Induced QCD with $N_c$ auxiliary bosonic fields

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21.06.2014
Contents

1. Motivation

2. The new weight factor
   Zirnbauers bosonic model

3. Numerical tests

4. Dual representation
   Integrating out the gauge degrees of freedom

5. Summary and Perspectives
1. Motivation
Limitations of LQCD – Why changing the gauge action?

Main problem for studies of the QCD phase diagram:

▶ Simulating QCD at (real) non-zero chemical potential. (sign problem)

Possible solutions:

▶ Use complex Langevin for simulations.
  

▶ Simulate on a Lefschetz thimble?
  
  [ Christoforetti et al, PRD 86 (2012); PRD 88 (2013) ]

▶ Dual variables and worm algorithms
  
  [ e.g. Delgado Mercado et al, PRL 111 (2013), Gattringer, Lattice 2013 ]

▶ Fermion bags
  
  [ e.g. Chandrasekharan, EPJA 49 (2013) ]

Typically it is the gauge action which makes it difficult to find solutions.
(see e.g. strong coupling solution to sign problem  

[ Karsch, Mütter, NPB 313 (1989) ]

Idea: Find an alternative discretisation of pure gauge theory which allows the use of strong coupling methods!

⇒ A gauge action which is linear in the gauge fields might do this job!
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**Motivation**

Induced QCD

This idea is not new!

**Ansatz:** Induce pure gauge dynamics using auxiliary fields.

- **Using fermionic fields:**
  - with standard (Wilson) fermions.  
    [Hamber, PLB 126 (1983)]
  - Standard fermions + 4-fermion current-current interaction.  
    [Hasenfratz, Hasenfratz, PLB 297 (1992)]

Need the limit $N_f \to \infty$, $\kappa \to 0$.

- **Using scalar fields:**
  - Spin model.  
    [Bander, PLB 126 (1983)]
    
Need the limit $N_s \to \infty$ and $g_s \to \infty$.

  - Adjoint scalar fields.  
    [Kazakov, Migdal, NPB 397 (1992)]
    
No “exact” pure gauge limit.

It is interesting since it allows a solution in terms of large $N_C$.

$\Rightarrow$ This is where our induced model offers improvement!
Lattice regularised path integrals – fixing notations

Expectation value of operator $O$:

$$\langle O \rangle = \frac{1}{Z} \int [dU][d\psi][d\bar{\psi}] \quad O \quad \omega_G[U] \quad \omega_F[\psi, \bar{\psi}, U]$$

- $\omega_F[\psi, \bar{\psi}, U]$: Quark weight factor.

Typically:

$$\omega_G[U] \quad \omega_F[\psi, \bar{\psi}, U] = \exp \left[ -S[\psi, \bar{\psi}, U] \right].$$

Basic demands:

- The discretised action has to reproduce the continuum Yang-Mills action.
- All weight factors should be gauge invariant.
2. The new weight factor
Zirnbauer’s weight factor

Consider the weight factor: \[ \omega_{\text{BZ}}[U] \sim \prod_p \left| \det \left( m_{\text{BZ}}^4 - U_p \right) \right|^{-2N_b} \]

Here:

- \( p \) is an index running over unoriented plaquettes \( U_p \).
- \( m_{\text{BZ}} \) is a real parameter with \( m_{\text{BZ}} \geq 1 \)
  (or more generally \( m_{\text{BZ}} \in \mathbb{C} \) with \( \text{Re}(m_{\text{BZ}}) \geq 1 \))
- \( N_b \) is an integer number
- we consider a hypercubic lattice

Does this weight factor have anything to do with continuum Yang-Mills theory?

Why is this weight factor interesting?
The naive pure gauge limit

There is one obvious way to establish a connection:

- Write the weight factor as:

\[ \omega_{BZ}[U] \sim \exp \left\{ -2 N_b \text{Re} \left[ \sum_p \text{Tr} \ln (1 - \alpha_{BZ} U_p) \right] \right\} \]

with \( \alpha_{BZ} = m_{BZ}^{-4} \)

- Expand the exponent in small \( \alpha_{BZ} \):

\[ \Rightarrow S_{BZ}[U] = -2 N_b \sum_p \left[ \alpha_{BZ} \text{ReTr} (U_p) + O(\alpha_{BZ}^2) \right] \]

- Comparison with the Wilson action \( S_W \sim \frac{\beta}{N_c} \sum_p \text{ReTr} (U_p) \):

Equivalent if \( \beta = 2N_b N_c \alpha_{BZ} \)!

\[ \Rightarrow \text{Pure gauge limit: } \alpha_{BZ} \rightarrow 0 \quad N_b \rightarrow \infty \quad (\text{so that } \beta \text{ fixed}) \]
Phases in the \((N_b, \alpha_{BZ})\) parameter space

Numerical tests confirm this asymptotic behaviour!

