]

\[
q_f = \frac{f}{\frac{-\pi w_f^2}{i\lambda} + 1} = \frac{\lambda f^2}{\lambda f + i\pi w_f^2}
\]

\[
1/q_f = 1/R_f + i\lambda/(\pi w_f^2)
\]
$$\frac{i\lambda}{\pi w_f^2} = \frac{i\pi w_i^2}{\lambda f^2}$$

(34)

$$w_f = \frac{f\lambda}{\pi w_i} = 2 \times 10^{-6}\text{m}$$

(35)

b. At the focus, the transverse field distribution is the Fourier transform of the field at the lens. If one wants to achieve a uniform intensity inside a circle, one needs a uniform field inside the same circle and therefore one needs to write a phase and amplitude at the lens which is the Fourier transform of this distribution. As given in the hint this corresponds to a Bessel function, whose phase and intensity vary with radial distance. One could accomplish this with a spatial light modulator such as a liquid crystal (similar to what is currently used for many TVs and computer monitors). The phase can be modulated directly and in combination with polarization optics can be used to modulate the intensity.
Two-level coupled atomic system: Consider a two-level atom with electronic eigenstates $|g⟩, |e⟩$ and associated energies $E_g = 0, E_e = \hbar \omega_0$ that is coupled to a weak oscillating laser field $E = E_0 \cos \omega t$ (you can assume $E_0$ to be real).

a. (5 pts) As a result of the coupling, the electron will be in a superposition $|\psi(t)⟩ = a_g(t)|g⟩ + a_e(t)|e⟩$. Find the differential equations for the amplitudes $a_g$ and $a_e$.

b. (5 pts) For further analysis, it is useful to make the transformation $a_g(t) = c_g(t)$ and $a_e(t) = c_e(t)e^{-i\omega t}$. By time averaging over rapidly oscillating terms $\propto e^{-i2\omega t}$, show that the Hamiltonian for the remaining slow dynamics of $c_g,e$ is given by

$$H = -\frac{\hbar}{2} \begin{pmatrix} 0 & \omega_R \\ -\omega_R & 2\delta \end{pmatrix}$$

(36)

where $\omega_R = \langle g|e|e⟩ \cdot E_0/\hbar$ is the Rabi frequency (assumed to be real), and $\delta = \omega - \omega_0$ is the laser detuning.

c. (5 pts) Derive the AC Stark shift of the ground state. Show that it is given by $\hbar \omega^2_R/(4\delta)$ in the limit of large detuning $\delta \gg \omega_R$.

d. (5 pts) Find the time evolution of the probabilities $P_g = |a_g|^2$ and $P_e = |a_e|^2$ for the resonant case $\delta = 0$, assuming that the atom is initially in the ground state $|g⟩$. How would you define a $\pi/2$ pulse, and how a $\pi$ pulse, and what is the physical meaning?

SOLUTION:

a. The total Hamiltonian $H = H_0 + H_I$, with $H_0 = \hbar \omega_0 |e⟩⟨e|$ and $H_I = -d \cdot E$ (where $d = e\mathbf{r}$), is given by

$$H = \hbar \begin{pmatrix} 0 & -\omega_R \cos \omega t \\ -\omega_R \cos \omega t & \omega_0 \end{pmatrix}$$

(37)

(note $\langle g|\mathbf{r}|g⟩ = \langle e|\mathbf{r}|e⟩ = 0$ because of parity) and thus

$$i\hbar \begin{pmatrix} \dot{a}_g \\ \dot{a}_e \end{pmatrix} = H \begin{pmatrix} a_g \\ a_e \end{pmatrix}$$

(38)

b. Utilizing $\cos \omega t = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})$ and the definitions for $c_{e,g}$, a straightforward calculation yields

$$i\hbar \begin{pmatrix} \dot{c}_g \\ \dot{c}_e \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & -\omega_R(1 + e^{-2i\omega t}) \\ -\omega_R(e^{2i\omega t} + 1) & 2(\omega_0 - \omega) \end{pmatrix} \begin{pmatrix} c_g \\ c_e \end{pmatrix}$$

