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SOLUTIONS. Comprehensive Examination, January 31, 2008

Astronomy 1

a)

\[ \rho_m(R) = \rho_{m0} \left( \frac{R}{R_0} \right)^{-3}, \]

\[ \rho_\Lambda(R) = \rho_{\Lambda0}. \]

b) Start with the Friedmann equation

\[ \ddot{R}^2 = \frac{8\pi G \rho R^2}{3}, \]

\[ = H_0^2 \Omega R^2, \]

\[ = H_0^2 \left[ \left( \frac{R}{R_0} \right)^{-3} \Omega_{m0} + \Omega_{\Lambda0} \right] R^2. \]

For \( R/R_0 \gg 1 \),

\[ \ddot{R} = H_0^2 \Omega_{\Lambda0} R^2. \]

So

\[ \dot{R} = H_0 \Omega_{\Lambda0}^{1/2} R. \]

So

\[ \frac{dR}{R} = H_0 \Omega_{\Lambda0}^{1/2} dt. \]

So

\[ \ln R = H_0 \Omega_{\Lambda0}^{1/2} t. \]

So

\[ R \propto \exp H_0 \Omega_{\Lambda0}^{1/2} t. \]

c) At \( R = R_{\text{max}} \), \( \dot{R} = 0 \), so

\[ \left( \frac{R_{\text{max}}}{R_0} \right)^{-3} \Omega_{m0} = -\Omega_{\Lambda0}. \]

So

\[ \frac{R_{\text{max}}}{R_0} = \left( \frac{\Omega_{m0}}{\Omega_{\Lambda0}} \right)^{1/3}. \]

d) Again starting from the Friedmann equation,

\[ dt = \frac{1}{H_0} \frac{dR}{R \left[ \left( \frac{R}{R_0} \right)^{-3} \Omega_{m0} + \Omega_{\Lambda0} \right]^{1/2}}. \]

So

\[ t = \frac{2}{H_0} \int_0^{R_{\text{max}}} \frac{dR/R_0}{\left[ \Omega_{m0} R_0/R + \Omega_{\Lambda0} (R/R_0)^2 \right]^{1/2}}. \]
**Astronomy 2**

a)

\[ \frac{1}{T_{\text{recomb}}} = N_e \alpha^{[2]} \]

\[ T_{\text{recomb}} = \frac{1.1 \times 10^8 \text{ years}}{N_e} = 11 \text{ years} \]

b)

\[ T_{\text{cascade}} \sim \frac{1}{\text{Typical Einstein } A \text{ - value}} = 10^{-7} \text{ to } 10^{-8} \text{ sec} \]

Students knowing something about the interstellar medium should know that typical radiative rates for the hydrogen lines are \(10^7 - 10^8 \text{ sec}^{-1}\).

c) One doesn’t need to remember the Stromgren formula to write the ionization balance equation. Recombination is a collision, and the rate is proportional to the product \(N_e N_p\). In steady state, Total number of recombinations per sec = Number of photoionizations per sec.

\[ \frac{4\pi}{3} R^3 \alpha^{[2]} N_e N_p = N_{\text{Lyman}} = 10^{48} \]

and hence \(R\), setting \(N_e = N_p\).

**Should there be a numerical value??**

**Astronomy 3.**

a) The source is thought to be a collapsing Chandrasekhar-mass white dwarf. There is no remnant. They blast 1.4 solar masses of iron and other debris out into the ISM.

b) Use the inverse-square law (or remember that \(m-M = 5 \log D - 5\)). The distance modulus is 35 mag; the apparent magnitude is 15.7

c) \(R = \frac{A_V}{E(B-V)} = 3.1\). The intrinsic colors of an A0 star are 0. \(A_V = 3.1\) mag. This is a lower limit, because the A0 star is inside the Galaxy, and there may be more extinction behind it.

d) The flux is \(1.98 \times 10^{-5} / 550\text{nm} \times \text{bandwidth (100 nm)} \times 2.512^{-15.7} = 1.89 \times 10^{-15} \text{erg/cm}^2/\text{s}\). Multiply by the area of the 1.3m (diameter) aperture and the system efficiency. You detect \(6.3 \times 10^9 \text{ erg/sec}\). Convert to counts using \(E = \frac{hc}{\lambda}\). The SN yields about 1700 counts/second.

e) You also detect 490 background counts/second, so there are about 2200 counts/sec in your detection cell.

