STONY BROOK UNIVERSITY
DEPARTMENT OF PHYSICS AND ASTRONOMY

Comprehensive Examination

August 28, 2013, 6:00pm - 10:30pm

General Instructions: Twelve problems are given; you should solve four problems. If you do more than four problems, you must choose which four should be graded, and only submit those four. You may do two problems from the same field only once, except for Astronomy, for which you can do up to four problems.

Each problem counts 20 points, and the solution should typically take less than 45 minutes.

Some of the problems cover multiple pages. Make sure you do all the parts of each problem you choose.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use a one page help sheet, a calculator, and with the proctor’s approval, a foreign language dictionary. No other materials may be used.

Some potentially useful information:

The atomic mass of hydrogen is 1.00794 amu.
The atomic mass of helium is 4.002602 amu.
1 amu is $1.66 \times 10^{-27}$ kg.
$c = 2.998 \times 10^8$ ms$^{-1}$.
$G = 6.673 \times 10^{-11}$ m$^3$kg$^{-1}$s$^{-2}$.
The solar luminosity is $3.85 \times 10^{26}$ W.
The mass of the Sun is $1.989 \times 10^{30}$ kg.
The radius of the Sun is $7.00 \times 10^8$ meters.
$\hbar = 1.055 \times 10^{-34}$ J
$e = 1.602 \times 10^{-19}$ C
$k_B = 1.38 \times 10^{-23}$ J/K
$mc^2 \simeq 0.5$ MeV
$hc \simeq 197$ MeV-fm.
Astronomy 1

Extremely sensitive limits to the density of smoothly-distributed intergalactic neutral H are obtained from measurements of the optical depth to the redshifted Lyα resonance transition in spectra of high-redshift quasi-stellar objects or QSOs. Because the redshift of the intergalactic medium varies continuously, the Lyα resonance transition gives rise to a continuous absorption process at wavelengths between the rest-frame wavelength \( \lambda_\alpha = 1215.67 \, \text{Å} \) of the Lyα resonance and the redshifted wavelength \( \lambda_\alpha(1 + z_{\text{em}}) \) of the Lyα resonance at the emission redshift \( z_{\text{em}} \) of the QSO. This method was applied by Gunn and Peterson in 1965 to a QSO of redshift \( z_{\text{em}} = 2.01 \) and has become known as the “Gunn–Peterson” effect. Observations of the redshifted Lyα spectral regions of QSOs of redshift as large as \( z_{\text{em}} \approx 6 \) yield upper limits to the Gunn-Peterson optical depth of \( \tau < \sim 0.05 \), which constrains the intergalactic medium to be very highly ionized at all redshifts \( z < 6 \). More recent observations of the redshifted Lyα spectral regions of QSOs of redshift in the range \( z_{\text{em}} = 6.28 \) to as large as \( z_{\text{em}} = 7.085 \) yield a detection of significant Gunn-Peterson optical depth (i.e. \( \tau \approx 1 \)), suggesting that “reionization” of the intergalactic medium occurred at a redshift around \( z \approx 6.5 \), presumably due to the formation of the first sources of ionizing radiation (i.e. massive stars, supernovae, or QSOs). This problem examines the Gunn–Peterson effect in an Einstein-de Sitter cosmological model of matter density parameter \( \Omega_m = 1 \), for which the relationship between cosmic time \( t \) and redshift \( z \) is

\[
t = \frac{2}{3H_0}(1 + z)^{-3/2},
\]

where \( H_0 = 70.4 \, \text{km s}^{-1} \, \text{Mpc}^{-1} \) is the Hubble constant. (Note: 1 pc = \( 3.09 \times 10^{18} \, \text{cm} \).)

a. (5 pts.) When deriving the Gunn-Peterson effect, the frequency dependence of the redshifted Lyα resonance is usually taken to be a delta function in frequency \( \delta[(1+z)\nu - \nu_\alpha] \), where \( \nu_\alpha = 2.466 \times 10^{15} \, \text{Hz} \) is the rest-frame frequency of the Lyα resonance, since any thermal or other broadening is negligible in comparison to the range of redshifts involved. Taking the integrated absorption cross section \( \sigma \) of the Lyα resonance to be

\[
\sigma = \frac{\pi e^2}{m_e c} f = 2.654 \times 10^{-2} f \, \text{cm}^2 \, \text{Hz}^{-1}
\]

where \( e \) is the electron charge, \( m_e \) is the electron mass, \( c \) is the speed of light, and \( f = 0.92 \) is the oscillator strength of the Lyα resonance, and using the proper distance element \( dl = cd\tau \) appropriate for an Einstein-de Sitter cosmological model, write down the infinitesimal optical depth \( d\tau \) that corresponds to some arbitrary infinitesimal redshift interval \( dz \) in terms of the comoving number density \( n_0 \) of neutral H.
b. (5 pts.) Integrate the result of part A with respect to redshift from redshift $z = 0$ to the emission redshift $z_{em}$ of the QSO to derive a relationship between the optical depth $\tau$ and the comoving number density $n_0$ of neutral H in terms of $\sigma$, $\nu_\alpha$, and $z$.

c. (5 pts.) Taking the upper limit to the Gunn-Peterson optical depth at redshift $z = 3$ to be $\tau < 0.05$, use the result of part B to set an upper limit to the comoving number density $n_0$ of neutral H at redshift $z = 3$. Compare this to the comoving number density $n_b$ of baryons

$$n_b = 1.1 \times 10^{-5} \Omega_b h^2 \text{ cm}^{-3},$$

where $\Omega_b h^2$ is the physical baryon density parameter and has a value set by WMAP of $\Omega_b h^2 = 0.02260$, to set a lower limit to the ionization of the intergalactic medium at redshift $z = 3$.

d. (5 pts.) Recent observations of a QSO of redshift $z = 7.085$ suggest a transmission profile at wavelengths just longward of redshifted Ly$\alpha$ at the emission redshift of the QSO that is consistent with a Ly$\alpha$ damping wing. Discuss the significance of these observations in constraining the epoch of reionization.

**Solution**

a. In terms of the infinitesimal column density $dN$,

$$d\tau = dN \sigma \nu = n_0 (1 + z)^3 d(1 + z) \sigma \delta[(1 + z) \nu - \nu_\alpha].$$

For an Einstein–de Sitter cosmological model,

$$c dt = \frac{c}{H_0} (1 + z)^{-5/2} dz.$$  

So

$$d\tau = n_0 (1 + z)^3 \sigma \delta[(1 + z) \nu - \nu_\alpha] \frac{c}{H_0} (1 + z)^{-5/2} dz$$

$$= n_0 \frac{c}{H_0} (1 + z)^{1/2} \sigma \delta[(1 + z) \nu - \nu_\alpha] dz.$$  

b. Integrate over redshift to get

$$\tau = \int d\tau$$

$$= \int_0^{z_{em}} n_0 \sigma \frac{c}{H_0} (1 + z)^{1/2} \delta[(1 + z) \nu - \nu_\alpha] dz.$$
Now let
\[ x = (1 + z)\nu \] (11)
and
\[ dx = \nu dz. \] (12)

Then for \( \lambda_\alpha < \lambda < (1 + z)\lambda_\alpha \)
\[ \tau = n_0\sigma \frac{c}{H_0} \frac{1}{\nu_\alpha} (1 + z)^{3/2} \] (13)
\[ = n_0 \frac{\pi e^2}{m_e c} f \frac{c}{H_0} \frac{1}{\nu_\alpha} (1 + z)^{3/2}. \] (14)

where \( \lambda_\alpha = c/\nu_\alpha \), otherwise
\[ \tau = 0. \] (15)

c. Evaluating equation (14) for \( \tau < 0.05, \nu_\alpha = 2.47 \times 10^{15} \) Hz, and \( z = 3 \) yields an upper limit to the comoving density of neutral H
\[ n_0 = 4.8 \times 10^{-14} \text{ cm}^{-3}. \] (16)

Comparing to the comoving density of baryons \( n_b = 2.49 \times 10^{-7} \) yields an upper limit the neutral fraction of the intergalactic medium of \( 1.9 \times 10^{-7} \).
d. At some wavelength sufficiently far from the resonance, the Ly\( \alpha \) damping wing is optically thin and therefore allows for a measurement of (rather than a limit to) the comoving density of neutral H.
Astronomy 2

