Group Theory for Physicists Tu, Th: 11:20-1:00

P. van Nieuwenhuizen, PHY680.2 Spring 2009

Group theory was developed by mathematicians in the 18th and 19th centuries, but applications to physics started only in the 20th century. It describes symmetries of objects such as crystals. Or symmetries of the action or the Hamiltonian, for example Lorentz symmetry, Poincaré symmetry, the $SU(3)$ flavor symmetry of quarks, the $SU(3) \times SU(2) \times U(1)$ symmetries of the Standard Model, the $SU(5)$, $SO(10)$ and $E(6)$ symmetries of Grand Unification Theories, the modular group and Kac-Moody algebras of string theory, etc. Or the parity, rotational, translational or permutational symmetries of wave functions in quantum mechanics. In the 1970s, a new kind of group was discovered by physicists, supergroups and superalgebras.

Group theory arose from studies in two areas in mathematics: algebraic equations and geometry. The quadratic, cubic and quartic algebraic equations can be solved by taking square and cubic roots.[1] But for the quintic equation ($ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$), no such result was known. Algebraic equations have symmetry groups, and if these groups are "solvable groups", the algebraic equations can be solved by roots. The symmetry group of the quintic equation is S_5, and S_5 is not solvable. In this way Abel proved that the quintic equation cannot be solved by roots.

In geometry, the concept of groups emerged when one studied what are now called transformation groups: transformations of an object which leave the object unchanged. For example, the symmetries of polyhedra form finite groups. The rotations of vectors after parallel transport along a closed curve form infinite groups, the so-called holonomy groups. Symmetries (isometries) of Riemannian manifolds are described by Killing vectors, and these Killing vectors form groups. For physicists a very important area of group theory is the construction of coset manifolds, with harmonics as eigenfunctions of Laplace, Dirac and other operators.

Many mathematicians contributed to the construction of the concept of groups: Euler, Lagrange, Gausz, Abel, Galois, Cayley, Sylow, Jordan, Hölder, Schreier, Lie, F. Klein, Cartan,

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[1]The cubic equation $\alpha x^3 + \beta x^2 + \gamma x + \delta = 0$ can be simplified by the substitution $x = t + d$ to $t^3 + at + b = 0$. Setting $t = y + k/y$ one gets $\left(y^3 + k^3/y^3 + b\right) + (3ky + 3k/y + ay + ak/y) = 0$. Choose k such that the second term vanishes ($3k + a = 0$), then solve the vanishing of the first term (a quadratic equation for y^3). The quartic equation can similarly be simplified to $x^4 + ax^2 + bx + c = 0$. Introduce a number d and write $\left(x^2 + d\right)^2 + \left((a - 2d)\,x^2 + bx + c - d^2\right) = 0$. Now choose d such that the second term becomes a pure square (the discriminant gives a cubic equation for d). Finally, solve $\left(x^2 + d\right)^2 = (2d - a)\,(x - \alpha)^2$.

Maurer, Schur, Frobenius, H. Weyl, Young, Dynkin, and others. In 1854 Cayley showed that every finite group is a subgroup of a permutation group S_n. The notion of a group is due to Galois (1830), but only in 1882 the modern definition of a group was given by W. von Dyck, a student of Felix Klein. Klein is known for his work on the "Erlanger Programm", which aimed at a synthesis of geometry as the study of the properties of spaces that are invariant under a given group of transformations. These are finite and infinite groups, and in 1982, the classification of all simple finite groups was completed.

In the early applications of group theory (in the years around 1930), two groups played a central role: the rotation group, and the permutation group S_n. The latter was intended for the quantum mechanics of n electrons, but the Pauli principle superseded this. Nowadays, the group S_n is used to construct Young tableaux for the classical groups. The rotation group was used to derive the Wigner-Eckart theorem. Not all physicists liked group theory in the beginning; Heisenberg called it "the group pest", presumably because it was used in those days as an alternative to a true dynamical theory of quantum physics. But others saw a great future. Wigner constructed the unitary irreducible representations of the Poincaré group. The quark model made the group SU(3) familiar, and later SU(4) and SU(6). In solid state physics, group theory was used to classify crystal structures. In atomic physics, group theory was used to diagonalize large interaction matrices. With the advent of nonabelian gauge theories, group theory rose to prominence among particle physicists because in these theories particles fall into multiplets which transform as matrix representation of nonabelian groups. The study of the properties of irreducible representations of simple and semisimple Lie groups (compact or noncompact versions) will be a major part of this course.

Contents

1. We begin with finite group theory. We present theorems on matrix representations. As an application we derive the Dirac matrices in n dimensions, in Euclidean and Minkowski space, and the charge conjugation matrices. As another application, we construct the multiplets of normal modes of polyhedra.

2. Then we derive the simple Lie algebras SU(n), SO(n), Sp(n) and F_4, G_2, E_6, E_7, E_8 using particular explicit matrix representations. We construct the corresponding groups, and discuss covering groups and various real forms.

3. From these defining matrix representations of simple Lie algebras we read off the roots, Cartan generators, and we introduce weights, and all other representations. This leads to Dynkin diagrams, conjugacy classes of Lie algebras, and the spinor representations for SO (n).

4. Next we introduce Young tableaux for SU (n), SO (n) and Sp (n). We first state the rules and give many examples. Then we use the group S_n to explain the construction of Young tableaux.

5. We also consider non-semisimple groups, in particular the Poincaré group, and their unitary irreducible representations. We end with the crystal groups.

6. Finally we construct coset manifolds, coset vielbeins and connections, the coset measure (Haar measure), torsions and curvatures, covariant Lie derivatives and spherical harmonics on coset manifolds.

Prerequisites: for the mathematical part none, the course is just fun and is accessible to any graduate student. However, for the many applications to various physical situations, familiarity with classical gauge field theories (Yang-Mills fields coupled to Dirac fermions) is useful.

There will be homework. The final exam consists of a written and an oral exam, and the grade will be based on these exams and the homework. Office hours: appointment can be made after each class.

First lecture: Tuesday, January 26 at 11:20 in the YITP common room. There and then the location and time of the next lectures will be determined.

Literature: we use H. Georgi, Lie Algebra in Particle Physics, and W. Ledermann, Introduction to the Theory of Finite Groups. Additional notes will be handed out in class.