EM1 (Weisberger). Consider a spherically symmetric charge distribution with density \( \rho(r) = C \exp(-r / r_0) \).

(a, 5 points) Calculate the total charge.

\[
Q = \int_0^\infty \rho d^3 r = 4\pi \int_0^\infty \rho(r) r^2 dr = 4\pi C r_0^3 \int_0^\infty e^{-x} x^2 dx = 8\pi C r_0^3,
\]

where the constant is obtained via integration by parts twice:

\[
\int_0^\infty e^{-x} x^2 dx = -\int_0^\infty x^2 e^{-x} dx = -2x^2 e^{-x} \bigg|_0^\infty + 2\int_0^\infty e^{-x} x dx = -2\int_0^\infty x e^{-x} dx = 2(\exp(2) - 2).
\]

(b, 7 points) Find and sketch the electric field distribution \( E(r) \).

Applying the Gauss law to a sphere of radius \( r \), we have (in Gaussian units):

\[
4\pi r^2 E = 4\pi Q(r), \quad Q(r) \equiv 4\pi \int_0^r \rho(r') r'^2 dr',
\]

so that

\[
E(r) \equiv \frac{1}{r^2} \int_0^r 4\pi \rho(r') r'^2 dr' = \frac{4\pi C r_0^3}{r^2} \int_0^r \exp(-r' / r_0) r'^2 dr' = \frac{4\pi C r_0^3}{r^2} r/r_0 \exp(-r / r_0).
\]

Integrating twice by parts (exactly as has been done above, but in finite limits), we get

\[
E(r) = \frac{4\pi C r_0^3}{r^2} \left[ 2 - \left( 2 + \frac{2r}{r_0} + \frac{r^2}{r_0^2} \right) \exp(-r / r_0) \right].
\]

A plot of this function is presented below. Naturally, \( E \) vanishes at \( r = 0 \) and tends to zero at large distances, albeit only slowly (as \( 1/r^2 \)), because the total charge \( Q \) is nonvanishing.

(c, 8 points) Find and sketch the electrostatic potential distribution \( \phi(r) \).

Since \( \vec{\nabla} \phi = -\vec{E} \), in our spherically-symmetric case we have \( d\phi(r) / dr = -E(r) \), so that with \( \phi(\infty) = 0 \) we get
\[ \varphi(r) = \int_r^\infty E(r) dr = 4\pi Cr_0^2 \int_{r/r_0}^\infty \left[ \frac{2}{x^2} \left( 1 - e^{-x} \right) - \left( \frac{2}{x} + 1 \right) e^{-x} \right] dx. \]

At integration by parts, the two terms proportional to \( \int \frac{e^{-x}}{x} \, dx \) (which would lead to a special function called exponential integral) cancel each other, and the result is simple:

\[ \varphi(r) = 4\pi Cr_0^2 \left[ 2 \frac{r_0}{r} \left( 1 - e^{-r/r_0} \right) - e^{-r/r_0} \right]. \]

The plot of this function is shown below. The potential is positive, and at \( r = 0 \), reaches a constant value \( \varphi(0) = 4\pi Cr_0^2 \).
EM2 (Abanov). A hollow waveguide of rectangular cross-section \( a \times b \) is used for the transmission of electromagnetic waves. Assume that the walls are perfect conductors.

(a, 5 points) For the lowest transverse electric (TE) mode, calculate and sketch the dispersion relation, i.e. the relation between frequency \( \omega \) and longitudinal wavenumber \( k \). (In TE modes, the electric field is perpendicular to the propagation direction at all points.)

The lowest TE mode is sketched below. The electric field is directed along the shortest wall of the waveguide (say, \( b < a \)), is constant along its direction and changes sinusoidally along the longer wall:

\[
\vec{E} = E\hat{n}_y, \quad E = E_0 \sin \frac{\pi x}{a} \cos(kz - \alpha x + \phi).
\]

Plugging this relation into the 3D wave equation

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) E = 0,
\]

we get the required dispersion relation

\[
\omega(k) = \sqrt{\omega_c^2 + (ck)^2}, \quad \omega_c = \frac{\pi c}{a}.
\]

The relation is plotted below. It is evident that the constant \( \omega_c \) plays the role of the lowest ("cutoff") frequency.

(b, 3 points) For \( a = 3 \text{ cm} \) and \( b = 1 \text{ cm} \), find the "cutoff" (lowest) frequency \( \omega_c \) of this mode.

Using the formula derived above, we get \( \omega_c = \pi \times 3 \times 10^{10}/3 = 3.14 \times 10^{10} \text{ s}^{-1} \), so that the corresponding \( f_c = \omega_c / 2\pi = 5 \text{ GHz} \).

(c, 7 points) For the same waveguide as in (b), find the frequency range in which the lowest TE wave is the only propagating mode.
Besides the lowest mode we have analyzed, called TE\textsubscript{10}, other lower-frequency modes are:

(i) A similar TE\textsubscript{20} mode, with two sine half-waves along wall \(a\). Since this doubles the wave vector in \(x\) direction, \(\pi/a \rightarrow 2\pi/a\), the cutoff frequency also doubles, to \(6.28 \times 10^{10} \text{ s}^{-1}\) (\(f_c = 10 \text{ GHz}\)).

