Group Theory for Physicists

P. van Nieuwenhuizen

PHY 680.01, Fall 2014; Tu, Th: 10:00 – 11:20

Group theory was developed by mathematicians in the 18th and 19th centuries, but applications to physics started only in the 20th century. It describes symmetries of objects such as atoms or molecules. Or symmetries of the action of field theories, for example Lorentz symmetry, Poincaré symmetry, the SU(3) flavor symmetry of quarks, the SU(3) × SU(2) × U(1) symmetries of the Standard Model, the SU(5), SO(10) and E(6) symmetries of Grand Unification Theories, the modular group and Kac-Moody algebras of string theory, etc. Or the parity, rotational, translational or permutational symmetries of wave functions in quantum mechanics. In the 1970s, a new kind of group was discovered by physicists, supergroups and superalgebras.

Group theory arose from studies in three areas in mathematics: number theory, algebraic equations, and geometry. Number theory is an area with few practitioners and high standards but it remains an area with fewer applications to physics. Groups enter here in such problems as Fermat’s observation (his “little theorem”) that \( n^p = n \mod p \), where \( n \) is a natural number, and \( p \) a prime or Wilson’s theorem that \((p-1)! \mod p = -1\). We shall only use these examples to extract the definition of an abstract group.

The quadratic, cubic and quartic algebraic equations can be solved by taking square and cubic roots. But for the quintic equation \((ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0)\), no such result was known. Algebraic equations have symmetry groups, and if these groups are “solvable groups”, the algebraic equations can be solved by roots. The symmetry group of the quintic equation is \(S_5\), and \(S_5\) is not solvable. In this way group theory proved that the quintic equation cannot always be solved by roots.

In geometry, the concept of groups emerged when one studied what are now called transformation groups: transformations of an object which leave the object unchanged. For example, the symmetries of polyhedra form finite groups. The rotations of vectors after parallel transport along a closed curve form infinite groups, the so-called holonomy groups. Symmetries (isometries) of Riemannian manifolds are described by Killing vectors, and these Killing vectors form groups. For physicists a very important area of group theory is the construction of coset manifolds, with harmonics as eigenfunctions of Laplace, Dirac and other operators.

Many mathematicians contributed to the construction of the concept of groups: Euler, Lagrange, Gausz, Abel, Galois, Cayley, Sylow, Jordan, Hölder, Schreier, Lie, F. Klein, Elie Cartan, Maurer, Schur, Frobenius, Weyl, Young, Dynkin, and others. The notion of a group was already formulated by Ruffini, Galois, Cauchy and Abel by 1830. In 1854 Cayley showed that every finite group is a subgroup of a permutation group \(S_n\), but only in 1882 the modern definition of a group was given by W. von Dyck, a student of Felix Klein. Klein is known for
his work on the ‘Erlanger Programm’, which aimed at a synthesis of geometry as the study of the properties of spaces that are invariant under a given group of transformations. There are finite and infinite groups, and in 1982, the classification of all simple finite groups was completed.

In the early applications of group theory (in the years around 1930), two groups played a central role: the rotation group, and the permutation group $S_n$. The latter was intended for the quantum mechanics of $n$ electrons, but the Pauli principle superseded this. Nowadays, the group $S_n$ is used to construct Young tableaux for the classical groups. The rotation group was used to derive the Wigner-Eckart theorem. Not all physicists liked group theory in the beginning; Heisenberg called it “the group pest”, presumably because it was used in those days as an alternative to a true dynamical theory of quantum physics. But others saw a great future. Wigner constructed the unitary irreducible representations of the Poincaré group. The quark model made the group SU(3) familiar, and later SU(4) and SU(6). In solid state physics, group theory was used to classify crystal structures. In atomic physics, group theory was used to diagonalize large interaction matrices. With the advent of nonabelian gauge theories, group theory rose to prominence among particle physicists because in these theories particles fall into multiplets which transform as matrix representations of nonabelian groups. The study of the properties of irreducible representations of simple and semisimple Lie groups (compact or noncompact versions) will be a major part of this course.

Contents

We have chosen to discuss those subjects in group theory which we have found to be central for modern theoretical physics. We give proofs of most theorems, but the emphasis is on applications.

1. We begin with finite group theory. First we define some basic concepts. Then we present theorems on matrix representations. As an application we derive the Dirac matrices in $n$ dimensions, in Euclidean and Minkowski space, and the charge conjugation matrices, as well as their reality properties. As another application, we construct the multiplets of normal modes of polyhedra and some molecules such as buckyballs.

2. Then we define the simple Lie algebras SU(n), SO(n), Sp(n) and $F_4$, $G_2$, $E_6$, $E_7$, $E_8$ using particular explicit matrix representations. We construct the corresponding Lie groups, and discuss covering groups and the various real forms.

3. From these defining matrix representations of simple groups we read off the roots, Cartan generators, and we introduce weights, and all other representations. This leads to Dynkin diagrams, conjugacy classes of Lie algebras, and the spinor representations for SO(n).

4. Next we introduce Young tableaux for SU(n), SO(n) and Sp(n). We first state the rules how to obtain them and how to multiply them, and give many examples. Then we use
the group $S_n$ to explain the construction of Young tableaux.

5. We also consider non-semisimple groups, in particular the Poincaré group and derive their unitary irreducible representations and Casimir operators by using the theory of induced representations.

6. We discuss some topological properties of compact and noncompact Lie groups.

7. Then we construct coset manifolds, coset vielbeins and connections, the coset measure (Haar measure), torsions and curvatures, covariant Lie derivatives and spherical harmonics on coset manifolds.

8. Finally we consider some interesting generalizations: Kac-Moody algebras, superalgebras, and Virasoro algebras.

Prerequisites: For the mathematical part, none; the course is just fun and is accessible to any graduate student. For some of the applications, some familiarity with classical gauge field theories (Yang-Mills fields coupled to Dirac fermions) is useful.

Evaluation: There will be homework. The final exam consists of a written and an oral exam, and the grade will be based on these exams and to a lesser degree on the homework.

Office hours: Appointment can be made after each class.