Problem F.1 (to be graded of 1,000 points). A meteorite with initial velocity $v_\infty$ approaches an atmosphere-free planet of mass $M$ and radius $R$.

(i) Find the condition on the impact parameter $b$ for the meteorite to hit planet’s surface.

(ii) If the meteorite barely avoids the collision, what is its scattering angle?

*Solutions:*

(i) The required condition is $b < b_{\text{max}}$, where $b_{\text{max}}$ is the impact’s parameter value corresponding to the equality $r_{\text{min}} = R$, where $r_{\text{min}}$ is the lowest point of meteorite’s orbit. In order to find $r_{\text{min}}$, we may use the conservation of meteorite’s angular momentum $L_z$ and its full energy $E$ that may be presented by Eqs. (3.43)-(3.44) of the lecture notes.

$$E = \frac{mv_r^2}{2} + U_{\text{ef}}(r), \quad U_{\text{ef}}(r) = U(r) + \frac{L_z^2}{2mr^2}.$$ 

Applying the energy conservation to the lowest point $r = r_{\text{min}} = R$ (where $\dot{r} = 0$) and the infinite point $r = r_\infty \to \infty$ (where $E = mv_\infty^2/2$), we get

$$U_{\text{ef}}(R) = \frac{mv_\infty^2}{2}, \quad \text{i.e.} \quad U(R) + \frac{L_z^2}{2mR^2} = \frac{mv_\infty^2}{2}. \quad (*)$$

In order to spell out the potential energy, we may use the fact that a massive, spherically-symmetric sphere produces a gravity field similar to that of a particle of the same mass, located in the sphere’s center, so that $U(R) = GMm/R$. (This fact may be readily proven by the gravitational analog of the Gauss theorem.) Using also Eq. (3.65), $L_z = bm v_\infty$, with $b = b_{\text{max}}$, we may rewrite Eq. (*) as

$$-\frac{GMm}{R} + b_{\text{max}}^2 \frac{mv_\infty^2}{2R^3} = \frac{mv_\infty^2}{2}.$$ 

From this equation, we readily get

$$b_{\text{max}} = R \left(1 + \frac{2GM}{Rv_\infty^2} \right)^{1/2}. \quad (***)$$

For very large initial velocities, $v_\infty^2 \gg GM/R$, $b_{\text{max}}$ tends to planet’s radius – as it should.

(ii) Plugging Eq. (***)) into Eq. (3.68b) of the lecture notes with $\alpha = GMm$ and $E = mv_\infty^2/2$, we get

$$\tan \frac{\theta}{2} = \frac{GM}{v_\infty b_{\text{max}}} = \frac{GM / v_\infty R}{\left[1 + (2GM / v_\infty^2 R) \right]^{3/2}}.$$ 

This result shows that at large initial velocities, the scattering angle is very small, but as the velocity is decreased well below $GM/R$ (that is of course just the 1st space velocity for this planet – see Problem 1.2), the right-hand part of this relation, and hence $\tan(\theta/2)$, tend to infinity, so that $\theta/2 \to \pi/2$, and hence the scattering angle $\theta$ tends to $\pi$ - the perfect backscattering.
The fact that the scattering angle may be arbitrary even for a planet of a substantial radius is used for spacecraft orbit manipulation.

Problem F.2 (600 points). A thick round cylindrical shell of mass \( M \) is rolling over a plane surface without slipping, with the center-mass velocity \( V \). Find its kinetic energy.

Solution: According to Eqs. (6.14)-(6.15) of the lecture notes,

\[
T = \frac{M}{2} V^2 + \frac{I}{2} \omega^2, \quad (*)
\]

where the angular velocity \( \omega \) may be readily calculated from the no-slippage condition (6.45): \( \omega = -V/b \). The moment of inertia \( I \) (relative to the center of mass) of the cylinder is

\[
I = \int_a^b \rho^2 (2\pi\rho) d\rho = \frac{1}{2} \pi \rho \left( b^4 - a^4 \right),
\]

(\( \rho \) is the density of cylinder’s material, \( l \) is length), while the mass is

\[
M = \int_a^b (2\pi\rho) d\rho = \pi \rho \left( b^2 - a^2 \right), \quad \text{so that} \quad I = \frac{M}{2} \left( \frac{b^4 - a^4}{b^2 - a^2} = \frac{M}{2} (b^2 + a^2). \right.
\]

Plugging these relations into Eq. (*)\), we finally get

\[
T = \frac{M}{2} V^2 \left( \frac{3}{2} \frac{a^2}{2b^2} \right).
\]

We should not be surprised that \( T \) apparently grows with the cavity radius \( a \), because such increase (at fixed density \( \rho \)) also reduces mass \( M \).