(39)

which, after neglecting the oscillating terms, leads directly to the specified form of the Hamiltonian.
c. Diagonalization of 2x2 matrix yields eigenvalues $\Omega_\pm = -\frac{\hbar}{2} \pm \frac{\hbar}{2} \sqrt{\delta^2 + \omega_R^2} \simeq -\frac{\hbar}{2} \pm \left( \frac{\hbar}{2} + \frac{\hbar \omega_R^2}{4\delta} \right)$ (Taylor expansion). The AC Stark shift of the ground state is the $\omega_R$-dependent change of $\Omega_-$.

d. The probabilities $P_j = |a_j|^2 = |c_j|^2$. For $\delta = 0$, can easily solve equations of motion for the $c_j$ (they decouple), to obtain $P_g = \cos^2 \frac{\omega_R}{2} t$ and $P_e = \sin^2 \frac{\omega_R}{2} t$ using the initial conditions $c_g(0) = 1$, $c_e(0) = 0$. For a pulse with duration $t^*$ such that $\omega_R t^* = \pi/2$ (“$\pi/2$ pulse”), end up with $P_1 = P_2$ (coherent superposition of states); for $\omega_R t^* = \pi$ (“$\pi/2$ pulse”), obtain population inversion.
This problem concerns properties of Bloch Functions.

a. (8 pts) Show that two Bloch functions $\Psi_1 = e^{i\vec{k}_1 \cdot \vec{r}} u_{\vec{k}_1}^*(\vec{r})$ and $\Psi_2 = e^{i\vec{k}_2 \cdot \vec{r}} u_{\vec{k}_2}^*(\vec{r})$ are orthogonal if $\vec{k}_1 - \vec{k}_2$ is not a reciprocal lattice vector.

b. (8 pts) Consider a crystal potential with inversion symmetry (i.e. $V(\vec{r}) = V(-\vec{r})$). Show that for most wave vectors in the first Brillouin zone, the Bloch functions cannot have a definite parity despite the crystal inversion symmetry. Hint: You need to show that $P\Psi_k(\vec{r})$ and $\Psi_k(\vec{r})$ are orthogonal, where $P$ is the parity operator ($P\Psi_k(\vec{r}) = \Psi_k(-\vec{r})$).

c. (4 pts) There are specific wave vectors for which the Bloch function can have definite parity. Find them and describe their locations in the first Brillouin zone.

SOLUTION:

a. Show that two Bloch functions $\Psi_1 = e^{i\vec{k}_1 \cdot \vec{r}} u_{\vec{k}_1}^*(\vec{r})$ and $\Psi_2 = e^{i\vec{k}_2 \cdot \vec{r}} u_{\vec{k}_2}^*(\vec{r})$ are orthogonal if $\vec{k}_1 - \vec{k}_2$ is not a reciprocal lattice vector.

Note I’m throwing the arrow from the vectors in the solution, but $\vec{k}, \vec{r}$ and $\vec{s}$ are vectors in the following. Define:

$$A = \int d^3r \Psi_{\vec{k}_1}^*(r) \Psi_{\vec{k}_2}(r)$$

$$= \int d^3r e^{-i\vec{k}_1 \cdot \vec{r}} u_{\vec{k}_1}^*(r)e^{i\vec{k}_2 \cdot \vec{r}} u_{\vec{k}_2}(r)$$

$$= \int d^3r e^{-i(k_1-k_2) \cdot \vec{r}} u_{\vec{k}_1}^*(r) u_{\vec{k}_2}(r)$$

Then, changing variables $\vec{r} = \vec{s} - \vec{R}_n$, with $\vec{R}_n$ a direct lattice vector, and knowing that $u_k$ has the lattice periodicity by definition:

$$A = e^{-i(k_1-k_2) \cdot \vec{R}_n} \int d^3s e^{-i(k_1-k_2) \cdot \vec{s}} u_{\vec{k}_1}^*(\vec{s} - \vec{R}_n) u_{\vec{k}_2}(\vec{s} - \vec{R}_n)$$

$$= e^{-i(k_1-k_2) \cdot \vec{R}_n} \int d^3s e^{-i(k_1-k_2) \cdot \vec{s}} u_{\vec{k}_1}^*(\vec{s}) u_{\vec{k}_2}(\vec{s})$$

Therefore: $A = e^{-i(k_1-k_2) \cdot \vec{R}_n} A$ which implies that $A = 0$ unless $e^{-i(k_1-k_2) \cdot \vec{R}_n} = 1$. Sp two Bloch functions are orthogonal ($A=0$) unless $(k_1 - k_2)$ is a reciprocal lattice vector.
b. Consider a crystal potential with inversion symmetry (i.e. \( V(\vec{r}) = V(-\vec{r}) \)). Show that for most wave vectors in the first Brillouin zone, the Bloch functions cannot have a definite parity despite the crystal inversion symmetry. Hint: You need to show that \( \mathbf{P} \Psi_k(\vec{r}) \) and \( \Psi_k(\vec{r}) \) are orthogonal, where \( \mathbf{P} \) is the parity operator (\( \mathbf{P} \Psi_k(\vec{r}) = \Psi_k(-\vec{r}) \)).