Observations of distant Type Ia supernovae indicate that (1) the Universe is expanding and (2) the rate of expansion is accelerating. In the context of a homogeneous, isotropic Universe described by the Friedmann equations

\[
\dot{R} = \frac{8\pi G \rho}{3} R^2 - kc^2
\]

and

\[
\ddot{R} = 4\pi G \left( \rho + \frac{3p}{c^2} \right) R,
\]

where \( R \) is the scale factor of the Universe, \( G \) is the gravitational constant, \( \rho \) is the density, \( p \) is the pressure, \( k \) is the dimensionless curvature constant, and \( c \) is the speed of light, these observations imply that either (1) there was a time in the past at which the Universe was arbitrarily small and dense (the “Big Bang” cosmological model) or (2) there was a time in the past at which the Universe transitioned from a contracting phase to an expanding phase at some minimum scale factor (the “bounce” cosmological model).

a. (5 pts.) Starting from the Friedmann equations, derive an expression for the Hubble parameter \( \dot{R}/R \) in terms of the current Hubble constant \( H_0 \) and the current scale factor \( R_0 \) for a cosmological model of arbitrary matter density parameter \( \Omega_m \) and vacuum energy density parameter \( \Omega_\Lambda \).

b. (5 pts.) Show that the expression for the Hubble parameter derived in part A admits zero, one, or two extrema (maxima or minima) of \( x \equiv R/R_0 > 0 \), depending on the values of \( \Omega_m \) and \( \Omega_\Lambda \). Further show that two extrema are obtained if and only if \( \Omega_m \) and \( \Omega_\Lambda \) satisfy

\[
\frac{2}{3}(\Omega_m + \Omega_\Lambda - 1)^{3/2} - \Omega_m(3\Omega_\Lambda)^{1/2} > 0.
\]

c. (5 pts.) Consider a cosmological model of \( \Omega_m = 0.3 \) (which is consistent with current measurements) and \( \Omega_\Lambda = 1.8 \). Verify that this model exhibits an extremum at \( x \approx 0.56 \), and show that this extremum is a maximum, i.e. that this is a bounce cosmological model.

d. (5 pts.) Recollecting the relationship between cosmological redshift \( z \) and scale factor \( R \), describe an observation that rules out the cosmological model of part C.

Solution
a. Start with
\[ \dot{R} = \frac{8\pi G \rho}{3} R^2 - kc^2. \quad (4) \]
Now write
\[ \rho = \rho_m + \rho_\Lambda \]
\[ = \frac{3H_0^2}{8\pi G} \left[ \Omega_m \left( \frac{R}{R_0} \right)^{-3} + \Omega_\Lambda \right] \quad (6) \]
and
\[ kc^2 = H_0^2 R_0^2 (\Omega_m + \Omega_\Lambda - 1). \quad (8) \]
Then
\[ \left( \frac{H}{H_0} \right)^2 = \frac{1}{H_0^2} \left( \frac{\dot{R}}{\dot{R}_0} \right)^2 = \Omega_m x^{-3} + \Omega_\Lambda - (\Omega_m + \Omega_\Lambda - 1)x^{-2}. \quad (9) \]

b. An extremum of \( x \) is obtained at \( H = \dot{R}/R = 0 \). Setting equation (9) equal to 0 yields
\[ \Omega_m + \Omega_\Lambda x^3 - (\Omega_m + \Omega_\Lambda - 1)x = 0. \quad (10) \]
Plot
\[ \Omega_\Lambda(x^3 - x) \quad (11) \]
and
\[ -\Omega_m + (\Omega_m - 1)x \quad (12) \]
versus \( x \) to show that there are zero, one, or two of \( x \), depending on the values of \( \Omega_m \) and \( \Omega_\Lambda \). There is one extremum of \( x \) when
\[ \frac{d}{dx}[\Omega_\Lambda(x^3 - x)] = \frac{d}{dx}[-\Omega_m + (\Omega_m - 1)x] \quad (13) \]
at
\[ \Omega_\Lambda(x^3 - x) = \Omega_m - (\Omega_m - 1)x \quad (14) \]
Equation (13) yields
\[ \Omega_\Lambda(3x^2 - 1) = \Omega_m - 1. \quad (15) \]
The value of \( x \) at the solution of equation (15) is
\[ x_0 = \left( \frac{\Omega_m + \Omega_\Lambda - 1}{3\Omega_\Lambda} \right)^{1/2} \quad (16) \]
Substituting \( x_0 \) from equation (16) for \( x \) in equation (14) then yields
\[ \frac{2}{3}(\Omega_m + \Omega_\Lambda - 1)^{3/2} - \Omega_m (3\Omega_\Lambda)^{1/2} = 0. \quad (17) \]
There are two extrema of $x$ when

$$\Omega_{\Lambda}(x^3 - x) < \Omega_m - (\Omega_m - 1)x$$  \hspace{1cm} (18)

at the value of $x_0$ from equation (16). Substituting $x_0$ from equation (16) for $x$ in equation (18) then yields

$$\frac{2}{3}(\Omega_m + \Omega_{\Lambda} - 1)^{3/2} - \Omega_m(3\Omega_{\Lambda})^{1/2} > 0.$$  \hspace{1cm} (19)

c. Evaluating the left-hand side of equation (19) for $\Omega_m = 0.3$ and $\Omega_{\Lambda} = 1.8$ yields $0.07$, which satisfies $0.07 > 0$. Evaluating the right-hand side of equation (9) for $\Omega_m = 0.3$, $\Omega_{\Lambda} = 1.8$, and $x = 0.56$ yields $0.00062$, which satisfies $(H/H_0)^2 \approx 0$. To show that this solution is a maximum, start with

$$\ddot{R} = 4\pi G(\rho + \frac{3p}{c})R.$$  \hspace{1cm} (20)

Now write

$$p_m = 0$$  \hspace{1cm} (21)

and

$$p_{\Lambda} = -\rho_{\Lambda}c^2.$$  \hspace{1cm} (22)

Substituting equations (7), (21), and (22) into equation (20) then yields

$$\frac{\ddot{R}}{R_0} = H_0^2 \left( -\frac{1}{2}\Omega_m x^{-2} + \Omega_{\Lambda}x \right).$$  \hspace{1cm} (23)

Evaluating the right-hand side of equation (23) for $\Omega_m = 0.3$, $\Omega_{\Lambda} = 1.8$, and $x = 0.56$ yields $0.53H_0^2$, so $\ddot{R} > 0$ and $x = 0.56$ is a minimum.

d. The relationship between cosmological redshift $z$ and scale factor $R$ is

$$1 + z = \left(\frac{R}{R_0}\right)^{-1} = x^{-1}.$$  \hspace{1cm} (24)

In the cosmological model of part C, the minimum scale factor is $x_{\min} = 0.56$, which means the maximum redshift is $z_{\max} = x_{\min}^{-1} - 1 = 0.79$. There are many galaxies and QSOs with redshifts $z > 0.79$. 


Astronomy 3

An exoplanet is discovered by the Doppler spectroscopy (wobble) method. A solar-type star \((M = 2 \cdot 10^{33} \text{ g}, R = 7 \cdot 10^{10} \text{ cm}, T_{\text{eff}} = 5800 \text{ K})\) is observed to have spectral features which undergo periodic shifts in wavelength of approximately 1 part in \(10^7\) amplitude \((\Delta \lambda / \lambda = 10^{-7})\) with a period of \(10^7\) s. The orbit appears to be circular. The exoplanet discovery is confirmed by observations of transits with the same period in which the brightness of the star temporarily decreases, with a maximum observed flux decrease of approximately 1%.

a. (12 pts.) Calculate the mass and the radius of the exoplanet.

b. (4 pts.) Is the planet likely to be largely composed of gas, water, ice or rock? Justify your answer.

c. (4 pts.) Neglecting atmospheric effects, estimate the average temperature of the exoplanet.

Solution

a. With the assumption that the exoplanet’s mass is small compared to the star’s mass, i.e., \(M_2 << M_1\), Kepler’s Third Law gives the semi-major axis \(a\) of the planet’s orbit:

\[
a^3 = \frac{G(M_1 + M_2)}{4\pi^2} P^2 \approx \frac{GM_1}{4\pi^2} P^2 \tag{1}\]

where \(P\) is the orbital period. This yields \(a \simeq 7 \times 10^{12}\) cm.

