4. Energy and Stress in an Electrostatic Field

a) Energy: work done to bring a small charge \( dq_i \) from \( \infty \) to \( r_i \)

\[
\delta W_i = - \int_{\infty}^{r_i} \mathbf{E} \cdot dq_i = (\phi(r_i) - \phi(\infty)) dq_i
\]

\[
\overset{r_i}{\underset{\infty}{\leftarrow}}
\]

Total work by many charges

\[
\delta W = \sum_i \delta W_i = \sum_i \phi(r_i) dq_i
\]

\[
= \int d^3r \, \phi(r) \, dq_i
\]

\[
\nabla \mathbf{E} = 4\pi \rho = \delta \rho = \frac{1}{4\pi} \nabla \delta \mathbf{E}
\]

\[
\Rightarrow \delta W = \int d^3r \, \phi(r) \, \frac{1}{4\pi} \nabla \delta \mathbf{E}
\]

Part integration

Surface term vanishes

\[
= - \int d^2r \, \delta \phi(r) \, 
\]

\[
= \frac{1}{4\pi} \int d^2r \, \mathbf{E} \cdot \delta \mathbf{E}
\]

\[
= \frac{1}{4\pi} \int d^2r \, \delta \mathbf{E}^2
\]

\[
W = \frac{1}{4\pi} \int d^3r \, (\mathbf{E}^2 - \mathbf{E}_0^2)
\]

to change the field strength from \( \mathbf{E}_0 \) to \( \mathbf{E}_f \)

\[
\mathbf{E} \] is changed by taking \( dq_i \) from \( \infty \) to \( r_i \)