We first need to show that if \( \Psi_k(r) \) is a Bloch function, then \( \mathbf{P} \Psi_k(\vec{r}) = \Psi_k(-\vec{r}) \) is also an eigenstate of the Hamiltonian:

\[
H \Psi_k(r) = E_k \Psi_k(r)
\]

\[
H = -\frac{\hbar^2}{2m} \nabla^2 + V(r)
\]

Therefore, making the change \( s = -r \) we have:

\[
H \mathbf{P} \Psi_k(\vec{r}) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] \Psi_k(-\vec{r})
\]

\[
= \left[ -\frac{\hbar^2}{2m} (-\nabla)^2 + V(-s) \right] \Psi_k(\vec{s})
\]

Since the crystal has inversion symmetry and therefore \( V(-s) = V(s) \), we have:

\[
H \mathbf{P} \Psi_k(\vec{r}) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] \Psi_k(-\vec{r})
\]

\[
= \left[ -\frac{\hbar^2}{2m} (\nabla)^2 + V(s) \right] \Psi_k(\vec{s})
\]

\[
= E_k \Psi_k(s) = E_k \Psi_k(-r) = E_k \mathbf{P} \Psi_k(r).
\]

This proves that \( \mathbf{P} \Psi_k(\vec{r}) \) is an eigenstate of \( H \) with eigenvalue \( E_k \).

Now we show that \( \mathbf{P} \Psi_k(\vec{r}) \) is a Bloch function with crystal wave vector \(-k\). We have \( \Psi_k(r) = e^{ikr}u_k(r) \), then:

\[
\mathbf{P} \Psi_k(\vec{r}) = e^{ik(-r)}u_k(-r)
\]

\[
= e^{i(-k)r}u_{-k}(r)
\]

where in the last sept we have defined \( u_{-k}(r) \equiv u_k(-r) \), the subscript indicating that \( u_{-k}(r) \) is the periodic part of a Bloch function with crystal wavevector \(-k\), and where we have applied the inversion symmetry of the lattice. Also \( u_{-k}(r) \) is periodic because \( u_k(r) \) is.

From part (a) we know that unless \( k \) and \(-k\) differ by a reciprocal lattice vector the \( \Psi_k(r) \) and \( \mathbf{P} \Psi_k(\vec{r}) \) are orthogonal. This proves that \( \Psi_k(\vec{r}) \) cannot possibly have definite parity.
c. There are specific wave vectors for which the Bloch function can have definite parity. Find them and describe their locations in the first Brillouin zone.

Comparing the expressions for $\Psi_k(\vec{r})$ and $\mathbf{P}\Psi_k(\vec{r})$, we see that if $k = -k + G$ (or $k = G/2$) then $\Psi_k(r)$ can have definite parity. This condition requires $k$ to be at the center of the Brillouin zone, at the centers of the zone boundary faces or, possibly, at other high-symmetry points.
The measured phonon dispersion curves of semiconducting GaAs are shown here, along three symmetry directions in the Brillouin zone (except one is suppressed.) The crystal structure is cubic, based on the fcc lattice. The lattice constant (of the “conventional cube”) is \( a = 5.65 \text{Å} \). The labels on the k axis are “K”, meaning \((2\pi/a)(0.75,0.75,0)\); “X” meaning \((2\pi/a)(1,0,0)\) or equivalently \((2\pi/a)(0,1,1) = (2\pi/a)(1,1,0)\); and “L” meaning \((2\pi/a)(1/2,1/2,1/2)\).

