### Introduction

- Simulations with a finite chemical potential typically lead to a severe sign problem \((\det D(x)) = \det(D(\mu) \rightarrow \det D(\mu = 0) \in \mathbb{C})\).
- Importance Sampling based Monte Carlo methods cannot be applied to path-integrals with a complex weight
  \[\langle A \rangle = \frac{1}{Z} \int A(x) |\det(D(x))| e^{-i\mathcal{S}[\phi]} dx.\]
- Stochastic quantization such as the Complex Langevin method offers an alternative allowing a simulation of QCD with non-zero chemical potential and a study of the phase diagram of QCD [1].

### Complex Langevin simulations - Basic concepts

- The expectation value of the operator \(A\) can be obtained by integrating along a path of the so-called Langevin time \(\tau\)
  \[\langle A \rangle = \int A(x(\tau)) \, d\tau.\]
- The Langevin evolution is achieved by a stochastic process in the degrees of freedom, generically denoted as \(x\)
  \[\frac{dx}{d\tau} = -\frac{\partial \mathcal{S}}{\partial x} + \eta(\tau).\]
  where the Gaussian random noise \(\eta(\tau)\) has to fulfill the following conditions
  \[\langle \eta(\tau) \rangle = 0 \text{ and } \langle \eta(\tau_1) \eta(\tau_2) \rangle = 2\lambda (\tau_1 - \tau_2).\]
- Real Langevin simulations converge to the expected result. For gauge theories simulations with complexified degrees of freedom are no longer in a compact group, \(SU(3) \rightarrow SL(3, \mathbb{C})\), and convergence is no longer guaranteed and needs to be checked [2].
- Empirical studies and theoretical arguments show that convergence is achieved, if the distribution is localised (compact) in the imaginary part.
- We use gauge-invariance to get gauge-links to the proximity of the SU(3) manifold, this method is known as gauge cooling [3]
  \[U_t(x) \rightarrow V(x)U_t(x)V^{-1}(x + \kappa \vec{x}) \quad U, V \in SL(3, \mathbb{C}).\]
- A measure for the distance of the gauge links from the SU(3) manifold is given by the unitarity norm, which can be used to judge convergence
  \[\text{unitnorm} = \text{Tr}((UU^\dagger - I)^2).\]

### Simulation Results - QCD in the limit of heavy quarks

- Here we present results for QCD in the limit of heavy quarks (no spatial hopping). The fermion determinant can be written in terms of the (conjugate) Polyakov loops \(P_x\) and \(P^{-1}_x\) as
  \[\det D(\mu) = \prod_x \det(1 + C P_x + C' P^{-1}_x),\]
  where the coefficients \(C\) and \(C'\) are defined as
  \[C = (2\kappa e^{i\eta})^N \quad \text{and} \quad C' = (2\kappa e^{-i\eta})^N.\]
  For the gluonic part we use the full Wilson gauge action.
- The unitarity norm for one simulation setup close to the transition is shown in the following figure:
- Quantities such as the Polyakov loop \(P_x\) can be extracted from the Langevin evolution shown in the following figure:

### Simulation Results - Phase structure

- The real part of the density (left) and the polyakov loop (right) as function of the temperature and chemical potential for fixed \(\beta = 5.7\). The black points are the result of a complex Langevin simulation for a given value of \(\mu\) and \(T\). The surface is the result of a cubic interpolation, in which the value of the density or the polyakov loop is encoded in the colour. We use the Wilson flow to set the scale \((\sigma = 0.187 \text{fm})\). The hopping parameter is \(\kappa = 0.12\), which leads to the critical chemical potential of \(\mu_c = -\ln(2\kappa) \approx 1.43\).

### Conclusions & Outlook

- Complex Langevin simulation can be used to study the phase diagram of QCD.
- Thermal transition is visible in the polyakov loop \(P_x\).
- Transition in \(\mu\) is studied in the density \(n\).
- Further (planned) improvements:
  - Extend simulations to different \(\beta\) values and improve the thermal transition.
  - Determine the order of the transition.
  - Include fully dynamical fermions (Staggered or Wilson)

### References