Problem: \(N_b \to \infty\) is unfeasible for applications with the bosonic theory! (introduced later)

⇒ What is the gain compared to the other theories?
Non-trivial pure gauge limit

Zirnbauers conjecture: [ Budczies, Zirnbauer, math-ph/0305058 ]

At fixed $N_b \geq N_c$ and $d \geq 2$ the theory has a continuum limit for $\alpha_{BZ} \to 1$ which reproduces continuum Yang-Mills theory.

(excluding the case $d = 2$ and $N_b = N_c$)

- This can be shown rigorously for $d = 2$ and $N_b > N_c$.

The proof for $U(N_c)$ is given in [ math-ph/0305058 ].

It is straightforwardly extended to $SU(N_c)$.

(we will not go through the details here)

(probably $N_b > N_c - 1$ is sufficient for $SU(N_c)$)

- For $d > 2$ the equivalence with Yang-Mills theory is only a conjecture and relies on the increase of the collective behaviour when going to $d > 2$. 
Phases in the \((N_b, \alpha_{BZ})\) parameter space

\[ \begin{align*}
\text{divergent} \\
\text{no continuum limit?}
\end{align*} \]

\[ \begin{align*}
\alpha_{BZ} \\
0 \\
1
\end{align*} \]

\[ \begin{align*}
N_c \\
0 \\
\infty
\end{align*} \]

\[ \begin{align*}
N_b
\end{align*} \]

\[ \Rightarrow \quad \text{We will now test this limit numerically!} \]
3. Numerical tests
Basic idea and setup

Consider the cheap case: SU(2) at $d = 3$!

Suitable observables for a first test:

- $T = 0$ observables:
  Quantities connected with the $q\bar{q}$ potential.

- $T \neq 0$ observables:
  Transition temperature and order of the transition.

Simulation setup:

- Wilson theory: Standard mixture of heatbath and overrelaxation updates.
- Induced theory: Local metropolis with random link proposal.
- Computation of $q\bar{q}$ potential: Lüscher-Weisz algorithm
  \[ \text{[ Lüscher, Weisz, JHEP 0109 (2010) ]} \]
- Scale setting: Sommer parameter $r_0$
  \[ \text{[ Sommer, NPB 411 (1994) ]} \]
Scale setting and matching

First step: Matching between $\alpha (\sim m^{-4})$ and $\beta$.

- Start with some information from $\langle U_p \rangle$.
- Compute $r_0$ in the interesting region:

$\Rightarrow$ Matching ($N_b = 2$): $\beta(\alpha) = \frac{2.47(1)}{1 - \alpha} - 2.70(3)$

Second step:
Simulate at similar lattice spacings and look at the static potential.

- Compare to high precision results obtained with the Wilson action.

[ BB, PoS EPS-HEP (2013) ]

- Here we use the prediction for the potential of an effective string theory for the flux tube as a method to look at its subleading properties.
  $\Rightarrow$ There are two non-universal parameters, $\sigma$ and $\bar{b}_2$ (boundary coeff.).

- An agreement of $\bar{b}_2$ means that the potential is identical up to 4-5 significant digits!
Results for $\bar{b}_2$

First result: $\sqrt{\sigma} r_0$ is equivalent in both theories!

Results for $\bar{b}_2$:

⇒ All results are in excellent agreement!
Finite $T$ properties

For $T = 0$ quantities comparison looks good!

So what about the finite temperature transition?

- For SU(2) and $d = 3$:
  Second order phase transition in the 2d Ising universality class.

  [Engels et al, NPPS 53 (1997)]

- We will test this at $N_t = 4$ first!

  ⇒ $N_t = 6$ is in progress.

- Scale setting via $r_0$ and the mapping obtained at $T = 0$. 
Phase transition at $N_t = 4$

Polyakov loop expectation value:

$\langle L \rangle$

$T \, T_0$

SU(2), $d = 3$, $N_t = 4$, $V = 32^2$
Phase transition at $N_t = 4$

Polyakov loop expectation value:

$\langle L \rangle$

$T \, r_0$

SU(2), $d = 3$, $N_t = 4$, $V = 48^2$

WPG
IPG $N_b = 2$
Phase transition at $N_t = 4$

Polyakov loop expectation value:

$$\langle L \rangle$$

$T / r_0$

$WPG$  $IPG$  $N_b = 2$

$SU(2), d = 3, N_t = 4, V = 64^2$
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Numerical tests