![Graph of phonon dispersion curves](image)

a. (3 pts) How many atoms are in the unit cell? You should explain how you deduce this from the data shown here.

b. (3 pts) In what sense is \((2\pi/a)(1,1,0)\) “equivalent to” \((2\pi/a)(1,0,0)\)?

c. (4 pts) The direction \(\Gamma\) to \(K\) in reciprocal space goes along \((110)\). The direction \(\Gamma\) to \(X\) goes along \((100)\). The direction \(\Gamma\) to \(L\) goes along \((111)\). At the \(L\) point, four normal mode frequencies are measured, 60, 200, 235, and 260 \((\text{in cm}^{-1})\). Between \(\Gamma\) and \(L\), how many distinct normal mode frequencies should occur and why? Draw the curves in your blue book from \(\Gamma\) to \(L\), and indicate any degeneracies.

d. (4 pts) At the point labeled \(\Gamma\), there are modes around 295 and 275 \text{cm}^{-1}. Explain what they are, and why their frequencies are relatively close to each other.

e. (3 pts) Name a material with the same, or at least a related crystal structure, where the splitting of these two modes would be zero, and another where the splitting would be larger than in GaAs.

f. (3 pts) What wavelength of infrared light will be resonantly absorbed in a measurement of the optical transmission through a thin film of GaAs?
SOLUTION:

a. Because there are $3n = 6$ branches of phonons.

b. $(2\pi/a)(1,1,1)$ is a reciprocal lattice vector of fcc, as is $(2\pi/a)(-1,-1,1)$. So the vector $k = (2\pi/a)(1,1,0)$ is equivalent to $(2\pi/a)(1,1,0) + (2\pi/a)(-1,-1,1) = (2\pi/a)(0,0,1)$. Finally, by rotational symmetry, $(2\pi/a)(0,0,1)$ is equivalent to $(2\pi/a)(1,0,0)$.

c. The experimental answer is shown in the figure below.

![Graph showing phonon branches](image)

It is not obvious that the upper two levels will cross as $k$ increases, so a diagram without crossing is equally acceptable. The 3-fold rotational symmetry of the (111) direction forces Longitudinal modes to lie exactly along (111), singly degenerate, and transverse modes to lie perpendicular, doubly degenerate. The lowest branch is the TA mode (doubly degenerate.) Next is LA (singly degenerate). Next is TO (doubly degenerate) and lower than LO (singly degenerate) near $\Gamma$, but crossing before reaching L.

d. The mode at 295 cm$^{-1}$ is “LO” (longitudinal optic). The modes at 275 cm$^{-1}$ are the “TO” doublet. They are split because the LO mode creates a long-range $E$-field and the TO modes do not. This $E$-field is small in GaAs, because it is not strongly polar. Therefore the splitting is small.

e. Silicon (non-polar) has zero splitting. NaCl (very polar) has bigger splitting.

f. The TO mode at $Q = 0$ interacts resonantly with light. This has wavelength equal to $1/275$ cm$^{-1} = 36.4$ $\mu$m.
Nuclear 1

a. (5 pts) Write down the radial dependence of the potential acting between the nucleons assuming the dominance of one pion exchange.

b. (5 pts) What is the mass of the pion? Which pion is lighter, the neutral or the charged? Why is it small on the typical hadron mass scale?

c. (10 pts) Write down the pion-nucleon Lagrangian (responsible for generating this potential) consistent with chiral symmetry

SOLUTION:

a. \( V(r) \sim \frac{1}{r} \exp(-m_\pi r) \)

b. \( m_\pi \approx 140 \text{ MeV (charged)}, \quad m_\pi \approx 135 \text{ MeV (neutral)} \). The mass is small because in the chiral limit of massless quarks, the pion is a Goldstone boson of the spontaneously broken chiral symmetry.

c. The pion-nucleon interaction is described by the Lagrangian

\[
L_{int} = \frac{2g}{f_\pi} \bar{\psi} \gamma_\mu \gamma_5 \psi \partial^\mu \phi
\]

where \( \psi \) is the spinor describing the nucleon of mass \( M \), \( \phi \) is the pseudoscalar field of the pion of mass \( m \), \( g \) is the pion–nucleon coupling constant, and \( f_\pi \) is the pion decay constant.
Nuclear 2

Consider three amplitudes of low energy pion-neutron scattering: \( M^+ \equiv M(\pi^+ n \rightarrow \pi^+ n) \), \( M^- \equiv M(\pi^- n \rightarrow \pi^- n) \), and \( M^{+0} \equiv M(\pi^+ n \rightarrow \pi^0 p) \).

a. (8 pts) Assuming isospin invariance, find the relation between these scattering amplitudes.

b. (12 pts) When the center-of-mass pion-nucleon energy is close to the mass of the \( \Delta \) resonance (isospin \( I = 3/2 \)), it dominates the scattering process. Find the ratio of the cross sections of the three scattering processes above in this energy range.