The velocity of the planet in it’s circular orbit is

\[
V_2 = 2\pi a / P \simeq 4.4 \times 10^6 \text{ cm/s}. \tag{2}\]

The Doppler shift gives the star’s orbital velocity:

\[
\frac{\Delta \lambda}{\lambda} = \frac{V_1 \sin i}{c} \simeq 10^{-7}, \tag{3}\]

where \(i\) is the inclination of the orbit with respect to the plane of the sky. However, the observation of transits implies that \(\sin i \simeq 1\), and we find \(V_1 \simeq 3000 \text{ cm/s}\). Momentum conservation now gives the planet’s mass:

\[
M_2 = M_1 V_1 / V_2 \simeq 1.4 \times 10^{30} \text{ g}. \tag{4}\]

The planet’s radius is estimated from the amplitude of the transit: \(R_2 / R_1 \simeq \sqrt{0.01}\) and \(R_2 \simeq 7 \times 10^9\) cm.
b. The planet’s average density is

\[ \rho_2 = \frac{3M_2}{4\pi R_2^3} \simeq 1 \text{ g/cm}^3. \]  

(5)

Although this is the density of ice or water, the planet must be composed of something less dense due to gravitational compression. Hence, this is a gaseous giant planet.

c. Assuming blackbodies for the Sun and the planet, the planet’s temperature in equilibrium will be

\[ T_2 = T_1 \sqrt{\frac{R_1}{2a}} \simeq 410 \text{ K}. \]

This is found by equating the power received by the planet \(4\pi \sigma R_1^2 T_1^4 \times \pi R_2^2/(4\pi a^2)\) with the power radiated by the planet \(4\pi R_2^2 \sigma T_2^4\).
Astronomy 4

Consider a uniform-composition hydrostatic atmosphere on the surface of a compact object. If the atmosphere is thin compared to the radius of the underlying star, then the plane-parallel approximation holds (we can neglect curvature and take the gravitational acceleration to be constant). Assume that the ideal gas law holds and the ratio of specific heats, $\gamma$, is constant.

Take the pressure and density at the base of the atmosphere to be $p_b$ and $\rho_b$ respectively.

a. (4 pts.) Derive the density as a function of height in the case of an isothermal atmosphere.

b. (3 pts.) Now instead assume that the atmosphere is isentropic, and again derive the density as a function of height.

c. (3 pts.) Imagine displacing a fluid element upwards in each atmosphere. Assume that there are no local heat sources in the element. Describe, qualitatively the motion of this fluid element in each of the two atmospheres above.

d. (6 pts.) Now denote our atmosphere’s hydrostatic pressure profile as $p_0(r)$ (it doesn’t matter what the temperature profile is, as long as it is hydrostatic). Consider a fluid element moving through this atmosphere. In the absence of any heat sources, and assuming that the Mach number is small, to good approximation we can say that the pressure in the fluid element remains $p_0$ as it advects through the atmosphere.

By considering the Lagrangian history of the pressure, show that this approximation implies a constraint on the velocity field:

$$\nabla \cdot (p_0^{1/\gamma} U) = 0$$

(1)

Hint: you can approach this either by considering $p_0$ as a function of $\rho$ (density) and $e$ (the specific internal energy), or alternately $p_0$ as a function of $\rho$ and $s$ (the entropy). This is called the pseudo-incompressible constraint in atmospheric flows.

e. (4 pts.) Finally, consider the isentropic atmosphere from part b, and imagine that the above constraint holds. Show that even in the presence of a random velocity field, the density at a specific point in the atmosphere remains unchanged.

Solution

a. Start with the equation of hydrostatic equilibrium:

$$\frac{dp}{dr} = -\rho |g|$$

(2)
An isothermal gas obeys \( p = \rho c^2 \), where \( c \) is the sound speed. Putting this into the equation of HSE, we have:

\[
\frac{d\rho}{\rho} = -\frac{|g|}{c^2} \, dr
\]

Integrating, we have

\[
\int_{\rho_0}^{\rho'} \frac{d\rho'}{\rho'} = -\int_{r_0}^{r} \frac{|g|}{c^2} \, dr'
\]

which gives

\[
\rho(r) = \rho_b e^{-\frac{|g|r}{c^2}}
\]

b. This procedure is similar, but now our equation of state can be expressed as \( p = K \rho^\gamma \), where the constant \( K = \frac{p_b}{\rho_b^\gamma} \).

Substituting this into the equation of HSE, we have:

\[
\rho^\gamma - 2 \, d\rho = -\frac{|g|}{\gamma K} \, dr
\]

Integration and eliminating \( K \) gives:

\[
\rho(r) = \rho_0 \left[ 1 - \frac{\gamma - 1}{\gamma} \frac{|g|\rho_0}{p_0} \frac{r}{p} \right]^{1/(\gamma - 1)}
\]

c. The isentropic atmosphere is convectively unstable—the fluid element will continue to rise. The isothermal atmosphere has higher entropy over lower entropy—this is stable against convection. The fluid element will oscillate and drive gravity waves.

d. Consider the Lagrangian derivative of \( p_0 \):

\[
\frac{Dp_0}{Dt} = \left. \frac{\partial p_0}{\partial \rho} \right|_e \frac{D\rho}{Dt} + \left. \frac{\partial p_0}{\partial e} \right|_\rho \frac{De}{Dt}
\]

For a constant \( \gamma \), we have \( p = \rho e(\gamma - 1) \), and

\[
\left. \frac{\partial p_0}{\partial \rho} \right|_e = e(\gamma - 1) = \frac{p}{\rho}
\]

\[
\left. \frac{\partial p_0}{\partial e} \right|_\rho = \rho(\gamma - 1)
\]

From the first-law of thermodynamics, we know:

\[
\frac{De}{Dt} = -p \frac{D(1/\rho)}{Dt} = \frac{p}{\rho^2} \frac{D\rho}{Dt}
\]

so we have:

\[
\frac{Dp_0}{Dt} = \frac{p}{\rho^\gamma} \frac{D\rho}{Dt}
\]
Mass conservation tells us that \( \frac{D\rho}{Dt} = -\rho \nabla \cdot U \), and for a time-independent atmosphere, we know that \( \frac{Dp_0}{Dt} = U \cdot \nabla p_0 \), giving

\[
\nabla \cdot U + \frac{1}{\gamma p_0} U \cdot \nabla p_0 = 0
\]

which is equivalent to

\[
\nabla \cdot (p^{1/\gamma} U) = 0
\]

Alternately, considering \( p_0 = p_0(\rho, s) \), we have

\[
\frac{Dp_0}{Dt} = \frac{\partial p_0}{\partial \rho} \left|_s \right. \frac{D\rho}{Dt} + \frac{\partial p_0}{\partial s} \left|_\rho \right. \frac{Ds}{Dt}
\]

Then since the first adiabatic exponent is

\[
\Gamma_1 = \left. \frac{d \log p}{d \log \rho} \right|_s
\]

and \( \Gamma_1 = \gamma \) for an ideal gas, we have

\[
\frac{Dp}{Dt} = \frac{p}{\rho} \gamma (-\nabla \cdot U)
\]

This is the same as above.

e. Mass conservation is

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho U) = 0
\]

Since the constraint holds, we have \( \rho = \rho_0 = K^{-1} p_0^{1/\gamma} \), and then

\[
\frac{\partial \rho}{\partial t} = -K^{-1} \nabla \cdot (p_0^{1/\gamma} U) = 0
\]

So in the Eulerian frame, the density does not change.
This problem concerns the detection of light.

a. (3 pts) A 1 nW beam of laser light with a wavelength of 800 nm is incident on a photodetector. What is the smallest possible r.m.s uncertainty in the determination of the power that can be made in one second?

b. (3 pts.) Same as part a.), but now with a wavelength of 8 µm.