Phase transition at $N_t = 4$

Polyakov loop expectation value:

$\langle L \rangle$

$T r_0$

SU(2), $d = 3$, $N_t = 4$, $V = 96^2$
Phase transition at $N_t = 4$

Polyakov loop susceptibility:

![Graph showing Polyakov loop susceptibility with phase transition at $N_t = 4$. The graph includes data points for WPG and IPG with $N_b = 2$, $SU(2)$, $d = 3$, $N_t = 4$, and $V = 32^2$.](image-url)
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Numerical tests

Phase transition at $N_t = 4$

Polyakov loop susceptibility:

![Graph showing Polyakov loop susceptibility](image-url)

- WPG
- IPG $N_b = 2$
- SU(2), $d = 3$, $N_t = 4$
- $V = 48^2$
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Numerical tests

Phase transition at $N_t = 4$

Polyakov loop susceptibility:

$\chi_L$

$WPG$

$IPG \; N_b = 2$

$SU(2), \; d = 3, \; N_t = 4$

$V = 64^2$
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Numerical tests

Phase transition at $N_t = 4$

Polyakov loop susceptibility:

\[ \chi_L \]

WPG
IPG $N_b = 2$

SU(2), $d = 3$, $N_t = 4$

$V = 96^2$
Phase transition at $N_t = 4$

Fit: $\ln(\chi_L) = C + \gamma/\nu \ln(N_s)$

Result for critical exponents: $\gamma/\nu = 1.74(2)(9)$

Black point: $\gamma/\nu = 1.70(4)$ (WPG)  [Engels et al, NPPS 53 (1997)]
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4. Dual representation
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Dual representation

The bosonic version

Now: Why is this weight factor interesting?

Bosonisation of the determinant:

\[ \omega_{\text{BZ}}[U] = \prod_p \left| \det \left( m_{\text{BZ}}^4 - U_p \right) \right|^{-2N_b} = \int [d\phi] \exp \left\{ -S_{\text{BZ}}[\phi, \bar{\phi}, U] \right\} \]

\[ S_{\text{BZ}}[\phi, \bar{\phi}, U] = \sum_{b=1}^{N_b} \sum_{p} \sum_{j=1}^{4} \left[ m_{\text{BZ}} \bar{\phi}_{b,p}(x_j) \phi_{b,p}(x_j) - \bar{\phi}_{b,p}(x_{j+1}) U(x_{j+1}, x_j) \phi_{b,p}(x_j) \right] \]

- $\phi$ are complex scalar fields
- $p$: index for oriented plaquette
- Scalar fields carry plaquette index $p$.  
  $\Rightarrow$ Propagate only locally opposite to the plaquette orientation.
- Gauge field only couples to bosons.  
  $\Rightarrow$ Can be modified more easily!
- $N_b$ defines the number of boson fields.
Modified version

Problem: This action is complex!
Solution: Rewrite determinant weight factor:

\[ \omega_{\text{BZ}}[U] \sim \prod_p \left[ \det \left( m_{\text{BZ}}^4 - U_p \right) \det \left( m_{\text{BZ}}^4 - U_p^\dagger \right) \right]^{-N_b} \]

\[ \sim \prod_p \left[ \det \left( \tilde{m} - \left\{ U_p + U_p^\dagger \right\} \right) \right]^{-N_b} \]

Now bosonize this determinant:

\[ S_B[\phi, \bar{\phi}, U] = \sum_{b=1}^{N_b} \sum_p \sum_{j=1}^4 \left[ m \bar{\phi}_{b,p}(x_j) \phi_{b,p}(x_j) - \bar{\phi}_{b,p}(x_{j+1}) U(x_{j+1}, x_j) \phi_{b,p}(x_j) - \bar{\phi}_{b,p}(x_j) U(x_j, x_{j+1}) \phi_{b,p}(x_{j+1}) \right] \]

Here: \( \tilde{m} = m_{\text{BZ}}^4 + m_{\text{BZ}}^{-4} \) and \( \tilde{m} = m^4 - 4m^2 + 2 \).
Integration over gauge fields

First step: Integration over the gauge degrees of freedom.