**SOLUTION:**
The isospin of the nucleon is \( I = 1/2 \), with the proton’s and neutron’s isospin projections \( I_z = +1/2, -1/2 \):

\[
p = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \quad n = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \end{pmatrix}
\]

(55)

The pions have isospin \( I = 1 \), with \( I_3 \) components as following

\[
\pi^+ = |11\rangle \quad \pi^0 = |10\rangle \quad \pi^- = |1 - 1\rangle
\]

(56)

Using the angular momentum addition, we can write

\[
\pi^+ n : |11\rangle \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 \\ 1 \\ 2 \\ 2 \end{pmatrix} + \frac{2}{\sqrt{3}} \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \end{pmatrix} + \frac{1}{\sqrt{3}} \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \end{pmatrix}
\]

(57)

\[
\pi^0 p : |10\rangle \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 \\ 1 \\ 2 \\ 2 \end{pmatrix} - \frac{2}{\sqrt{3}} \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \end{pmatrix} - \frac{1}{\sqrt{3}} \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \end{pmatrix}
\]

\[
\pi^0 n : |10\rangle \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 \\ 2 \\ 2 \\ 2 \end{pmatrix} - \frac{3}{\sqrt{3}} \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \end{pmatrix}
\]

\[
\pi^- n : |1 - 1\rangle \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 \\ 2 \\ 2 \\ 2 \end{pmatrix} + \frac{2}{\sqrt{3}} \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \end{pmatrix}
\]

\[
\pi^- p : |1 - 1\rangle \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 \\ 2 \\ 2 \\ 2 \end{pmatrix} - \frac{3}{\sqrt{3}} \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \end{pmatrix}
\]

(57)

If we denote with \( M_3 \) and \( M_1 \) the amplitude for the \( I = 3/2 \) and \( I = 1/2 \) process respectively, then we get

\[
M_{\pi^- n \rightarrow \pi^0 p} = \frac{\sqrt{2}}{3} M_3 - \frac{\sqrt{2}}{3} M_1
\]

\[
M_{\pi^- n \rightarrow \pi^- n} = M_3
\]

\[
M_{\pi^- p \rightarrow \pi^0 n} = \frac{\sqrt{2}}{3} M_3 - \frac{\sqrt{2}}{3} M_1
\]

(58)
When the $\Delta$ isobar dominates the scattering, then only $\mathcal{M}_3$ contributes, therefore the ratio of cross sections becomes

$$\sigma_{\pi^- n \to \pi^0 p} : \sigma_{\pi^- n \to \pi^- n} : \sigma_{p^+ \to \pi^0 n} = 2 : 9 : 2$$

(59)
High Energy 1

In the Standard Model (SM) electroweak symmetry is believed to be broken by the vacuum expectation of the Higgs boson. In the SM the Higgs field is a doublet with weak hypercharge $1/2$. However, in principle a different representation could be used for electroweak symmetry breaking (EWSB). In this problem we investigate the possibility that EWSB comes from a scalar triplet Higgs with weak hypercharge of 1.

a. (4 pts) We will assume that $\Phi$ gets a vacuum expectation value (VEV) by some potential that we will not specify here. Given that $\Phi$ obtains a VEV, write the corresponding term in the Lagrangian which gives rise to gauge boson masses in the SM.

b. (4 pts) The Pauli matrices in the triplet representation are listed below, and the generator of electric charge is given by $Q = \tau^3 + Y$. What is the possible VEV of $\Phi$, which we will assume has magnitude $v'$, such that electric charge is not broken?

$$
\tau^1 = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\quad
\tau^2 = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & -i & 0 \\
i & 0 & -i \\
0 & i & 0
\end{pmatrix}
\quad
\tau^3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}
$$

c. (12 pts) Using your results from part a) and part b) find the gauge boson mass matrix for the gauge bosons $W^a\mu$ and $B^\mu$ for $SU(2) \times U(1)$ (where $a = 1, 2, 3$), with corresponding gauge couplings $g$ and $g'$.

solutions on next page
SOLUTION:

a. The general form of the mass term is $|D_\mu \Phi|^2$.

b. The charges of the $\phi$ components are $(2, 1, 0)$ so the vacuum expectation value can only be $(0, 0, v')$.

