Magnetic properties of QCD matter: lattice results and implications

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XQCD, 19.6.14
Based on

JHEP 1202 044 [1111.4956]
PoS(Lattice11) 192 [1111.5155]
PRD 86 (12) 071502 [1206.4205]
PRD 86 (12) 094512 [1209.6015]
PoS(Conf X) 197 [1301:5826]
JHEP 1304 130 [1303.1328]
PoS(Lattice13) [1310.8145]
PRL 112 (14) 042301 [1311.2559]
arXiv:1406.0269

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- Motivation
- Simulation details
- The QCD phase diagram(s)
- The equation of state: magnetization and pressure
- Spin contribution
- Paramagnetic squeezing
- Summary
Man-made magnets

neodymium magnets, € 5.00 at amazon.de, up to $10^4$ G.

non-permanent at condensed matter labs, up to $4.5 \cdot 10^5$ G.

Highest pulsed field: $2.8 \cdot 10^7$ G

non-destructive: $10^6$ G.
Magnetars

$10^{13} \, \text{G} < B < 10^{15} \, \text{G}$ near surface but even larger in the interior.

$(100 \, \text{MeV})^2 \approx 1.69 \cdot 10^{18} \, \text{eG} = 1.69 \cdot 10^{14} \, \text{eT}$ ($e \approx 0.3$ in natural units.)
(Almost) anything is possible.

\[ B = 10^{18} \text{G}, \quad 10^{23} \text{G} \approx (150 \text{ MeV})^2, \quad (50 \text{ GeV})^2 \]

at strong, electroweak phase transitions?
Collider experiment: non-central heavy ion collisions

Two very big currents at very short distances produce extremely strong magnetic fields.

LHC: $eB \lesssim 0.3 \text{ GeV}^2 \approx 5 \cdot 10^{19} \text{ eG}$, RHIC: $eB \lesssim 0.04 \text{ GeV}^2 \approx 6 \cdot 10^{18} \text{ eG}$. This is bigger than $m_{\pi}^2 \approx 0.02 \text{ GeV}^2$!

Fields have life time of only $\sim 0.1 \text{ fm} = 10^{-16} \text{ m} \approx 5 \cdot 10^{-24} \text{ s} \ (c = 1)$. 
Magnetic background field on the lattice

4-potential \( (A_\nu) = (0, Bx, 0, 0) \implies B = (0, 0, B) \)

Lattice: multiply links \( U_\nu \) with \( u_\nu = e^{iaqA_\nu} \in U(1) \)

\[
\begin{align*}
  u_y(n) &= e^{ia^2qBn_x} \\
  u_x(n) &= 1 & n \neq N_x - 1 \\
  u_x(N_x - 1, n_y, n_z, n_t) &= e^{-ia^2qBN_xn_y} \\
  u_\nu(n) &= 1 & \nu \neq x, y
\end{align*}
\]

The magnetic flux through the \( x-y \) plane is constant:

\[
\exp \left( i q \int_{F} d\sigma B \right) = \exp \left( i q \int d_{x_\nu} A_\nu \right) = e^{ia^2N_xN_yqB}
\]

Flux quantization due to the finite volume + boundary conditions:

\[
a^2N_xN_y \cdot qB = 2\pi N_b \quad N_b \in \mathbb{Z}
\]
Flux quantization
Implementation and limitations

- $B$ is invariant under $N_b \leftrightarrow N_b + N_x N_y$ (periodicity)
- Lattice field is unambiguous if $0 < N_b < N_x N_y / 4$
- Apply quantization for smallest charge $q = q_u / 2 = -q_d = e / 3$
- Typical lattice spacings:
  - Maximal $B$: $qB_{\text{max}} = \pi / (2a^2)$
    \[
    \sqrt{eB} \approx 1 \text{ GeV} \rightarrow 10^{20} \text{ e Gauss}
    \]
  - Minimal $B$: $qB_{\text{min}} = 2\pi T^2 (N_t / N_s)^2$
    \[
    \sqrt{eB} \approx 0.1 \text{ GeV} \rightarrow 10^{18} \text{ e Gauss}
    \]

Phenomenologically interesting range!
Renormalization I

By definition, $qB$ and quark masses do not change the lattice spacing. Does $qB$ induce any new divergencies? Does it acquire an anomalous dimension like the quark mass? Other renormalization factors may even become $qB$-dependent?