(ii) A similar TE\textsubscript{01} mode, but with the electric field directed along another wall \((b)\). Evidently, for this mode the cutoff frequency is \(\pi c/b \approx 9.42 \times 10^{10} \text{ s}^{-1}\), i.e. \(f_c = 15 \text{ GHz}\) (For our parameters, this is exactly equal to \(f_c\) of TE\textsubscript{30} mode.)

(iii) The lowest possible TM mode, TM\textsubscript{11}, has an even higher cutoff:
\[
\omega_c = \pi c \sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = \pi \times 3 \times 10^{10}/1.054, \text{ i.e. } f_c \approx 15.81 \text{ GHz}.
\]

So, the range in which the TE\textsubscript{10} is the only propagating mode is from 5 to 10 GHz.

(d, 5 points) Now let the waveguide be excited, at its end, by a source of frequency \(\omega_c/2\). Calculate the ratio of electric field amplitudes at the distances 5 cm and 10 cm from the end.

At distances well beyond \(a\) and \(b\), the effects of higher modes may be neglected. Hence we may again use Eq. (*) for the TE\textsubscript{10} mode, but in this case it yields imaginary \(k\), meaning that the field decays along the waveguide: \(E \propto \exp(-\kappa z)\), with
\[
\kappa = |k| = \frac{1}{c} \sqrt{\omega_c^2 - \omega^2} = \frac{\sqrt{3} \omega_c}{2c} = \frac{\sqrt{3} \pi}{2a} \approx 0.907 \text{ cm}^{-1}.
\]

As a result, the required ratio equals to \(\exp(-0.907 \times 5) \approx 0.0107\).
EM3 (Weisberger). A radiating electric dipole consists of a rod of length \( l \) with charge \( +q \) at one end and charge \( -q \) at the other end. The rod lies in the \([x, y]\) plane and rotates about the \( z \)-axis with angular velocity \( \omega \).

(a, 4 points) Calculate the dipole moment.

By definition, \( \vec{p} = \sum q_i \vec{r}_i \), so that in our case the dipole vector is aligned with the rod (i.e. rotates in the \( x-y \) plane) and has magnitude \( p = 2lq \).

(b, 8 points) Calculate the angular distribution of the radiation power, \( dP / d\theta \).

The main result of the electric dipole radiation theory is

\[
\frac{dP}{d\Omega} = \frac{\omega^4}{4\pi c^2} \left| (\vec{n} \times \vec{p}) \times \vec{n} \right|,
\]

where \( dP \) is the instant power emitted into the small body angle \( d\Omega \), and \( \vec{n} \) is the unit vector in the radiation direction. Selecting the coordinate system so that vector \( \vec{n} \) is in the \( xz \) plane (see Fig. below), we have \( \vec{n} = \vec{n}_x \sin \theta + \vec{n}_z \cos \theta \), and can always select the time origin so that \( \vec{p} = p(-\vec{n}_x \cos \omega t + \vec{n}_z \sin \omega t, \vec{n}_y \sin \omega t) \).

\[
\vec{n} \times \vec{p} = p \begin{vmatrix} \vec{n}_x & \vec{n}_y & \vec{n}_z \\ \sin \theta & 0 & \cos \theta \\ \cos \omega t & \sin \omega t & 0 \end{vmatrix} = p \left( -\vec{n}_x \cos \omega t \sin \omega t + \vec{n}_z \cos \omega t + \vec{n}_z \sin \theta \sin \omega t, \vec{n}_y \sin \theta \sin \omega t \right).
\]

From here,

\[
(\vec{n} \times \vec{p}) \times \vec{n} = p \begin{vmatrix} \vec{n}_x & \vec{n}_y & \vec{n}_z \\ -\cos \theta \sin \omega t & \cos \theta \cos \omega t & \sin \theta \sin \omega t \\ \sin \theta & 0 & \cos \theta \end{vmatrix} = p \left[ \vec{n}_x \cos^2 \theta \cos \omega t + \vec{n}_y \sin \omega t - \vec{n}_x \cos \theta \sin \theta \cos \omega t \right].
\]

so that

\[
(\vec{n} \times \vec{p}) \times \vec{n} \right| = p^2 \left[ \cos^4 \theta \cos^2 \omega t + \sin^2 \omega t + \cos^2 \theta \sin^2 \omega t \cos^2 \omega t \right] = p^2 \left[ \cos^2 \theta \cos^2 \omega t + \sin^2 \omega t \right].
\]

Thus the average (over time) radiation power

\[
\frac{dP}{d\Omega} = \frac{\omega^4 p^2}{8\pi c^2} \left( 1 + \cos^2 \theta \right).
\]

Since \( d\Omega = \sin \theta d\theta d\phi \), and the average radiation distribution is axially-symmetric, for the power radiated into a small polar angle range \( d\theta \) we get
\[
\frac{dP}{d\theta} = \frac{\omega^4 p^2}{4 c^3} \left(1 + \cos^2 \theta\right) \sin \theta.
\]

(c, 8 points) Find the total radiation power \( P \).

Integrating over \( \theta \), we get

\[
P = \int_0^\pi \frac{dP}{d\theta} d\theta = \frac{\omega^4 p^2}{4 c^3} \int_0^\pi \left(1 + \cos^2 \theta\right) \sin \theta d\theta = \frac{2}{3} \frac{\omega^4 p^2}{c^3}.
\]