c. The appropriate term in the Lagrangian is $|(gW^{a\tau a}_\mu + g'B_\mu) \ast (0, 0, v')|^2$. Substituting this into the mass matrix in the $W_1, W_2, W_3$ and $B$ space gives the solution. Doing this gives

$$m^2_{ij} = \begin{pmatrix}
\frac{g^2v'^2}{2} & \frac{ig^2v'^2}{2} & 0 & 0 \\
-\frac{ig^2v'^2}{2} & \frac{g^2v'^2}{2} & 0 & 0 \\
0 & 0 & g^2v'^2 & -gg'v'^2 \\
0 & 0 & -gg'v'^2 & g'^2v'^2
\end{pmatrix}$$

$i = (W^{1\mu}, W^{2\mu}, W^{3\mu}, B_\mu)$
The usual three generation neutrino oscillations model describes the mixing matrix between interaction and energy eigenstates of the neutrino using three angles and a complex phases. These together with two mass squared differences, $\Delta m^2_{21} = m^2_2 - m^2_1 \approx 7.6 \times 10^{-5} \text{eV}^2$ and $\Delta m^2_{31} = m^2_3 - m^2_1$, $|\Delta m^2_{31}| \approx 2.4 \times 10^{-3} \text{eV}^2$, fully specify the oscillation. Within this model, the mixing angle $\theta_{13}$ has yet to be determined, but it can be measured using $\bar{\nu}_e$ from a nuclear reactor. For distance less than about 5 km, the survival property is approximately

$$P(\bar{\nu}_e \to \bar{\nu}_e) = 1 - \sin^2 2\theta_{13} \sin^2 1.27\Delta m^2_{31} \frac{L}{E}$$ (60)

where $L$ is measured in meters, $E$ is measured in MeV, and $\Delta m^2_{31}$ is measured in eV$^2$. This problem works through the conceptual design an experiment to measure $\sin^2 2\theta_{13}$.

In three generation neutrino oscillations, the $\bar{\nu}_e$ will oscillate $\bar{\nu}_\mu$ or $\bar{\nu}_\tau$, but the total number of neutrinos is constant. However, the neutrino energy is below the production threshold for the $\mu^+$ and $\tau^+$, so the rate of charged current neutrino interactions will vary according to $P(\bar{\nu}_e \to \bar{\nu}_e)$. Put another way, the oscillation signature will be a deviation from the geometric $1/r^2$ dependency. The deviation depends on the travel distance and the neutrino energy. For a fixed distance and energy,

$$\sin^2 2\theta_{13} = \frac{1 - W_{\bar{\nu}_e}/W_{\text{geom}}}{\sin^2 1.27\Delta m^2_{31} \frac{L}{E}}.$$ (61)

where $W_{\bar{\nu}_e}$ is the observed $\bar{\nu}_e$ interaction rate, and $W_{\text{geom}}$ is the expected rate absent oscillations.

For the purposes of this question, assume that: 1) a nuclear reactor can be treated as a point source emitting $R_{\bar{\nu}_e} = 3 \times 10^{20} \bar{\nu}_e/s$; 2) the neutrinos can be treated as a 4 MeV mono-energetic flux, 3) the uncertainty on the absolute reactor flux is 5%, and 4) the cross section at 4 MeV for $p \bar{\nu}_e \to n e^+$ is $\sigma_{\bar{\nu}_e} = 4 \times 10^{-48} \text{m}^2/\bar{\nu}_e$ with a 3% uncertainty.

a. (3 pts) How far from the reactor will the maximum oscillation effect due to $\theta_{13}$ happen?

b. (3 pts) Based on the assumptions above, what is the fractional uncertainty on the neutrino interaction rate due to the flux and cross section uncertainty?

c. (3 pts) What is the size of uncertainty required to distinguish $\sin^2 2\theta_{13} = 0.05$ from zero with a $5\sigma$ significance? How large can the statistical uncertainty be if the statistical and systematic uncertainties are equal in size?

d. (3 pts) How could you design an experiment so that the large uncertainties on the flux and cross section “cancel” in the final measurement?
e. (4 pts) How many neutrino interactions will you need to record to get the required statistical accuracy?

f. (4 pts) Assuming ideal detectors (i.e. with an efficiency of 100% and no background) and an exposure of $10^8$ s, approximately how much target mass does your proposed experiment require to achieve the needed statistical accuracy?