c. (4 pts.) Real photodetectors suffer from limitations both in their efficiency for detecting incident photons and their own intrinsic noise sources. Among the best photodetectors are photodiodes. Photodiodes are semiconductor devices that convert light into an electric current which can subsequently be measured with an ammeter. For example, a commercial large area photodiode from THORLabs inc. lists a “responsivity” of $R = 0.4 \text{ A/W}$ and a “dark current” of $i_D = 20 \text{ nA}$, so that the current out of the photodiode is given by:

$$i_{PD} = RP + i_D$$

where $P$ is the incident power. Repeat your estimates in part a.) using this real detector. Assume the dark current shot noise dominates over all other sources of noise inherent to the detection system.

d. (5 pts.) Now consider a coherent detection, or “heterodyne” scheme in which a weak optical signal is interfered with another, more intense laser beam (the local oscillator or “LO”) with a slight frequency offset $\Delta\omega$ before detection. The electric field at the detector is now the superposition of the two fields.

$$E(t) = E_{sig} \cos(\omega t) + E_{LO} \cos([\omega + \Delta\omega]t)$$

where $E_{sig}$ and $E_{LO}$ are constants. Calculate the time dependent light power recorded by a detector with bandwidth much slower than the optical carrier frequency $\omega$ but much larger than the small frequency offset $\Delta\omega$. Express your answer in terms of the individual powers of the two beams $P_{sig}$ and $P_{LO}$.

e. (5 pts.) Estimate the r.m.s. uncertainty in determining the power of the 1 nW beam of part a. using the coherent detection scheme with an intense local oscillator with power of 1 mW and the real detector of part c.). Again use a one second measurement time. *Hint: Rephrase your answer form part 1 in terms of a noise per unit bandwidth.*

Solution
a. The photon energy is $hc/\lambda = 2.5 \times 10^{-19}$ J so in 1 second $N = 4 \times 10^9$ photons are collected. So the RMS uncertainty in this is $\sqrt{N} = 6.3 \times 10^4$ photons or a corresponding uncertainty $\Delta P = 15$ fW.

b. Now the photon energy is 10 times lower so for the same power there will be 10 times as many photons and the corresponding uncertainty is reduced by $\sqrt{10} \approx 3$ so now we have $\Delta P = 5$ fW.

c. The 1 nW beam will give a 400 pA photocurrent or $2.5 \times 10^9$ electrons/second. The shot noise on this in 1 s DC measurement will then be $\Delta N = \sqrt{2.5 \times 10^9}$ corresponding to an RMS noise of $i_{NP} = 8$ fA. Similarly, the shot noise on the dark current is $i_{ND} = 56$ fA. The uncertainties add in quadrature so

$$i_N = \sqrt{i_{ND}^2 + i_{NP}^2} = 57 \text{ fA}$$

So the noise is dominated by the shot noise of the dark current, not the photocurrent, and the RMS uncertainty in the optical power is $57 \text{ fA}/(0.4 \text{A/W}) = 140$ fW.

d. Using complex notation.

$$E(t) = e^{-i(\omega + \Delta \omega)t} E_{LO} \left[ 1 + \frac{E_{sig}}{E_{LO}} e^{i\Delta \omega t} \right]$$

The first exponential represent the fast optical carrier for which the detector will only see the time average. The rest of the expression represents a slowly varying envelope to this carrier that the detector can see the variation of. The power $P(t)$ will be proportional to square modulus of the amplitude. Since we can express our answer in terms of the input powers of the signal and LO beams we do not need to keep track of prefactors. The result after a very little algebra is the familiar equation for two beam interference:

$$P(t) = P_{LO} + P_{Sig} + 2\sqrt{P_{LO}P_{Sig}} \cos(\Delta \omega t)$$

e. Now the strength of our nW beam is encoded in the AC heterodyne beat signal at the photodetector. The beat signal has power $P_{beat} = 2\sqrt{P_{LO}P_{sig}}$ so the signal is related to the beat strength by

$$P_{sig} = \frac{1}{4P_{LO}}P_{beat}^2$$

Now since the power in the signal is so much weaker than the local oscillator, the noise is completely dominated by the noise of the local oscillator! The dark current of the photodetector is now also irrelevant. Observed in the Fourier domain (i.e. with a spectrum analyzer), the peak at $\Delta \omega$ will sit on the noise floor provided by
the shot noise of the local oscillator. The uncertainty in measuring the beat power is given by the uncertainty in the local oscillator power. The photon shot noise per unit bandwidth can be written as

\[ \Delta P_{\text{beat}} \approx \Delta P_{\text{LO}} = \sqrt{2h\nu P_{\text{LO}} B} \]  \hspace{1cm} (7)

Where \( B \) is the measurement bandwidth. If you don’t believe me, take the standard expression for the shot noise of a current and take a detector with a quantum efficiency of one. A 1 second measurement time on an AC signal corresponds to a 1 Hz noise bandwidth. Note that there is a subtlety in the factor of \( \sqrt{2} \) that appears in this equation versus what we calculated in the DC case. The noise bandwidth of a 1 second DC (baseband) measurement is actually only 0.5 Hz (see Phillip C. D. Hobbs’ Book: “Building Electro-Optic Systems, Making it All Work” chapter 13 for some discussion of this subtlety). In any event, differentiating the first equation, the uncertainty in the signal power is now given by:

\[ \Delta P_{\text{sig}} = \frac{P_{\text{beat}}}{2P_{\text{LO}}} \Delta P_{\text{LO}} = \sqrt{2P_{\text{sig}}h\nu B} = 21fW \]  \hspace{1cm} (8)

If you are not impressed, you should be. The heterodyne detection scheme has acted as a noiseless amplifier, recovering the signal’s photon shot noise limited performance even with a noisy detector! It’s no wonder that all radio receivers employ heterodyne detection.
In this problem we will analyze the Purcell effect, i.e. how the radiative properties of atoms can be modified by placing them inside a high finesse optical cavity that is resonant with an atomic transition.

a. (4 pts.) The spontaneous emission of a two-level atom (states $|g\rangle, |e\rangle$) at optical wavelengths is described by the Einstein coefficient $A_{eg} = \frac{1}{2\pi \epsilon_0^2 e^2 \omega_0^2 mc^3} f$, where $m$ is the electron mass, $\omega_0 = (E_e - E_g)/\hbar$ is the atomic transition frequency and $f$ is the so-called oscillator strength. Demonstrate that in the quantum mechanical case, the latter is given by $f = \frac{2m\omega_0}{3\hbar} |x_{eg}|^2$, where $\hat{x}$ denotes the electron displacement.

b. (4 pts.) Using the information above, calculate the spatially averaged transition dipole moment $|\mu_{eg}|^2$ in terms of the Einstein coefficient.

c. (4 pts.) Derive an expression for the spontaneous emission rate for a two-level atom in free space $W_{fs}$.

d. (4 pts.) Determine the quality factor ($Q$) of a high finesse optical cavity of length $L$, refractive index $n$, and mirror reflectivity $R \ll 1$.

e. (4 pts.) Calculate the spontaneous emission rate for the same two-level atom but now inside the resonant cavity $W_{cav}$, and derive the Purcell factor $W_{cav}/W_{fs}$.

Solution

a. A simple argument relies on comparing the quantum electron oscillator equation $\frac{d^2}{dt^2} \langle x \rangle + \omega_0 \langle x \rangle = \frac{2e\omega_0}{\hbar} \langle x_{eg} \rangle (\langle x_{eg} \rangle \cdot E)$ with the Newtonian equation for the electron displacement, $\frac{d^2z}{dt^2} + \omega_0^2 z = \frac{e}{m} E$.

b. Because of symmetry arguments, $|\mu_{eg}|^2 = \frac{1}{3} e^2 |x_{eg}|^2 = \left( \frac{\pi \epsilon_0 \hbar c^3}{\omega_0^3} \right) A_{21}$.

c. We have to use Fermi’s Golden Rule $W = \frac{2\pi}{\hbar} |\mu_{eg}|^2 N(\omega)$ and the well-known formula for the density of photon states in k-space for a volume $V$, $N(k) dk = \frac{V}{(2\pi)^3} 4\pi k^2 dk$. Using the dispersion relation in free space $\omega(k) = ck$, we have $dk = d\omega/c$. We thus obtain the density of states in frequency space as $N_{fs}(\omega) = \frac{4\pi V \omega^3}{(2\pi c)^3} \tau$, and can directly calculate $W_{fs}$.

d. The quality factor of a cavity is defined as $Q = \omega_0 \tau$, in which $\omega_0$ is the resonant mode frequency and $\tau$ is the photon decay time of the cavity. We have to consider a pulse of $N$ photons entering the cavity. After a half-round trip, which takes a time $\Delta t = nL/c$, the quantity $\Delta N = (1 - R)N$ defines the number of photons that are lost upon reflection at a cavity mirror. The photon loss rate can then be calculated as $\frac{dN}{dt} \sim \frac{\Delta N}{nL/c} = \frac{c(1-R)}{nL} N$. From here we can estimate the decay time of the cavity as $\tau = \frac{nL}{c(1-R)}$ and thus we can obtain $Q$. 

