Classical Mechanics I

A particle is launched from afar toward a repulsive Coulomb scattering center (potential energy $U = \frac{\alpha}{r}, \alpha > 0$), with initial energy $E$ and impact parameter $b$. Find

a) (10 points) the nearest approach distance $r_{\text{min}}$.

b) (10 points) the magnitude of particle’s speed in the nearest approach point.
A uniform solid semisphere of radius $R$ and mass $m$ is placed on a horizontal surface with finite (non-zero) friction coefficient.

a) (3 points) Find the center of mass of the hemisphere.

b) (7 points) Find the moment of inertia of the hemisphere with respect to an axis PX lying on the horizontal surface (see Figure).

c) (10 points) Find the period of small oscillations of the semisphere.
A ball is bouncing vertically and perfectly elastically in a standing elevator. The maximal height of bouncing motion is \( h_0 \). The upward acceleration of the elevator is changing very slowly from 0 to \( g/2 \). Find the new maximal height of bouncing motion.
Solution I: Kepler problem (Kostya Likharev)

a) The point of the nearest approach may be found from the equation $U_{\text{eff}}(r_{\text{min}}) = E$, where

$$U_{\text{eff}}(r) = U(r) + \frac{L_z^2}{2mr^2} = \frac{\alpha}{r} + \frac{Eb^2}{r^2},$$

since $L_z^2 = (mv_i b)^2 = 2mEb^2$. Solving the resulting quadratic equation, we get

$$r_{\text{min}} = \frac{\alpha}{2E} + \sqrt{\left(\frac{\alpha}{2E}\right)^2 + b^2} > b.$$  

(The negative sign before the square root would give a non-physical, negative value for $r_{\text{min}}$.)

b) The simplest way to find the velocity $v_x$ in the nearest approach point is to realize that its vector is perpendicular to the radius-vector and hence $L_z = mv_x r_{\text{min}}$. Since $L_z = \text{const} = b\sqrt{2mE}$, we get

$$v_x = \sqrt{\frac{2E}{m}} \frac{b}{r_{\text{min}}} = v_i \frac{b}{r_{\text{min}}} < v_i.$$
Solution II: Semisphere (Sasha Abanov)

a) The density of the semisphere is $\rho = m/(\frac{2}{3}\pi R^3)$. The center of mass of the semisphere is obviously on the axis of symmetry of the semisphere. Introducing the vertical coordinate $z$ directed from the center of the semisphere downwards we have:

$$z_{cm} = \frac{1}{m} \int_0^R z \rho \pi (R^2 - z^2) \, dz = \frac{3}{2R^3} \int_0^R z \pi (R^2 - z^2) \, dz = \frac{3}{8} R.$$ 

b) The moment of inertia of the solid ball with respect to its axis of symmetry is $\frac{2}{5} mR^2$. Therefore, the moment of inertia of the semisphere with respect to the axis going through the point $O$ and orthogonal to the axis of symmetry is $I_O = \frac{2}{5} mR^2 = I_{cm} + mz_{cm}^2$. We have $I_{cm} = \frac{2}{5} mR^2 - mz_{cm}^2$. On the other hand the moment of inertia relative to the instantaneous axis of rotation $PY$ is $I_{PY} = I_{cm} + m(R - z_{cm})^2 = \frac{2}{5} mR^2 - mz_{cm}^2 + m(R - z_{cm})^2 = \frac{13}{20} mR^2$.

$$I_{PY} = \frac{13}{20} mR^2.$$ 

c) If $\phi$ is the angular deviation of the semisphere from its equilibrium position we have $I \ddot{\phi} = -mgz_{cm} \cos \phi$ and the period of small oscillations $T = \frac{2\pi}{\sqrt{\frac{I}{mgz_{cm}}}} = 2\pi \sqrt{\frac{26R}{15g}}$.

$$T = 2\pi \sqrt{\frac{26R}{15g}}.$$
Solution III: Bouncing ball (Sasha Abanov)

The trajectory of the bouncing ball in the phase space is given by parabola \( \frac{p^2}{2m} + mgx = E = mgh \) or by \( p = m\sqrt{2g(h-x)} \) and by the straight line \( x = 0 \). The adiabatic invariant is the area of the phase space bounded by that trajectory

\[
I = \int p \, dx = 2 \int_0^h m\sqrt{2g(h-x)} \, dx = \frac{4\sqrt{2}}{3} mg^{1/2}h^{3/2}.
\]

In the moving elevator \( g \to g_{eff} = a + g \) and we have

\[
I = (a + g)^{1/2}h^{3/2} = \text{const}.
\]

Comparing adiabatic invariant \( I \) with its initial value we obtain \( (a + g)^{1/2}h^{3/2} = g^{1/2}h_0^{3/2} \) and

\[
h = h_0 \left( \frac{g}{g + a} \right)^{\frac{1}{3}}.
\]

We substitute \( a = g/2 \) and obtain the final result

\[
h = h_0 \left( \frac{2}{3} \right)^{\frac{1}{3}}.
\]