**SOLUTION:**

a. For a 4 MeV mono-energetic neutrino flux, the Equation 60 predicts that the first minimum in the electron anti-neutrino survival probability will be found at $\frac{\pi}{2} = 1.27\Delta m^2_{31} L/E$, or $L = 2061$ m. The maximum oscillation effect will be found by placing a detector about 2 km from the reactor core where the oscillation effect will create a 5% reduction in $W_{\bar{\nu}_e}$, the measured rate of $\bar{\nu}_e$ interactions, relative to $W_{geom}$, the predicted number of $\bar{\nu}_e$ interactions absent oscillations.

b. The systematic uncertainty is 5% and 3% respectively for the number of produced neutrinos and the neutrino cross section. The interaction rate in a detector, $W$, is proportional the flux times cross section, $W_{\bar{\nu}_e} \propto \phi_{\nu} \sigma_{\bar{\nu}_e}$. Assuming the uncertainties on the flux and cross section are uncorrelated, the contribution to the fractional uncertainty on the rate will be $\sqrt{\sigma^2_\phi + \sigma^2_\sigma} = 5.8\%$. A more conservative assumption would be that the flux and cross section uncertainties are fully correlate. Under this assumption, the contribution to the fractional uncertainty would be $\sigma_\phi + \sigma_\sigma = 8\%$. Either set of assumptions will get full credit.

c. To distinguish a value of $\sin^2 2\theta_{13} = 0.05$ from zero by $5\sigma$, the total uncertainty, $\sigma_t$, will need to be less than 0.01. If the uncertainty is equally split between statistical and systematic uncertainty and assuming that the statistical and systematic uncertainty are uncorrelated, then $\sigma_t^2 = \sigma_{stat}^2 + \sigma_{sys}^2$, so the statistical contribution to the uncertainty will be $\sigma_{stat} = 0.007$.

d. This answer is more detailed than I expect from the students, but covers the full answer.

Unfortunately, due to the uncertainty in the flux and cross section, the theoretical prediction for the number of interactions, $W_{geom}$, has large uncertainty. However, the systematic uncertainty in $W_{geom}$ can be controlled by placing a second detector close to the reactor before the $\bar{\nu}_e$ oscillate. For example the “near detector” could be place within 20 m of the reactor core (or about 1% of the expected oscillation length) where it would measure the (mostly) unoscillated $\bar{\nu}_e$ flux.
Based on the considerations above, the experimental configuration should consist of two detectors. A near detector located close to the reactor core (<100 m) which would measure the rate of \( \bar{\nu}_e \), \( W_{\text{near}} \), and a far detector located at 2 km from the core which would measure the rate \( W_{\text{far}} \). The expected flux, \( W_{\text{geom}} \), is then simply \( W_{\text{geom}} = W_{\text{near}} \frac{r_{\text{near}}^2}{r_{\text{far}}^2} \) where \( r_{\text{near}} \) and \( r_{\text{far}} \) are the distances from the reactor to the near and far detectors. The measured value of \( \sin^2 2\theta_{13} \) is

\[
\sin^2 2\theta_{13} = 1 - \frac{W_{\text{far}}}{W_{\text{near}}} \frac{r_{\text{far}}^2}{r_{\text{near}}^2}
\]

and the theoretical uncertainty in the number of neutrinos produced by the reactor flux and the neutrino cross section drop out.

The near and far detectors will actually measure the number of charged current anti-neutrino interactions over some time period. The rates are

\[
W \propto \frac{N}{cTM}
\]

where \( N \) is the number of measured interactions, \( T \) is the exposure time, \( \epsilon \) is the detector efficiency, and \( M \) is the detector target mass. To minimize the uncertainty, the experimental design should require that the near and far detectors are almost identical, and have overlapping exposures. In this case,

\[
\sin^2 2\theta_{13} = 1 - \left( \frac{N_{\text{far}}}{N_{\text{near}}} \right) \left( \frac{\epsilon_{\text{near}}}{\epsilon_{\text{far}}} \right) \left( \frac{T_{\text{near}}}{T_{\text{far}}} \right) \left( \frac{M_{\text{near}}}{M_{\text{far}}} \right) \left( \frac{r_{\text{far}}^2}{r_{\text{near}}^2} \right)
\]

and each of the ratios, \( \frac{\epsilon_{\text{near}}}{\epsilon_{\text{far}}} \), \( \frac{T_{\text{near}}}{T_{\text{far}}} \), and \( \frac{M_{\text{near}}}{M_{\text{far}}} \) will be approximately equal to one. If the near and far detectors are constructed to be “identical,” many of the detector systematic errors will be correlated and cancel in the ratio.