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e. The key is to obtain the density of states in frequency space inside the cavity. This distribution \( N_{\text{cav}}(\omega) \) is given by a normalized Lorentzian of width \( \Delta \omega = 1/\tau \) centered around the resonance frequency \( \omega_0 \), i.e. \( N_{\text{cav}} = (2/\pi \Delta \omega) [(\omega - \omega_0)^2 + (\Delta \omega)^2]^{-1} \). Thus, on resonance, \( N_{\text{cav}}(\omega_0) = 2/(\pi \Delta \omega) = \frac{2Q}{\pi \omega_0} \). This leads to the Purcell factor

\[
\frac{W_{\text{cav}}}{W_{fs}} = \frac{N_{\text{cav}}}{N_{fs}} = \frac{2Q/\pi \omega_0}{V \omega_0^2/2\pi^2 c^3} = \frac{4\pi c^3 Q}{\omega_0^3 V}\]  

(1)
Consider a one dimensional lattice (i.e. chain) with a spin 1 (i.e. spin triplet, denoted \[ \mathbf{1} \]) degree of freedom on each lattice site, with Hamiltonian

\[ H = \sum_i \mathbf{\vec{S}}_i \cdot \mathbf{\vec{S}}_{i+1} \]

This is just a Heisenberg interaction, which gives an energy penalty for nearest neighbor spins pointing in the same direction. \( H \) has an interesting many-body ground state, realizing emergent fractionalization of the spin quantum number (the so-called “Haldane phase”). Although \( H \) itself is difficult to solve, it can be simplified in an unlikely way, by adding a quartic interaction:

\[ H_{2\text{-sites}} = \sum_i \left[ \mathbf{\vec{S}}_i \cdot \mathbf{\vec{S}}_{i+1} + g \left( \mathbf{\vec{S}}_i \cdot \mathbf{\vec{S}}_{i+1} \right)^2 \right] \]

a. (4 pts.) First consider just two sites, \( i \) and \( i + 1 \). What is the dimension of the Hilbert space of the two site system, and what is its Clebsch-Gordon decomposition into spin multiplets of \( \mathbf{\vec{S}}_{2\text{-sites}} = \mathbf{\vec{S}}_i + \mathbf{\vec{S}}_{i+1} \)?

b. (4 pts.) Find the value of \( g \) at which \( H_{2\text{-sites}} = \mathbf{\vec{S}}_i \cdot \mathbf{\vec{S}}_{i+1} + g \left( \mathbf{\vec{S}}_i \cdot \mathbf{\vec{S}}_{i+1} \right)^2 \) is degenerate between the lowest two spin multiplets of \( \mathbf{\vec{S}}_{2\text{-sites}} \) found in the decomposition in part (a) (hint: re-write the Hamiltonian in terms of \( S^2_i \), \( S^2_{i+1} \) and \( S^2_{2\text{-sites}} \) What is the resulting ground state degeneracy?

c. (4 pts.) Writing out the ground state wave-functions found in part (b) in the usual \( S^z \) eigenbasis \( \{ | -1 \rangle_i, |0\rangle_i, |1\rangle_i \} \) of the spin 1’s is not particularly enlightening. However, for the moment let us think of the spin 1 on each site \( i \) as being made up from the fusion of two fictitious spin \( \frac{1}{2} \)'s (recall the Clebsch-Gordon rule \( \left[ \frac{1}{2}, \frac{1}{2} \right] \times \left[ \frac{1}{2}, \frac{1}{2} \right] = \left[ 0 \right] + \left[ 1 \right] \)):

\[ | -1 \rangle_i \rightarrow | \downarrow \rangle_{i,1} | \downarrow \rangle_{i,2} \]

\[ |0\rangle \rightarrow \frac{1}{\sqrt{2}} \left( | \uparrow \rangle_{i,1} | \downarrow \rangle_{i,2} + | \downarrow \rangle_{i,1} | \uparrow \rangle_{i,2} \right) \]

\[ |1\rangle \rightarrow | \uparrow \rangle_{i,1} | \uparrow \rangle_{i,2} \]

First, re-write \( H_{2\text{-sites}} \) in terms of the Pauli spin matrices \( \mathbf{\vec{\sigma}}_{i,1}, \mathbf{\vec{\sigma}}_{i,2}, \mathbf{\vec{\sigma}}_{i+1,1}, \mathbf{\vec{\sigma}}_{i+1,2} \), using \( \mathbf{\vec{S}}_i = \mathbf{\vec{\sigma}}_{i,1} + \mathbf{\vec{\sigma}}_{i,2} \) and similarly for \( \mathbf{\vec{S}}_{i+1} \).

d. (4 pts.) Using the expression for \( H_{2\text{-sites}} \) derived in part (c), show that

\[ |\psi\rangle = | \uparrow \rangle_{i,1} \left( | \uparrow \rangle_{i,2} | \downarrow \rangle_{i+1,1} + | \downarrow \rangle_{i,2} | \uparrow \rangle_{i+1,1} \right) | \uparrow \rangle_{i+1,2} \]
is one of its four degenerate ground states. What are the three others?

e. (4 pts.) $|\psi\rangle$ is obtained by putting the right spin $\frac{1}{2}$ from site $i$ and the left spin $\frac{1}{2}$ from spin $i+1$ into a spin singlet. Intuitively, this works because the remaining two spin $\frac{1}{2}$’s can at most form a spin 0 or spin 1, but not the energetically costly spin 2. Using this intuition as guidance, write down the ground states of the system on $N$ sites ($i = 1, \ldots, N$), still considering each site as being made up of two fictitious spin $\frac{1}{2}$’s. How do you interpret the ground state degeneracy?

Solution

a. Each site is a spin $s = 1$, so has dimension $2s + 1 = 3$. The two site problem Hilbert space then has dimension $3^2 = 9$. Its decomposition into spin multiplets is the usual Clebsch-Gordon formula $[1] \times [1] = [0] + [1] + [2]$.

b. Re-writing $H_{2\text{-sites}}$ in terms of $S_i^2, S_{i+1}^2$ and $S_{\text{total}}^2$, and using the fact that $S_i^2 = S_{i+1}^2 = s(s+1) = 2$ we get

$$H_{2\text{-sites}} = \frac{g}{4} (S_{\text{total}}^2)^2 + (\frac{1}{2} - 2g) S_{\text{total}}^2 + 4g - 2$$

$S_{\text{total}}^2 = 0$ on $[0]$ and $S_{\text{total}}^2 = 2$ on $[1]$, so plugging these into the above equation and setting the corresponding energies equal, we obtain $g = \frac{1}{3}$. The ground state degeneracy is the dimension of the singlet plus the dimension of the triplet $= 4$.

c. Standard substitution.

d. The others are obtained by changing $|\uparrow\rangle_{i,1}$ to $|\downarrow\rangle_{i,1}$, or $|\uparrow\rangle_{i+1,2}$ to $|\downarrow\rangle_{i+1,2}$, or both.

e. Pair $(i,2)$ and $(i+1,1)$ into singlets for all $i$. The ground state degeneracy of $4 = 2 \times 2$ is interpreted as dangling spin $\frac{1}{2}$’s at sites $i$ and $N$. 19
Consider a square metallic plate of conductivity $\sigma$ and thickness $d$, where $d << a$ and $a$ is the side of the square. Contacts are placed on the four corners, which we shall designate upper left, upper right, lower left, and lower right.

a. (5 pts.) A source of constant current $I$ is connected to the upper left and upper right contacts, and the voltage is measured between the lower left and lower right contacts. How will the voltage depend on the thickness $d$ and on the linear size $a$? Explain your answer in terms of simple physical arguments.