Rewrite the partition function as a product of Itzykson-Zuber integrals:

\[ Z = \int d[\phi] \mathcal{F}[\phi, \bar{\phi}] \prod_{x, \mu} \int dU_\mu(x) e^{\frac{1}{2} \text{Tr} \left[ U_\mu(x) \mathcal{V}_\mu(x)[\phi, \bar{\phi}] + U_\mu^\dagger(x) \mathcal{V}_\mu^\dagger(x)[\phi, \bar{\phi}] \right]} \]

With \[ \mathcal{F}[\phi, \bar{\phi}] = \exp \left\{ - \sum_{b=1}^{N_b} \sum_p \sum_{j=1}^{4} m_{\bar{\phi}_b,p(x_j)} \phi_{b,p}(x_j) \right\} \]

and \[ \mathcal{V}_\mu(x)[\phi, \bar{\phi}] = 2 \sum_{b=1}^{N_b} \sum_{\nu \neq \mu} \left[ \phi_{b,\bar{\nu}(x,\mu,\nu)}(x_j(\mu,\nu,0,1)) \bar{\phi}_{b,\bar{\nu}(x,\mu,\nu)}(x_j(\mu,\nu,0,0)) \right. \]

\[ + \left. \phi_{b,\bar{\nu}(x-\hat{\nu},\mu,\nu)}(x_j(\mu,\nu,1,1)) \bar{\phi}_{b,\bar{\nu}(x-\hat{\nu},\mu,\nu)}(x_j(\mu,\nu,1,0)) \right] \]
Integration over gauge fields – IZ integrals

Need to solve integrals \[ I = \int dU \ e^{\text{Tr}[U \ \nu + U^\dagger \ \nu^\dagger]} \].

For U(N_c) they are known. [e.g. Brower, Rossi, Tan, PRD23 (1981)]

For SU(N_c): \[ I \sim \frac{1}{\Delta(\lambda^2)} \sum_{\xi=0}^{\infty} \varepsilon_\xi \cos(\xi \ \varphi) \det(A_\xi(\lambda)) \]

\begin{itemize}
  \item $\varepsilon_\xi$: Neumann’s factor; $\varepsilon_\xi = \begin{cases} 1 & \text{for } \xi = 0 \\ 2 & \text{for } \xi > 0 \end{cases}$
  \item $\varphi$: Phase of the determinant $\det(\nu)$
  \item $\lambda_i^2$: eigenvalues of the $N_c \times N_c$ matrix $\frac{1}{4} \nu \nu^\dagger$
  \item $\Delta(\lambda^2)$: Vandermonde determinant
  \item $A_\xi(\lambda)$: $N_c \times N_c$ matrix; $(A_\xi(\lambda))_{ij} = \lambda_i^{j-1} I_{\xi+j-1}(\lambda_i)$ with $I_m(z)$ modified Bessel function of the first kind (and $z \in \mathbb{R}$).
\end{itemize}

⇒ Looks difficult, but the sum in $I$ converges numerically very fast.
Full QCD

Now consider also fermionic fields, e.g. with a staggered type action:

\[ S_F = \sum_x \left\{ \sum_\mu \left[ \bar{\psi}(x) \alpha_\mu(x) U_\mu(x) \psi(x + \hat{\mu}) + \bar{\psi}(x + \hat{\mu}) \tilde{\alpha}_\mu(x) U^\dagger_\mu(x) \psi(x) \right] + m_q \bar{\psi}(x) \psi(x) \right\} \]

Most promising idea: Expand weight factor \( \exp(-S_F) \) in grassmann variables.
- Introduce dual variables \( b_{\mu,ab}(x), b^\dagger_{\mu,ab}(x) \) and \( n_a(x) \).
- Integral over grassmann fields leads to constraints for those variables.
- Integrate out the gauge fields.

Resulting dual partition function:

\[ Z_{\text{dual}} = \sum_{(b, b^\dagger, n)} \mathbb{I}_{(b, b^\dagger, n)} m_q N \int [d\bar{\phi}][d\phi] F(\phi, \bar{\phi}) \prod_{x, \mu} w(b(x, \mu), b^\dagger(x, \mu), \partial V) I_\mu(x, \phi, \bar{\phi}) \]

Problem: The dual theory has a sign problem!
Summary and Perspectives

▶ We have investigated a possible alternative discretisation of continuum pure gauge theory.
▶ While for \( d = 2 \) it can be shown that the theory has the correct continuum limit this is not guaranteed if \( d > 2 \).
▶ Numerical tests show good agreement with simulations using Wilson’s gauge action, both for \( T = 0 \) and \( T \neq 0 \).
▶ In its original formulation with auxiliary boson fields the theory has a sign problem. ⇒ We introduced a modified version without sign problem.
▶ Pass to a dual theory via a direct integration over gauge fields:
  ▶ Leads to a theory formulated in terms of auxiliary bosonic fields.
  ▶ When fermions are include one can expand the action in grassmann variables and integrate over the fermionic degrees of freedom and the gauge fields.
  ▶ However, the resulting dual representation has a sign problem.
  ▶ Is it possible to find a formulation without sign problem?
▶ Explore other analytical methods ...
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Thank you for your attention!