$qB$ breaks rotational symmetry and isospin symmetry. $qB$ modifies the free dispersion relation:

$$E(B) = \sqrt{p_z^2 + m^2 + 2n|qB|}$$

So maybe something sinister or complicated is happening? Fortunately not:

▶ For an external field there are no internal photon lines in Feynman diagrams $\rightarrow$ no new type of divergent diagram.
▶ $B$ couples to a conserved current $A_\nu \bar{\psi} \gamma_\nu \psi$.
▶ Protected by U(1) gauge invariance: $(e \cdot B)_r = e \cdot B$. 
Renormalization II

Electric charge renormalization:

\[ Z_e(\mu) = 1 + 2b_1 e^2 \ln \mu a , \quad e^2 = Z_e^{-1}(\mu) e_r^2(\mu) , \quad B^2 = Z_e(\mu) B_r^2(\mu) \]

(Free) energy density at zero temperature:

\[ f_B = f_0 + \Delta f = f_0 + \frac{2b_1(eB)^2 \ln(\mu a)}{2} + \frac{B_r^2(\mu)}{2} + c(eB)^4 \cdot \text{finite} \]

\[ B^2/2 \]

Renormalization of the free energy density \( f_B \) at \( T \geq 0 \):

**subtract the \( T = 0 \) \((eB)^2\)-term in the limit of small \( eB \)**

(so that \( \mu = \Lambda_H = O(\Lambda_{QCD}, m_\pi) \) but \( \Lambda_H \gg \sqrt{eB} \)).

The QED \( \beta \)-function coefficient \( b_1 = b_1(a^{-1}) \) depends on the lattice spacing, due to QCD-corrections.

The (trivial) background energy density \( B_r^2/2 \) is not included in the simulated Lagrangian.
Simulation and analysis details

Symanzik improved gauge action, $N_f = 2 + 1$ stout smeared staggered quarks at physical masses (Budapest-Wuppertal) action.

- Simulate at various $T$ and $N_b$.
- Fit all points by a 2D spline function.
- Keep physical $B$ fixed.
- Study finite volume effects with $N_s/N_t = 3, 4, 5$
- Extrapolate to the continuum limit with $N_t = 6, 8, 10$
  \[ = 1/(aT_c) \]
$T = 0$ simulation points
Observables

- Partition function for three flavors ($\mu_f = 0$ in the simulation)

$$Z = \int [dU] e^{-\beta S_g} \prod_{f=u,d,s} [\det M(q_f \cdot B, m_f, \mu_f)]^{1/4}$$

- Observables

$$\bar{\psi}\psi_f = \frac{T}{V} \frac{\partial \log Z}{\partial m_f} , \quad \chi_f = \frac{T}{V} \frac{\partial^2 \log Z}{\partial m_f^2} , \quad c_s^2 = \frac{T}{V} \frac{\partial^2 \log Z}{\partial \mu_s^2}$$

- Cancel additive divergences of $\bar{\psi}\psi$ by computing

$$\Sigma_{u,d}(B, T) = \frac{2m_{ud}}{M_\pi^2 F^2} \left[ \bar{\psi}\psi_{u,d}(B, T) - \bar{\psi}\psi_{u,d}(0, 0) \right] + 1 ,$$

$$\Delta \Sigma_{u,d}(B, T) = \Sigma_{u,d}(B, T) - \Sigma_{u,d}(0, T) .$$
Magnetic catalysis: \( T = 0 \)

\[
\Delta f = \int_{m_{\text{phys}}}^{\infty} dm \Delta \bar{\psi} \psi, \quad \mathcal{P}[X] = (eB)^2 \lim_{eB \to 0} \frac{X}{(eB)^2}
\]

\[
\mathcal{P}[\Delta f] + (1 - \mathcal{P})[\Delta f] = \int_{m_{\text{phys}}}^{\infty} dm \left[ \mathcal{P}[\Delta \bar{\psi} \psi] + (1 - \mathcal{P})[\Delta \bar{\psi} \psi] \right]
\]

\[
\sim b_1 \quad \sim -M_r
\]

\[
\implies M_r > 0 \quad \text{at } T = 0.
\]
Polyakov line and chiral condensate for physical quarks

S Borsányi et al. JHEP 1009 (10) 073

\[ T_c(\bar{\psi}\psi) = 155(3)(3) \text{ MeV} = 1.80(5) \cdot 10^{12} \text{ K.} \]

(Cross-over: other quantities may have different pseudocritical temperatures.)

\[ \not \exists \text{ order parameter but less than one in } 2.1 \cdot 10^{21} \text{ electrically charged particles differ by more than } e/6 \text{ from a multiple of } e! \]
The QCD phase diagram in the $B-T$ plane