b. (5 pts.) The current source is connected to the upper left and lower right contacts, and the voltage is measured between the lower left and upper right contacts. A magnetic field of magnitude $H$ is applied perpendicular to the plate. How does the voltage scale with $d$, $a$ and $H$? (Assume that the geometry of the square and the contacts is ideal, as described.)

c. (5 pts.) The plate is placed between a source and a detector of microwaves. The incident ($P_i$) and transmitted ($P_t$) power are measured, and the microwave transmission coefficient $T$ is defined as $T = P_t/P_i$. For $d << \lambda$, where $\lambda$ is the wavelength of the radiation, it is found that

$$T = (1 + \sigma Q)^{-2},$$

where $Q$ depends on the thickness and universal constants (permeability of vacuum, $\mu_0 = 12.56 \times 10^{-7}$ Vs/Am, permittivity of vacuum, $\varepsilon_0 = 8.845 \times 10^{-12}$ As/Vm). Up to a numerical factor, what is $Q$ as a function of $d$, $\mu_0$, and $\varepsilon_0$?

d. (5 pts.) If we want to make films of bismuth ($\sigma \sim 1 \times 10^6 \ \Omega^{-1} m^{-1}$) with a microwave transmission coefficient of approximately 50%, what should be the thickness of the film? (Order of magnitude is sufficient for the answer.)

Solution

a. Current flows a distance proportional to $a$ through a cross section proportional to $ad$, so the resistance $R \propto a/(ad\sigma) = 1/d\sigma$.

b. Hall effect, $V_{\text{Hall}} \propto H/d$, independent of $a$.

c. By dimensional analysis, $Q$ must be inverse conductivity, i.e., length times resistance. The impedance of free space is $(\mu/\varepsilon_0)^{1/2} = 377 \Omega$, so $Q \approx d(\mu/\varepsilon_0)^{1/2}$.

The transmission has this perhaps unexpected form because the radiation that is not transmitted is reflected, not absorbed.
d. To get $T \approx 1/2$, $R = 1/d\sigma \approx 377\Omega$, so $d \approx (377\Omega \times 10^6\Omega^{-1}m^{-1})^{-1} \approx 3 \text{ nm}$. The connection to part a) is now evident, because one gets a partially transmitting sample if the measured sheet resistance is on the order of 377Ω.
Nuclear 1

The Standard Model at $T \simeq 5$ GeV:

a. (4 pts.) State an order of magnitude of the QCD and Electroweak phase transition temperatures. Describe qualitatively the low and high temperature phases for both theories.

b. (8 pts.) Consider the electroweak plasma that existed in the early universe at a temperature of $T \simeq 5$ GeV:
   1. List the quarks, leptons, neutrinos, and gauge bosons in the standard model.
   2. Estimate the pressure, $p(T)$, in a high temperature Standard Model plasma with $T \simeq 5$ GeV using the integrals given below.

c. (4 pts.) Estimate the total number of quarks and anti-quarks per unit volume, $n_{q+\bar{q}}(T)$, using the integrals given below.

d. (4 pts.) Now we will estimate the electron mean free path. The $2 \leftrightarrow 2$ process which gives the largest scattering rate is electron-quark and electron-anti-quark scattering since these species are quite abundant.

   1. Draw some typical leading order Feynman diagrams for $e + q \rightarrow e + q$ and $e + \bar{q} \rightarrow e + \bar{q}$. Assuming that all of the momenta in these diagrams are of order the temperature, use dimensional reasoning to give an estimate the typical cross section in terms of the coupling constants and the temperature.

   2. Give a numerical estimate for the mean free path of an electron in this plasma using results from the previous sections.

We note the familiar integrals:

$$B_n \equiv \int_0^\infty dx \frac{x^n}{e^x - 1}, \quad F_n \equiv \int_0^\infty dx \frac{x^n}{e^x + 1}, \quad (1)$$

are $[B_1, B_2, B_3, B_4] = [\frac{\pi^2}{6}, 2.40, \frac{\pi^4}{15}, 24.8]$, and $[F_1, F_2, F_3, F_4] = [\frac{\pi^2}{12}, 1.80, \frac{7\pi^4}{120}, 23.3]$ respectively.

Solution

a. For QCD $T \sim 160$ MeV. The low temperature phase is described by a massive hadronic gas, and the high temperature phase is described by the QGP. The phase transition happens when the hadrons begin to overlap.

For the Electroweak phase transition $T \sim 90$ GeV, Anything of order the $Z$ mass is fine. The low temperature phase is the one we know with massive $W$ and $Z$. The high temperature phase has zero mass gauge bosons and and approximately zero mass leptons (up to thermal corrections).
b. 1. $u,d,c,s,b,t,\, e,\mu,\tau,\, \nu_e,\nu_\mu,\nu_\tau,\, g,\, Z,\gamma, W^+, W^-$

2. For $T \simeq 5$ GeV the $u,d,c,s$ quarks are nearly massless. The bottom quark is also present, and probably contributes a bit. The top contribution is exponentially small, $e^{-m_t T}$ and can be neglected. All of the leptons contribute including the $\tau$. The contributions of the $Z, W^\pm$ can be neglected since they are very heavy compared to the temperature. Straightforward black body physics shows that pressure is

$$ p = \frac{\pi^2}{90} T^4 \times \text{dof} $$

for massless bosonic degrees of freedom, and $7/8$ of this for fermionic degrees of freedom. In counting dof below we will incorporate the $7/8$ into the degrees of freedom.

Here the quark degrees of freedom for 4.5 light flavors (treating the bottom as a $1/2$ a light quark)

$$ \text{dof} - \text{quarks} = \text{flavor} \times \text{color} \times \text{helicity} \times \text{anti-quarks} \times \frac{7}{8} $$

$$ = 4.5 \times 3 \times 2 \times 2 \times \frac{7}{8} = 47.25 $$

The gauge bosons

$$ \text{dof} - \text{glue} = \text{color} \times \text{helicity} $$

$$ = 8 \times 2 = 16 $$

$$ \text{dof} - \gamma = \text{helicity} $$

$$ = 2 $$

The electroweak leptons

$$ \text{dof} - \text{neutrinos} = \text{species} \times \text{helicity} \times \text{anti-neutrinos} \times \frac{7}{8} $$

$$ = 3 \times 1 \times 2 \times \frac{7}{8} = 5.25 $$

$$ \text{dof} - \text{leptons} = \text{species} \times \text{helicity} \times \text{anti-leptons} \times \frac{7}{8} $$

$$ = 3 \times 2 \times 2 \times \frac{7}{8} = 10.5 $$

Thus the total is

$$ \text{total} = 47.25 + 16 + 2 + 5.25 + 10.5 = 81. $$

Makes one think that a large $N_f$ approximation might not be bad over a large range.

Thus we find

$$ p = \frac{\pi^2}{90} T^4 \times 81 \simeq 5500 \text{ GeV}^4 $$

Substituting $T = 5$ GeV

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c. For the density of quarks we have

\[ n_q = \text{dof} \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{p/T} + 1} \]

\[ = \text{[dof]} T^3 \frac{1.8}{2\pi^2} \]  

(15)

(16)

Here

\[ \text{dof} = \text{species} \times \text{helicity} \times \text{color} \]  

\[ = 4.5 \times 2 \times 3 = 27 \]  

(17)

(18)

and we have

\[ n_q = n_{\bar{q}} \]  

(19)

So

\[ n_{q+\bar{q}} = 4.9T^3 \]  

(20)

d. Estimating the electron mean free path

(a) For a typical quark (or anti-quark graph) the cross section is

\[ \sigma \sim \alpha_{EM}^2 / T^2 \]  

(21)

d. Thus the mean free path is

\[ \frac{1}{n_{q+\bar{q}}\sigma} \simeq \frac{1}{4.9\alpha_{EM}^2 T} \simeq 760 \text{GeV}^{-1} \simeq 150 \text{fm} \]  

(22)

where in the last step we took \( \alpha_{EM} = 1/137 \).
Many scattering experiments require that the particle type of a produced particle be identified uniquely. One method of particle identification is to measure both the energy (or momentum) and velocity of a particle and then calculate its mass. Such an identification becomes increasingly difficult at the highest energies since the velocity cannot exceed $c$.