\(S = 1700 \times T\) source counts (\(T\) is the observing time).

\(B = 490 \times T\) background counts.

Assume Gaussian statistics.

The uncertainty is \(\sqrt{(S+B)T}\) on the source+background, and \(\sqrt{BT}\) in the subtracted background. The errors add in quadrature, so that \(S/N = ST / \sqrt{(S+2B)T}\).

1% accuracy means \(S/N = 100\).

Re-arrange the equation. \(T = (S/N)^2 (S+2B)/S^2\). \(T = 9\) seconds.

In the absence of background, \(T = 6\) seconds (\(10^4\) counts give \(S/N = 100\)).
Astronomy 4.

a) The luminosity function is constant, so the mean absolute magnitude is the average of the bright cutoff \( M_* \) due to the luminosity function and the faint cutoff \( m_L = 5 \log \frac{r}{10 \text{ pc}} \) due to the limiting apparent magnitude of the sample.

b) Use \( m = 5 \log \frac{r}{10 \text{ pc}} + \langle M \rangle \), and rearrange.

c) The observed regression of \( m \) on \( \log v \) has the same regression coefficient (2.5) as that expected for \( \log r \). \( v \) and \( r \) must therefore be proportional to each other. If the Malmquist bias is ignored, the expected regression coefficient is 5 instead of 2.5. This means that \( v \) is proportional to \( r^2 \).

Atomic, Molecular, and Optics 1

(a) Either angular momentum \( L = mvr \) is quantized in units of \( \hbar \), or constructive interference of de Broglie waves around the circumference of the Bohr orbit, \( (\lambda_{dB}/2\pi r = \text{integer}) \). Derivation is simple.

(b) The new quantum numbers are the orbital angular momentum \( \ell \) and its projection on the spherical coordinate \( z \)-axis, \( m_\ell \). They are the indices of the spherical harmonics that are the solutions of the angular part of the Schrödinger equation.

c) The fine structure shifts are a result of the magnetic moment \( \vec{\mu}_e \) of the electron (which is proportional to its spin \( \vec{s} \)) interacting with the magnetic field \( \vec{B} \) caused by the classical proton "current" around the electron as seen in the rest frame of the electron. Either rest frame is acceptable. This field is proportional to the angular momentum \( \vec{l} \) of the electron associated with its motion around the much heavier proton. The fine structure can then be calculated through the classical formula
\[
E = -\vec{\mu}_e \cdot \vec{B} \propto \vec{s} \cdot \vec{l}.
\]
- If they expand the square root from the Dirac eigenvalue correctly, they also get credit.

d)
In (b) the quantum number $m$ is not shown because all states of the same $l$ but different $m$ have the same energy. They are degenerate.

In (c) the fine structure is shown for only one level but the general scheme is the same for all states with $l > 0$. Each level is split into two as shown. The size of the splitting, however, is not to scale. It's typically $10^4$ times smaller than the electronic spacing, namely $10^9$ Hz at $n = 2$ and decreases as $1/n^3$.

The standard notation is $n^{2s+1}L_j$, where $n$ is the principal quantum number, $s$ is the spin and it's $1/2$ for hydrogen so $2s+1$ is always $2$, and $L$ indicates the angular momentum using $S \Rightarrow l = 0$, $P$ for $l = 1$, $D$ for $l = 2$. Also, $j = l + 1/2, l - 1/2$, etc. but with only two values possible, there are only two levels.