Problem F.3 (1,000 points). Two thin rods of the same length and mass have been made of the same elastic, isotropic material. The cross-section of one of them is a circle, while another one is an equilateral triangle - see Fig. on the right. Which of the rods is more stiff for bending along its length? Quantify the relation. Does the result depend on the bending plane orientation?

Solution: According to the analysis of Sec. 7.5, the rod stiffness may be characterized by its curvature radius which is in turn proportional to the “moment of inertia”

\[
I_y = \int_A y^2 d^2 r,
\]

where \( y \) is the distance of the point from the neutral plane which passes through the cross-section’s center of mass – see the dashed line in the Fig. above. For a circle, it was calculated in Homework Problem 7.9, but let me illustrate another way to get the same result:
\[ I_c = \int_0^R \int_0^{2\pi} r^2 \, dr \, d\varphi \, \sin^2 \varphi = \int_0^R \int_0^{2\pi} r^3 \, dr \, \sin^2 \varphi \, d\varphi = \pi \int_0^R r^3 \, dr = \frac{\pi}{4} R^4, \]

which is of course independent of the bending direction. For an equilateral triangle with side \(a\), the moment of inertia has been calculated in Problem 6.1:

\[ I_T = \frac{\sqrt{3}}{96} a^4, \]

and also does not depend on the direction of axis \(y\), i.e. the bending plane orientation.

In order to have the same rod mass using the same material, the areas of these two cross-sections have to be equal:

\[ \pi R^2 = \frac{\sqrt{3}}{4} a^2. \]

From here,

\[ \frac{I_c}{I_T} = \frac{(\pi/4)R^4}{(\sqrt{3}/96)a^4} = \frac{\pi}{4} \frac{96}{\sqrt{3}} \left( \frac{\sqrt{3}}{4\pi} \right)^2 = \frac{9}{2\pi \sqrt{3}} \approx 0.827 < 1. \]

Hence, the round cross-section gives the rod a slightly lower bending stiffness than the triangular cross-section.

Problem F.4 (1,000 points). An incompressible fluid with density \(\rho\) and viscosity \(\eta\) flows down, driven by its own weight (rather than a pressure gradient), through a vertical gap of thickness \(t\) between two wide, parallel plates. Find the stationary velocity distribution and fluid discharge (per unit width).

\[ \text{Solution:} \quad \text{With the coordinate choice shown in Fig. on the right, fluid velocity has only one component: } v = n_z v_z(x), \quad \text{for which the Navier-Stokes equation (8.44) with } \nabla P = 0 \text{ and } f = -n_z \rho g \text{ gives} \]

\[ -\rho g + \eta \frac{d^2 v_z}{dx^2} = 0. \]

Solving this simple equation with boundary conditions \(v_z(\pm t/2) = 0\), we get

\[ v_z = -\frac{\rho g}{2\eta} \left( \frac{t^2}{4} - x^2 \right), \quad \text{giving } |v_z|_{\max} = \frac{\rho g t^2}{8\eta}. \]

From here, the fluid discharge per unit width is

\[ \frac{Q}{w} = \rho \int_{-t/2}^{+t/2} |v_z| \, dx = \frac{\rho^2 g t^3}{12\eta}. \]
Problem F.5 (400 points). Is deterministic chaos possible in our “testbed” problem shown in Figs. 1.5 and 2.1 of the lecture notes (copied on the right)? What if an additional periodic external force is applied to the bead? Explain your answers.

Solution: This system is described by one differential equation of the second order (see Eq. (2.25) of the lecture notes) which is equivalent to two equations of the first order, for example

\[
\begin{align*}
\dot{\theta} &= \frac{p}{mR^2}, \\
\dot{p} &= mR^2 \omega^2 \sin \theta \cos \theta - mgR \sin \theta.
\end{align*}
\]

Hence, the situation is the same as in the system discussed in Problems 9.1 and 9.2: deterministic chaos is impossible without an external force, but is possible in its presence. Actually, the simple, externally-driven pendulum, whose chaotic dynamics was discussed in Sec. 9.2, is just a particular case (for \(\omega = 0\)) of the testbed problem with additional sinusoidal external force, i.e. ring’s rotation does not create a qualitative difference in this aspect.