The device sketched below is a “proximity-focussing” Čerenkov detector. As shown in the figure below, the Čerenkov light generated in the radiator section is eventually deposited upon the photon detector in a ring-shaped region whose radius determines the velocity of the particle. Throughout this problem we shall assume that particles always enter the detector at normal incidence.

a. (4 pts.) Derive the angle of the Čerenkov radiation (in the radiator) as a function of the velocity, $\beta$, of the particle. (HINT: Čerenkov radiation is the optical analogy to the sonic boom).

b. (6 pts.) Using your result from part a, calculate the mean radius, $R$, and the width, $\Delta R$ of the band which that contains all the photons. Use the approximation $D >> d$. 

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c. (6 pts.) Assume that a single ring contains 20 detected photons and that the photon
detector’s spatial resolution is \(\ll \Delta R\). Determine the velocity resolution \(\delta \beta\) when
\(n\beta = 1.1\).

d. (4 pts.) Compare this performance with a conventional “Time-of-Flight” detector
having a time resolution of 100 psec and a flight path of one meter. Give an example
for which the conventional detector is a superior choice.

Solution

a. In a time \(t\), the particle moves a distance \(vt\). Light from the particle moves a
distance \((c/n)t\). These two distances are part of a right-triangle with \(vt\) (larger)
as the hypot. and \((c/n)t\) as one leg. Thus the Čerenkov angle (between the hypot
and leg) is given by:

\[
\cos(\theta) = \frac{vt}{c} = \frac{1}{n\beta}
\]

b. Snell’s Law relates the angle of the Čerenkov light with the angle of the light
between the radiator and the focal plane:

\[
n \sin(\theta_c) = \sin(\theta_2)
\]

\[
\sin(\theta_c) = \sqrt{1 - \left(\frac{1}{n\beta}\right)^2}
\]

\[
\sin(\theta_2) = n \sqrt{1 - \left(\frac{1}{n\beta}\right)^2} = \sqrt{n^2 - \beta^{-2}}
\]

When \(D \gg d\), we approximate that:

\[
R = D \tan(\theta_2) = D \frac{\sin(\theta_2)}{\sqrt{1 - \sin^2(\theta_2)}}
\]

\[
R = D \sqrt{\frac{n^2 - \beta^{-2}}{1 - (n^2 - \beta^{-2})}} \quad (2.1)
\]

\[
\Delta R = d \tan(\theta_c) = d \sqrt{\frac{1 - \left(\frac{1}{n\beta}\right)^2}{\frac{1}{n^2}}} = d \sqrt{(n\beta)^2 - 1} \quad (2.2)
\]

c. If the fill width of the band containing the photons has a width \(\Delta R\), then the \(\sigma\) of
a single photon’s measurement of the radius is

\[
\sigma_{\text{phot}} = \frac{\Delta R}{\sqrt{20}}.
\]

When 20 photons participate in the measurement, then the uncertainty in the
radius when \(n\beta = 1.1\) becomes

\[
\delta R = \frac{\sigma_{\text{phot}}}{\sqrt{20}} = \frac{\Delta R}{\sqrt{240}} = d \sqrt{\frac{(n\beta)^2 - 1}{240}} = 0.0296cm
\]

We transform the uncertainty in radius into an uncertainty in the velocity by taking
the derivative:

\[
\delta R = \left(\frac{dR}{d\sin(\theta_2)}\right)\left(\frac{d\sin(\theta_2)}{d\beta}\right) \delta \beta
\]
From equation 2.1:
\[
\frac{dR}{d\sin(\theta_2)} = \frac{D}{(1-\sin^2(\theta_2))^{\frac{3}{2}}} = 16.83
\]
\[
\frac{d\sin(\theta_2)}{d\beta} = \frac{1}{\beta^3 \sqrt{n^2 - \beta^{-2}}} = 5.03
\]
\[\delta \beta = 0.000349\]

d. For the simple counter,
\[v = \frac{L}{T}; \beta = \frac{L/c}{T}\]
\[\delta \beta = -\frac{L/c}{T^2} \delta T\]

However, since \[T = \frac{L/c}{\beta}\]:
\[\delta \beta = \frac{\beta^2}{L/c} \delta T\]

Putting in the numbers:
\[\frac{L}{c} = 3.3 \times 10^{-9} \text{sec} \delta T = 0.1 \times 10^{-9} \text{sec}\]
\[\delta \beta = 0.021\]

This is about 2 orders of magnitude worse than the Čerenkov counter!

HOWEVER!!!! If the particles to be measured are too slow (\[\beta < 1/n\]), then there is no Čerenkov light and the simple scintillator counter is better!
High Energy 1

Generating and detecting Neutrinos. Since the revolutionary discovery of the neutrino oscillations in the atmospheric neutrinos by the Super-Kamiokande (SuperK) experiment in 1998, there have been several accelerator based neutrino oscillation experiments (K2K, MINOS, OPERA and T2K) that confirmed the SuperKs observation and made precision measurements of neutrino oscillation parameters using man-made neutrinos. All these experiments essentially use the same method of generating neutrinos (mostly pure muon-neutrino beams) using accelerators. Briefly describe the method of generating neutrino beams employed by these experiments step-by-step by answering the questions below.

Note: In giving your answers to this problem, make sure to indicate the specific flavor of the neutrinos whenever you refer to neutrinos, e.g. \( \nu_e, \nu_\mu, \nu_\tau \) and \( \nu_x \) for all three flavors. Also distinguish between neutrinos and anti-neutrinos. Use Feynman diagrams whenever possible.

a. (2 pts.) What is the beam particle that is accelerated?

b. (4 pts.) What is the mechanism that generates the muon-neutrinos?

c. (3 pts.) It is essential to generate the neutrinos in a “beam” form so that they can be aimed at a detector in a location far away (\( \gg 100 \) km). This focusing is done by using so-called “neutrino horns”. Describe how neutrino horns work.

d. (4 pts.) This “focusing” mechanism also helps to generate mostly pure muon-neutrinos. However, the resultant neutrino beam contains some small amount of other types of neutrinos in the beam. What are the two dominant types of neutrinos among these residual neutrinos and where do they come from?

e. (4 pts.) Recently, the T2K experiment has produced evidence for the appearance of electron-neutrinos from a muon-neutrino beam, which requires distinguishing electron-neutrino interaction events from muon-neutrino interaction events in the SuperK (water Čerenkov) detector. Briefly describe the difference in the Čerenkov light patterns from these interactions and identify the cause(s) that create these differences.

f. (3 pts.) A \( \pi^0 \) produced via neutrino Neutral Current interaction (NC\( \pi^0 \)) can mimic an electron-neutrino Charged Current Quasi-Elastic (CCQE) interaction event, which is the signal for the \( \nu_e \) appearance experiment. Explain how a NC\( \pi^0 \) event can mimic an electron-neutrino CCQE event. The NC\( \pi^0 \) events are one of the two major backgrounds to the electron-neutrino appearance measurements.