**Atomic, Molecular, and Optics 2**

(a) The slit is placed at the entrance to act as a point source. The first mirror produces a broad beam of parallel light at the diffraction grating, so each wavelength emerges as a broad beam of well defined angle. The second mirror images that light onto the detector. Accordingly, the slit and the detector are each located a distance $f$ from their respective mirrors. The slit is imaged onto the detector and therefore the resolution depends on the slit size. The smaller the slit, the better the resolution of the spectrometer.
(b) The resolving power of a grating can be calculated by considering the angular width of a principal fringe and the angular width given by the grating equation, $n\lambda = h \sin \theta$ for some wavelength spread. The former comes from considering the change in $\theta$ when there is one extra wavelength $\lambda$ on the light from one side of the grating relative to the other.

$$\Delta \sin \theta = \frac{\lambda}{hN}.$$ Differentiating that yields the intrinsic angular width,

$$\Delta \theta_N = \frac{\lambda}{Nh \cos \theta}.$$ Compare to the derivative of the grating equation to extract the width in $\lambda$:

$$\Delta \theta_\lambda = \frac{n \Delta \lambda}{h \cos \theta}.$$ Here $\lambda$ is the wavelength, $N$ is the number of grooves, $n$ is the diffraction order, and $h$ is the groove spacing. Combining these equations yields:

$$\frac{\lambda}{\Delta \lambda} = nN.$$ For a 50mm grating with 2000 lines/mm, this yields about $10^5$.

(c) For a rough estimate, it is admissible to assume that the spectrum is Gaussian. The time-bandwidth product $\delta \nu \delta \tau = 0.44$ for a Gaussian pulse. One must convert the width of the spectrum from wavelength to frequency first ($\delta \nu = -c \delta \lambda / \lambda^2$) and then one finds that the minimum duration for a pulse with a bandwidth of 30nm centered at 780nm is 30 fs. This corresponds to a flat or linear spectral phase.

**High Energy Physics 1**

Collider experiments usually have a layered structure. First comes the central particle tracker,
which is a low mass device (mostly gas) detecting the ionization of charged particles as they pass and measuring their trajectories, usually in a magnetic field. This tracker includes a vertex detector, which is very close to the interaction point in order to detect displaced vertices, vertices not coming from the primary interaction due to decay.

Next comes the calorimetry, first electromagnetic and then hadronic. The electromagnetic calorimeter uses a high Z material such as lead to stimulate radiation and pair production, hence electromagnetic showers - which rapidly absorb all of the energy of electrons and photons. The hadronic calorimeter uses a high density material in an attempt to contain all the energy of hadrons and electrons in the total calorimeter. The energy is measured in samples throughout both calorimeters.

Except for an occasional hadron which gets through both calorimeters without interacting, the particles which emerge from the hadron calorimeter are muons (which do not radiate strongly because of their high mass and do not have strong interactions), and neutrinos (which interact only by the weak interaction). Muons are measured by additional tracking devices outside the calorimeter.

a) electron: Electrons are identified by their characteristic shower shape in the EM calorimeter and by their track in the central tracker. The energy from the calorimeter is compared to the momentum from the tracker.

b) photon: Photons are also identified by their shower shape in the EM calorimeter and by the lack of a track. The energy is measured by the EM calorimeter.

c) muon: Muons are identified by their track in the central tracker, only ionization energy in the calorimeter and by a muon track outside the calorimeter. The muon’s momentum (hence energy) is measured in the central tracker and also the outer muon spectrometer.

d) tau lepton: Tau leptons decay rapidly to electrons, muons or hadrons. The electron and muon decays also involve neutrinos, so you can focus on low energy electrons or muons but need to realize there will be missing energy. The hadron decays involve narrow jets which often can be distinguished from quark jets. In this case the energy of the tau is measured using the jets.

e) light quark: A light quark turns into a hadronic jet (spray) of particles. So you need to detect clumps of energy or jets in the calorimeter, for which you measure the energy.

f) b−quark: You need to detect a hadronic jet as above but also select b−quarks by looking for displaced vertices, from the decay of b−quarks to either leptons or hadrons. Jet energy is measured as above.

g) neutrino: The neutrino essentially always escapes the detector. But it is the only standard model particle which escapes the detector completely so it can be detected by measuring all the energy in the event and calculating what is missing. Some energy always escapes down the beam pipe but the energy transverse to the beam should add (as vectors - momentum) to zero. So the missing transverse energy can be detected and is used as the signature of a neutrino.

h) SUSY LSP: A SUSY LSP also does not interact in the detector and is detected by its missing transverse energy, like the neutrino. Sometimes it can be distinguished from the neutrino by the context (other particles) of the event.
High Energy Physics 2

a. The $W$ and $Z$ bosons, all quarks, and the leptons $e, \mu, \tau$.