e. From above, the statistical uncertainty must be, \( \sigma_{\text{stat}} < 0.007 \). Since the number of recorded neutrino interactions will be large, the uncertainty can be calculated assuming Gaussian errors. In this case, the uncertainty will be

\[
\sigma_{\text{stat}} = \frac{N_{\text{far}}}{N_{\text{near}}} \left( \frac{\epsilon_{\text{near}}}{\epsilon_{\text{far}}} \right) \left( \frac{T_{\text{near}}}{T_{\text{far}}} \right) \left( \frac{M_{\text{near}}}{M_{\text{far}}} \right) \left( \frac{r_{\text{far}}^2}{r_{\text{near}}^2} \right) \sqrt{\frac{\sigma^2_{\text{far}}}{N_{\text{far}}} + \frac{\sigma^2_{\text{near}}}{N_{\text{near}}}}
\]

and since the number of neutrino interactions will be Poisson distributed, \( \sigma^2_{\text{far}} = N_{\text{far}} \) and \( \sigma^2_{\text{near}} = N_{\text{near}} \) provide estimators of the statistical uncertainty. The statistical uncertainty reduces to

\[
\sigma_{\text{stat}} = \frac{N_{\text{far}}}{N_{\text{near}}} \left( \frac{\epsilon_{\text{near}}}{\epsilon_{\text{far}}} \right) \left( \frac{T_{\text{near}}}{T_{\text{far}}} \right) \left( \frac{M_{\text{near}}}{M_{\text{far}}} \right) \left( \frac{r_{\text{far}}^2}{r_{\text{near}}^2} \right) \sqrt{\frac{1}{N_{\text{far}}} + \frac{1}{N_{\text{near}}}}
\]
For an experimental configuration with identical near and far detectors with the same $T$, $\epsilon$, and $M$, the far to near ratio will be approximately $N_{far}/N_{near} = r_{near}^2/r_{far}^2$ resulting in an approximate uncertainty equal to

$$\sigma_{stat} \approx \sqrt{\frac{1}{N_{far}} + \frac{1}{N_{near}}}$$

(67)

This can be further reduced to

$$\sigma_{stat} \approx \sqrt{\frac{1}{N_{far}} + \frac{r_{near}^2}{r_{far}^2 N_{far}}}$$

(68)

The second term is small compared to the first and can be ignored, so we find that at least

$$N_{far} \approx \frac{1}{\sigma_{stat}^2} = \left(\frac{1}{0.007}\right)^2 = 20408 \approx 21000$$

(69)

interactions must be measured in the far detector. If the near detector has the same mass as the far detector, it will measure more than 8 million interactions.

f. For the identical detectors proposed above, the statistical uncertainty is controlled by the number $\bar{\nu}_e$ interactions observed at the far detector. Since the effect of neutrino oscillations is expected to be small, it can be ignored in this estimate and we find that $N_{\bar{\nu}_e} = \sigma_{\bar{\nu}_e} \phi_\nu T N_p$ where $N_{\bar{\nu}_e}$ is the number of electron anti-neutrino interactions, $\sigma_{\bar{\nu}_e}$ and $\phi_\nu$ are defined above, $T$ is the exposure time, and $N_p$ is the number of protons in the target. This means that

$$N_p = \frac{N_{\bar{\nu}_e}}{\sigma_{\bar{\nu}_e} \phi_\nu T} = \frac{N_{\bar{\nu}_e} r_{far}^2}{\sigma_{\bar{\nu}_e} R_{\bar{\nu}_e} T} = \frac{(21000 \bar{\nu}_e)(2000 \text{ m})^2}{(4 \times 10^{-48} \text{ m}^2)(3 \times 10^{20} \bar{\nu}_e/\text{s})(10^8 \text{ s})} = 7 \times 10^{29}$$

(70)

Since the proton mass is $1.7 \times 10^{-27} \text{ kg}$ there must be 1190 kg of protons. Any practical detector will not be all hydrogen, and the actual target will be about half protons so the far detector should have a mass of about 2000 kg (this final correction is not required to get full credit).