Solution
a. Proton. (See below for detailed explanations.)

b. First, an accelerated proton beam, typically 10’s of GeV, is bombarded on a target (aluminum, graphite, etc.), which produces copious hadronic showers. The end products of these hadronic showers are mostly pions (and some kaons). The muon-neutrinos are then generated from the decays of the charged pions, which decay with almost 100% to muons and muon neutrinos. $\pi^+ \rightarrow \mu^+ \nu_\mu$

c. Although the name says neutrino horn, the horn obviously does not focus neutrinos directly. Rather it focuses charged pion and kaons. The horn is essentially a magnet with a toroidal field which is created by a electric current circulating through inner the conductor and outer conductor of the horn. (See the sketch below.) The sketch shows simplified cut-away side view of a neutrino horn. In the top portion of the horn, current flows counter-clockwise producing a B-field coming out of the paper and in the bottom portion, the opposite. This makes a continuous circular (toroidal) B-field. In this configuration the proton beam hits the target from the left and the emanating positive charged particles are bent inward (focused) and the negative charged particles are swept away. Then the focused positive pions decay and in turn produce a focused neutrino beam.

d. As can be seen from the above sketch, although the negative pions are mostly swept away, there can still be some residual negative pions remaining in the beam direction. These negative pions decay creating anti-muon neutrinos, via: $\pi^- \rightarrow \mu^- \text{ anti-}\nu_\mu$. The $\mu^+$ from the $\pi^+$ decay also decays via: $\mu^+ \rightarrow e^+ \nu_e \text{ anti-}\nu_\mu$. There is also a small amount of $\nu_e$ from kaon decays (so-called $k_{e3}$ decays). Thus, anti-$\nu_\mu$ and $\nu_e$ are the two dominant types of neutrinos among these residual neutrinos.

e. Čerenkov light is generated when a charged particle travels in a medium faster than the speed of light in that medium. The Čerenkov photons are emitted at about 42 degrees w.r.t. the particle momentum axis for a relativistic charged particle. Electrons traveling in water develop an EM shower resulting multiple electrons (and positrons), each of which emits Čerenkov light at slightly different directions.
Furthermore, each of these electrons experiences much larger multiple scattering than muons, since electrons are about 200 times lighter than muons. Thus, the Čerenkov light is emitted in different directions as the electron changes direction due to multiple scattering. Muons on the other hand travel mostly straight without scattering and without generating EM showers. Then, the final observable effects in the detector are that the Čerenkov rings generated by electrons have generally very “fuzzy” edges, while the rings generated by muons have generally “sharp” edges.

f. The NC$\pi^0$ interaction produces only single $\pi^0$ and nothing else in the event. The $\pi^0$ decays to two photons immediately (electro-weak decay, lifetime $< 10^{-15}$ s). Each of these photons, then, creates an EM shower and a corresponding Čerenkov ring. However, since the $\pi^0$ is boosted when it is created, the opening angle between the two photons can be small, resulting in only one (overlapped) observed Čerenkov ring, which mimics an electron-neutrino CCQE event. Also, the decay can occur along the pion momentum axis, resulting in one of the photons to be boosted forward, generating a clear Čerenkov ring and the other boosted backward producing no observable Čerenkov ring, which in turn mimics an electron-neutrino CCQE event.
The Higgs boson. Recently, a new particle that is a strong candidate for the Higgs boson of the Standard Model (SM) was discovered at CERN. This new particle has a mass of 126 GeV and is seen to decay into two photons (and also into other particles).

a. (8 pts.) How do the W and the Z bosons get their mass in the SM? Derive the first two formulas (you need not to derive the formula for $e$)

$$m_W = \frac{g_2 v}{2}, \quad m_Z = \frac{v}{2} \sqrt{g_2^2 + g_1^2}, \quad \text{and} \quad e = \frac{g_2 g_1}{\sqrt{g_2^2 + g_1^2}}$$

of the SM at tree level. Taking $m_W \sim 80$ GeV and $m_Z \sim 90$ GeV, deduce the approximate value $v \sim 240$ GeV, using $\alpha_{QED} = e^2 / 4\pi = 1/137$.

b. (6 pts.) Write down the potential for the Higgs field in the SM, and derive a formula for the mass $m_H$ of the Higgs boson at the tree level. If $m_H = 126$ GeV, what is the value of the self-coupling constant $\lambda$ in the Higgs potential?

c. (6 pts.) Show that the spin $J$ of the new particle cannot be $J = 1$ (Yang’s theorem). Hint: go to the rest frame of the new particle, and write down all possible terms in the decay amplitude in terms of the momentum $k$ of one of the photons and the polarization vectors $\epsilon_1$, $\epsilon_2$, $\epsilon_\phi$ of the two photons and the new particle. Use that the photon polarization vectors are transversal, $\epsilon_i \cdot k = 0$. Treat the cases that $\phi$ has positive or negative parity separately, and require Bose symmetry.

Solution

a. The $W$ and $Z$ bosons get their masses from the Higgs effect.

From the coupling of the Higgs doublet to the $SU(2) \times U(1)$ gauge fields

$$\mathcal{L} = -\left| \left( \partial_\mu + ig_2 \frac{\tau_a}{2} W^a_\mu + ig_1 Y B_\mu \right) \left( \phi^+ \phi^0 \right) \right|^2 \quad (1)$$

with hypercharge $Y = 1/2$ for the Higgs doublet, one obtains after setting $\phi^0 = \frac{\sigma(x) + y}{\sqrt{2}}$ the following mass terms

$$\mathcal{L}(\text{mass}) = -\frac{1}{8} v^2 \left( g_2 W^3_\mu - g_1 B_\mu \right)^2 - \left( 2 g_2 v \right)^2 W^+_\mu W^{-\mu} \quad (2)$$

where $W^+_\mu = (W^1_\mu - i W^2_\mu)/\sqrt{2}$. Hence $m_W = \frac{1}{2} g_2 v$. The $Z_\mu$ field is

$$Z_\mu = \frac{g_2 W^3_\mu - g_1 B_\mu}{\sqrt{g_2^2 + g_1^2}} \quad (3)$$
so \( m_Z = \frac{1}{2} \sqrt{g_2^2 + g_1^2} v \). The photon field is then the orthogonal combination

\[
A_\mu = \frac{g_1 W_\mu^3 + g_2 B_\mu}{\sqrt{g_2^2 + g_1^2}}
\]

(4)

NB. The electric constant \( e \) follows from

\[
\mathcal{L} = \left( \frac{i}{2} g_2 W_\mu^3 + \frac{i}{2} g_1 B_\mu \right) \left[ (\phi^+)^* \overleftrightarrow{\partial}_\mu \phi^+ \right] \\
\sim ieA_\mu \left[ (\phi^+)^* \overleftrightarrow{\partial}_\mu \phi^+ \right] + \text{term with } Z_\mu
\]

(5)

Decomposing \( \frac{1}{2} g_2 W_\mu^3 + \frac{1}{2} g_1 B_\mu = k Z_\mu + e A_\mu \), one finds

\[
k g_2 + e g_1 = \frac{1}{2} g_2 \sqrt{g_2^2 + g_1^2} \\
-k g_1 + e g_2 = \frac{1}{2} g_1 \sqrt{g_2^2 + g_1^2}
\]

(6)

whose solution is

\[
e = \frac{g_1 g_2}{\sqrt{g_2^2 + g_1^2}}
\]

(7)

Then

\[
\sin^2 \theta_w = \frac{g_1^2}{g_1^2 + g_2^2} = 1 - \left( \frac{m_W}{m_Z} \right)^2 \approx 0.2
\]

(8)

and \( \alpha(\text{QED}) = \frac{1}{4\pi} (g_2^2 \sin^2 \theta_w) \), hence \( g_2 \approx \frac{2}{3} \) and finally \( v = \frac{2m_W}{g_2} \approx 240 \text{ GeV} \).

b.

\[
V = -\mu^2 \left( \frac{\sigma + v}{\sqrt{2}} \right)^2 + \lambda \left( \frac{\sigma + v}{\sqrt{2}} \right)^4
\]

(9)

The terms linear in \( \sigma \) cancel if \(-\mu^2 + \lambda v^2 = 0\). The mass term for \( \sigma \) is then

\[
-\frac{1}{2} \mu^2 \sigma^2 + \frac{3}{2} \lambda v^2 \sigma^2
\]

(10)

so the Higgs mass is \( m_H^2 = 2\lambda v^2 \). With \( v = 240 \text{ GeV} \) this gives \( \lambda = \frac{m_H^2}{2v^2} \approx \frac{1}{8} \).

c. In the rest frame of the new particle with positive parity, the amplitude can only be proportional to

\[
A = (\epsilon_1 \cdot \epsilon_2)(\epsilon_\phi \cdot k)
\]

(11)

because \( \epsilon_1 \cdot k = \epsilon_2 \cdot k = 0 \). But \( A \) is not invariant under \( \epsilon_1 \leftrightarrow \epsilon_2 \) and \( k \rightarrow -k \). For negative parity there are two terms possible,

\[
A = (\vec{\epsilon}_1 \times \vec{\epsilon}_2 \cdot \vec{k})(\epsilon_H \cdot k) \quad \text{and} \quad A = \vec{\epsilon}_1 \times \vec{\epsilon}_2 \cdot \vec{\epsilon}_H
\]

(12)

but again Bose symmetry is violated.