The Yukawa interaction $\mathcal{L} = \lambda \bar{\psi}_e \phi \psi_\ell$ where $\phi = \phi_0 + \varphi$ gives a mass $m_e = \lambda \phi_0$ to the electron. (Then the Higgs field couples as $\mathcal{L} = \frac{n}{\phi_0} \bar{\psi}_e \psi_\ell \varphi$, so proportional to the electron mass).

An explicit mass term $m_e \bar{\psi}_e \psi_e$ breaks $SU(2)$ gauge symmetry.

An experimental lower bound on the Higgs mass is about 114 GeV.

b. The right-handed component of the neutrino must be a scalar under $SU(3) \times SU(2) \times U(1)$ to preserve gauge invariance.

c. If at $t = 0$ the wave function of the muon is $\psi_\mu(x) = \cos \theta \psi_I(x) + \sin \theta \psi_{II}(x)$, where $\psi_I$ and $\psi_{II}$ are eigenstates of the Hamiltonian, then at later time $t$ one has

$$\psi_\mu(x, t) = \cos \theta \psi_I(x) e^{-\frac{i}{\hbar} E_{II} t} + \sin \theta \psi_{II}(x) e^{-\frac{i}{\hbar} E_I t},$$

where $E_I$ and $E_{II}$ are the energies of $\psi_I$ and $\psi_{II}$. The amplitude to find a tau lepton at time $t$ is then

$$\langle \sin \theta \psi_I(x) - \cos \theta \psi_{II}(x) | \cos \theta \psi_I(x) e^{-\frac{i}{\hbar} E_{II} t} + \sin \theta \psi_{II}(x) e^{-\frac{i}{\hbar} E_I t} = \sin \theta \cos \theta \left(e^{-\frac{i}{\hbar} E_I t} - e^{-\frac{i}{\hbar} E_{II} t}\right)$$

which can be rewritten as $\frac{1}{2} \sin 2\theta \ e^{-\frac{i}{\hbar} E_{II} t} \left(e^{\frac{i}{\hbar} \Delta E t} - e^{-\frac{i}{\hbar} \Delta E t}\right)$ where $\Delta E = E_{II} - E_I$. The probability is then $P = \sin^2 2\theta \sin^2 \frac{\Delta E t}{2\hbar}$.

Condensed Matter 1

a. We introduce the cyclotron frequency $\omega_B = \frac{eB}{mc}$ and the magnetic length $l = \frac{\hbar}{eB}$ to simplify notation. Straightforwardly we obtain $H_k \psi_k(y) = E \psi_k(y)$ with

$$H_k = \frac{\hbar^2}{2m} \left(-i \partial_y\right)^2 + \frac{m \omega_B^2}{2} (y - k l)^2 + \frac{m \omega_0^2}{2} y^2. \quad (15)$$

Simple transformation of (15) gives

$$H_k = \frac{\hbar^2}{2m} \left(-i \partial_y\right)^2 + \frac{m \omega_B^2 + \omega_0^2}{2} (y - y_k)^2 + \frac{m}{2} \frac{\omega_B^2 \omega_0^2}{\omega_B^2 + \omega_0^2} k^2 l^4 \quad (16)$$

with

$$y_k = \frac{\omega_B^2}{\omega_B^2 + \omega_0^2} k l^2. \quad (17)$$

It is easy to see that the first two terms of (16) for fixed $k$ are the Hamiltonian of harmonic oscillator centered at $y_k$ with an oscillator frequency $\sqrt{\omega_B^2 + \omega_0^2}$. The energy is offset by the third term, so the energy levels are given by