Low energy effective models of QCD predict(ed):

- increasing pseudocritical temperature $T_c(B)$
- increasing strength $1/W(B)$ Mizher et al 10

Supported by (P)NJL models, large-$N_c$ arguments, low-dimensional models, Schwinger-Dyson equations

Gatto et al 11, Johnson et al 09, Alexandre et al 01, Klimenko et al 92, Kanemura et al 98

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AJ Mizher et al PRD 82 (10) 105016 R Gatto, M Ruggieri PRD 83 (11)
Chiral condensate (continuum limit)

Inflection point moves to lower $T$ with larger $B$!

$\Rightarrow T_c(B)$ decreases with $B$.

Hard to predict in models that are based on $T < T_c$ degrees of freedom.
Quark mass dependence

\[ \Delta \Sigma_u(B, T) \sim \bar{u}u(B, T) - \bar{u}u(0, T) : \text{condensates are significantly different!} \]

Shift in the \( \chi_u \) peak positions!

⇒ dramatic dependence on the light quark mass!

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Width of the transition (continuum limit)

At $B = 0$: broad crossover. What happens at $B > 0$?
Height of the peak increases.
However when normalized to the same height...

... not much changes. No indication of a critical end point!
The effect is negligible for RHIC.
The temperature reduces by less than $5 - 10\%$ for LHC.
The effect might have been significant in the early universe.
Inverse magnetic catalysis (continuum limit)
... and gluonic (inverse) catalysis
Thermodynamics in an external magnetic field

Free energy density
\[ f = -\frac{T}{V} \ln Z \]

Interaction measure
\[ I = -\frac{T}{V} \frac{d \ln Z}{d \ln a} = \epsilon - 3p = \langle T_{\mu\mu} \rangle \]

With magnetic field
\[ f = \epsilon - Ts = \epsilon^{\text{total}} - Ts - \underbrace{\mathbf{M} \cdot eB}_{\text{field}} \]

Is the pressure isotropic?

For \( qB \) fixed
\[ \langle T_{xx} \rangle = \langle T_{yy} \rangle = \langle T_{zz} \rangle = -p \]
\[ p_x = p_y = p - \mathbf{M} \cdot eB, \quad p_z = p \]

We pursue two approaches:
- Macroscopic determination of \( \mathbf{M} \), using generalized integral method.
- Microscopic determination from \( T_{\mu\mu} \) (non-trivial renormalization).
Microscopic (anisotropy) method

\[ \mathcal{M} \cdot \mathbf{eB} = p_x - p_z = -\frac{T}{V} \left[ \left. \frac{\partial \beta \xi g_0}{\partial \xi} \right|_{\xi=1} \langle -P_x + P_z \rangle + \left. \frac{\partial (\beta / \xi g_0)}{\partial \xi} \right|_{\xi=1} \langle -\hat{P}_x + \hat{P}_z \rangle \right. \\
\left. + \left. \left. \frac{\partial \xi f_0}{\partial \xi} \right|_{\xi=1} \sum_f \langle \bar{\psi}_f D_x \gamma_x \psi_f - \bar{\psi}_f D_z \gamma_z \psi_f \rangle \right] \right. \\
\]
Volume: $V = L_x L_y L_z = a^3 N_s^3$, temperature: $T = 1/(aN_t)$, magnetization:

$$
\mathcal{M} = -\frac{1}{V} \frac{\partial \mathcal{F}}{\partial (eB)}, \quad \chi_B = -\frac{1}{V} \frac{\partial^2 \mathcal{F}}{\partial (eB)^2} \bigg|_{B=0}.
$$

At constant $\Phi = L_x L_y eB$, $eB$ will change as $L_x$ or $L_y$ are compressed:

$$
p_z = -\frac{1}{V} L_z \frac{\partial \mathcal{F}}{\partial L_z} = -\frac{1}{V} \left( L_x \frac{\partial \mathcal{F}}{\partial L_x} + eB \frac{\partial \mathcal{F}}{\partial (eB)} \right) = p_x + eB \cdot \mathcal{M}
$$

$$
p_z \xrightarrow{V \to \infty} -f = -\frac{\mathcal{F}}{V} = \frac{T}{V} \log Z = \frac{1}{N_s^3 N_t a^4} \log Z.
$$

$\Delta f = -\Delta p$ known $\Rightarrow$ obtain $\mathcal{M}$ etc., as derivatives with respect to $eB$.

$$
\frac{\partial [a^4 \Delta f (am_f, \Phi, T, \beta)]}{\partial (am_f)} = -a^3 \Delta \bar{\psi}_f \psi_f
$$

Obviously, $\Delta p(\text{am}_f = \infty, \Phi, T, \beta) = -\Delta f (\text{am}_f = \infty, \Phi, T, \beta) = 0$. 