$$E_{k,n} = \hbar \sqrt{\omega_B^2 + \omega_0^2} \left(n + \frac{1}{2}\right) + \frac{m}{2} \frac{\omega_B^2 \omega_0^2}{\omega_B^2 + \omega_0^2} k^2 l^4. \quad (18)$$
b. The filled states are defined by the condition $E_{k,n=0} = \mu$ which gives $-k_0 < k < k_0$ with
\[
\mu = \frac{1}{2} \hbar \sqrt{\omega_B^2 + \omega_0^2} + \frac{m}{2} \frac{\omega_B^2 \omega_0^2}{\omega_B^2 + \omega_0^2} k_0^2 l^4.
\]
(19)

c. The oscillator states are centered at $y = y_k$ with $y_k$ from (17). Therefore, the filled states have their centers in the range $-y_0 < y < y_0$ with
\[
y_0 = \frac{\omega_B^2}{\omega_B^2 + \omega_0^2} k_0 l^2
\]
(20)
at a given chemical potential.

d. The edge states have momentum $\hbar k_0$ and energy $E_{k_0,n=0}$. We find the velocity by differentiating energy over momentum and obtain $\pm v_0$ at $y = \pm y_0$ respectively with
\[
v_0 = \frac{1}{\hbar} \left. \frac{\partial E_{k,n=0}}{\partial k} \right|_{k=k_0} = \frac{m}{\hbar} \frac{\omega_B^2 \omega_0^2}{\omega_B^2 + \omega_0^2} k_0 l^4 = \frac{\omega_B^2}{\omega_B^2 + \omega_0^2} y_0 = \frac{E_0}{B}.
\]
(21)

Here $E_0$ is the confining electric field at the boundary $cE_0 = m\omega_0^2 y_0$ and the latter result is, of course, the drift velocity in crossed magnetic and electric fields.

**Condensed Matter 2**

a) Primitive lattice vectors: $\mathbf{a}_1 = a \hat{x}$ and $\mathbf{a}_2 = 2a \hat{y}$

Basis: Three black flies at $\mathbf{d}_1 = 0$, $\mathbf{d}_2 = \frac{1}{4} \mathbf{a}_1 + \frac{1}{4} \mathbf{a}_2$, and $\mathbf{d}_3 = \frac{1}{2} \mathbf{a}_1 + \frac{1}{2} \mathbf{a}_2$, and one white fly at $\mathbf{d}_4 = \frac{3}{4} \mathbf{a}_1 + \frac{3}{4} \mathbf{a}_2$

Reciprocal lattice: $\mathbf{K} = h \mathbf{b}_1 + k \mathbf{b}_2$ where $h$ and $k$ are integers. The condition $\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi \delta_{ij}$, requires that $\mathbf{b}_1 = \frac{2\pi}{a} \hat{x}$ and $\mathbf{b}_2 = \frac{2\pi}{a} \hat{y}$.

b) Diffraction spots require $\mathbf{K} = \mathbf{k} - \mathbf{k}'$ where $|\mathbf{k}| = |\mathbf{k}'|$. To find the spot with minimum $k$, we draw an Ewald circle and do some geometry.
\[
K_{11} = \sqrt{b_1^2 + b_2^2} = \sqrt{(2b_2)^2 + b_2^2} = \sqrt{5}b_2
\]

\[
\cos \theta = \frac{K_{11}/2}{k_{\text{min}}} = \frac{b_1}{K_{11}}
\]

\[
k_{\text{min}} = \frac{K_{11}^2}{2b_1} = \frac{5b_2^2}{4b_2} = \frac{5\pi}{4\alpha}
\]

\[
\lambda = \frac{2\pi}{k_{\text{min}}} = 16\text{cm} \rightarrow \text{microwaves}
\]

c) Compute the structure factor for \(K_{11}\)

\[
I \sim |S|^2 \quad S = \sum_j f_j e^{iK_{11} \cdot d_j} \quad K_{11} = b_1 + b_2
\]

\[
S = f_B (1 + e^{i\pi} + e^{i2\pi}) + f_W e^{i3\pi} = f_B - f_W
\]

If \(f_B = f_W\) then \(S = 0\) and the diffraction spot vanishes.