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Generalized integral method

\[
\Delta p_z(\Phi, T, \beta) = -N_t^4 \sum_f \int_{am_f}^{\infty} d(am_f) a^3 \Delta \bar{\psi}_f \psi_f / T^4
\]

\[
\mathcal{P} \left[ \frac{m_f \Delta \bar{\psi}_f \psi}{(eB)^2} \right] \rightarrow \left\{ \begin{array}{c} b_{1f} \quad (m_f \rightarrow \infty) \\ b_{1f} / (16 N_c) \quad (\chiPT) \end{array} \right.
\]

\[
\Delta p_{z,r} = (1 - \mathcal{P})[\Delta p_z]
\]
Susceptibility and permeability

half-half method: L Levkova, C DeTar, PRL 112 (14) 012002
finite diff. method: C Bonati, F Negro et al, PRD89 (14) 054506

(Linear) magnetic susceptibility and permeability ($f = \frac{1}{2} B_r^2 (1 - e_r^2 \chi_B)$):

$$\chi_B = \left. \frac{\partial M}{\partial (eB)} \right|_{B=0}, \quad B = H + e M, \quad \mu = \frac{B}{H} = \frac{1}{1 - 4\pi \alpha_{em} \chi_B}.$$
Spin contribution to the magnetic susceptibility

Decomposition:

\[ \chi_B = \sum_f \chi_{B,f}, \quad \chi_{B,f} = \chi_{B,f}^S + \chi_{B,f}^L, \]

\[ \chi_{B,f}^S = \frac{q_f/e}{2m_f} \left. \frac{\partial}{\partial(eB)} \langle \bar{\psi}_f \sigma_{xy} \psi_f \rangle \right|_{eB=0} = \left( \frac{q_f/e}{2m_f} \right)^2 \tau_f, \]

\[ \sigma_{\mu\nu} = \frac{1}{2i} [\gamma_\mu, \gamma_\nu]. \]

\[ \langle \bar{\psi}_f \sigma_{xy} \psi_f \rangle = q_f B \cdot \langle \bar{\psi}_f \psi_f \rangle \cdot \xi_f \equiv q_f B \cdot \tau_f \]

\[ \tau_f \] undergoes additive and multiplicative renormalization:

\[ \tau^r_f \equiv \left( 1 - m_f \frac{\partial}{\partial m_f} \right) \tau_f \cdot Z_T \equiv \tau_f Z_T - \tau^\text{div}_f, \quad \tau^\text{div}_f = m_f \left( \frac{1}{2\pi^2} + \cdots \right) \]
The tensor coefficient

\( \overline{\text{MS}} \)-scheme at 2 GeV for physical quark masses:

\[
\begin{align*}
\tau^r_u &= -40.7(1.3) \text{ MeV}, \quad \tau^r_d = -39.4(1.4) \text{ MeV}, \quad \tau^r_s = -53.0(7.2) \text{ MeV}.
\end{align*}
\]

Fundamental property of QCD vacuum.
Contributes, e.g., to the hadronic light-by-light correction to muon \( g - 2 \) !
Pressure anisotropy

Magnetization $\sim eB \Rightarrow \text{force density (pressure gradient)} \sim (eB)^2$. 
$\mathbf{B}(x,y)$

paramagnetic squeezing

$\rightarrow$ effect on elliptic flow $v_2$?
Summary

- Phase diagram: crossover with decreasing $T_c(B)$
  - complex, non-monotonic dependence in $\bar{\psi}\psi(B, T)$ (inverse catalysis)
  - similar for the gluonic contribution to the interaction measure
  - no critical endpoint at $\sqrt{eB} \lesssim 1$ GeV

- Once the QED running coupling (magnetic catalysis) is accounted for, the response at $T = 0$ and $T > 100$ MeV is paramagnetic. Pauli-diamagnetism and Landau-paramagnetism !!!

- At low temperatures and low fields (magnetars?) QCD is slightly diamagnetic.

- As a by-product the tensor coefficient was precisely determined (spin-diamagnetism).

- Can pressure anisotropy affect the elliptic flow for heavy ion collisions?

- Equation of state computed.