**Nuclear Physics 1**

(a) \(H_{\text{weak}}\) breaks \(P\) and \(PC\), but the other two terms are invariant.

(b) \(\langle \alpha + |H_0|\alpha^- \rangle = \langle \alpha + |PH_0P|\alpha^- \rangle = -\langle \alpha + |H_0|\alpha^- \rangle\). Note that we used \(P^2 = 1\).

(c) \(P_1(S) = \int d\alpha d\alpha^- \delta(2\sqrt{\alpha^2 + \alpha^-^2} - S) e^{-\frac{1}{2}(\alpha^2 + \alpha^-^2)} \sim S\). You can see this by scaling out \(S\).

(d) \(P_2(S) = \int d\alpha d\alpha d\alpha^- d\alpha^- \delta(2\sqrt{\alpha^2 + \alpha^-^2} + \alpha^-^2 - S)e^{-\frac{1}{4}((\alpha^2 + \alpha^-^2))} \sim S^2\). Follows from scaling out \(S\).

(e) In fact, the effect of a perturbation is determined by the norm of the operator so the effect is enhanced by roughly the number of components of an eigenfunction, say a factor of 100. If we would have 10000 spacings then it would be possible to detect a PC breaking effect of order \(10^{-4}\).

**Nuclear Physics 2**

a) The charged particle moves in a circular trajectory of radius \(R = p/qB\). The analysis of the system is simplest using a coordinate system whose origin is at the center of the particle’s path (as
shown in the figure). In this case we see that the coordinates of the particle’s entrance and exit from the field are:

\[
(x_1, y_1) = (R \cos \phi_1, R \sin \phi_1) \quad (22)
\]

\[
(x_2, y_2) = (R \cos \phi_2, R \sin \phi_2) \quad (23)
\]

The angles \( \theta_1 \) and \( \theta_2 \) are essentially complementary to the \( \phi_1 \) and \( \phi_2 \):

\[
\phi_1 = \theta_1 + \frac{3\pi}{2} \quad (24)
\]

\[
\phi_2 = \theta_2 + \frac{3\pi}{2} \quad (25)
\]

\[
x_1 = R \cos \left( \theta_1 + \frac{3\pi}{2} \right) = R \sin \theta_1 \quad (26)
\]

\[
x_2 = R \cos \left( \theta_2 + \frac{3\pi}{2} \right) = R \sin \theta_2 \quad (27)
\]

\[
x_2 - x_1 = L \quad (28)
\]

\[
\sin \theta_2 - \sin \theta_1 = \frac{L}{R} = \frac{qBL}{p} \quad (29)
\]

b)

\[
\sin \theta_2 - \sin \theta_1 \sim \theta_2 - \theta_1 \quad (30)
\]

\[
\sqrt{2}\delta \theta = -qBL \frac{\delta p}{p^2} \quad (31)
\]

\[
\frac{\delta p}{p} = \frac{\sqrt{2}\delta \theta}{qBLp} \quad (32)
\]

The fractional momentum resolution worsens linearly with momentum in a standard magnetic spectrometer.
c) Electrons: Bremsstrahlung is the dominant mechanism. A primary electron will very rapidly radiate its energy as bremsstrahlung photons. These photons will pair convert. These electrons will generate bremsstrahlung, etc... This is called an electromagnetic shower.
Muons: These muons are too massive to experience bremsstrahlung. They do not participate in the strong interaction and so their principal energy loss mechanism is $dE/dx$ via ionization of electrons from the surrounding material.

d) The number of charged particles produced in a sampling calorimeter is proportional to the incident energy of the initial particle. Hence:

\[
\begin{align*}
N_{\text{charged}} & \propto E \\
\delta N_{\text{charged}} & \propto \delta E \\
\delta N_{\text{charged}} & = \sqrt{N_{\text{charged}}} \\
\delta E & \propto \sqrt{E} \\
\frac{\delta E}{E} & \propto \frac{1}{\sqrt{E}} \\
\frac{\delta E}{E} & = \text{Constant} \sqrt{E}
\end{align*}
\]