Axial symmetry in the chiral symmetric phase

Swagato Mukherjee

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Axial symmetry in QCD

massless QCD Lagrangian is invariant under

\[ U_A(1): \psi(x) \rightarrow e^{-i\alpha(x)\gamma_5} \psi(x) \Rightarrow \partial_\mu J_5^\mu = 0, \quad J_5^\mu = \bar{\psi} \gamma_\mu \gamma_5 \psi \]

at the classical level

quantum fluctuations: Adler-Bell-Jackiw anomaly

\[ \partial_\mu J_5^\mu = \frac{1}{16\pi^2} \text{Tr} F_\mu \tilde{F}^{\mu \nu} \]

can be written as a total derivative

't Hooft:

global breaking of $U_A(1)$ ← non-trivial topology of the gauge field

non-perturbative phenomena

generates localized fermionic zero modes → index theorem
Axial symmetry in QCD

chiral broken phase:

\[ \langle \bar{\psi} \psi \rangle \neq 0 \]

axial symmetry is broken by the vacuum

not invariant under \( U_A(1) \)

chiral symmetric phase:

mechanism axial symmetry breaking can be studied directly

configurations with non-trivial topologies are suppressed due to color screening with increasing \( T \)

with increasing \( T \), \( U_A(1) \) breaking becomes smaller

can be compared with dilute instanton gas type approximations
This talk

LQCD studies of axial symmetry in the chiral symmetric phase:

- signatures of $U_A(1)$ in two-point correlation functions
- mechanism of $U_A(1)$ breaking through Dirac eigenvalue spectra
  → role of zero & near-zero fermionic modes, chirality, localization

mix-&-match of several LQCD studies:

- staggered (HISQ) fermions:

- domain wall fermions (DWF):
  P. Hegde, M. Cheng et. al. [HotQCD]: arXiv:1205.3535 [hep-lat];
  Z. Lin, H-T Ding et. al. [RBC-LLNL]: arXiv:1309.4149 [hep-lat];
  C. Schroeder, Z. Lin et. al. [HotQCD]: arXiv:1402.5175 [hep-lat]

- overlap fermions:
  S. Sharma, V. Dick et. al.: arXiv:1311.3943 [hep-lat], unpublished
  V. Dick: poster sessions
Lattice trivia

staggered (HISQ):
- emergence of correct axial anomaly is subtle, only in $a \to 0$ limit

DWF:
- 5-d formulation
- exact chiral symmetry & correct anomaly for $a \neq 0$ but in the limit of infinite 5th dimension

overlap:
- 4-d formulation with exact chiral symmetry & correct anomaly for $a \neq 0$
- computationally very expensive → show results for Dirac eigenvalue spectra measured on gauge configurations generated with HISQ

$$D = \frac{1 + \epsilon(D_w)}{2}$$

$$\epsilon(H) = \frac{H}{\sqrt{H^\dagger H}}$$
Degeneracies in two point correlation functions

\[ \chi_{5,\text{con}} \equiv \langle \overline{q} \gamma_5 q \rangle \quad \chi_{5,\text{disc}} \equiv \langle \overline{q} \gamma_5 q \rangle \]

\[ \begin{align*}
\chi_5,\text{con} & \sim \pi : \overline{q} \gamma_5 \frac{\tau}{2} q \\
\chi_5,\text{disc} & \sim \sigma : \overline{q} q \\
\chi_\text{con} + \chi_\text{disc} & \propto 2 \\
\chi_\text{con} - \chi_\text{disc} & \propto 2
\end{align*} \]

\[ \text{susceptibility: } \chi_i = \frac{T}{V} \int d^4x \langle O_i(x) O_i^\dagger(0) \rangle \]

con: (quark-line) connected

disc: (quark-line) disconnected
Measure of axial symmetry breaking from 2-pt functions

\[ \chi_{5,\text{con}} \quad \pi : \bar{q} \gamma_{5/2} q \quad \sigma : \bar{q} q \quad \chi_{\text{con}} + \chi_{\text{disc}} \times 2 \]

\[ \chi_{\text{con}} \quad \delta : \bar{q} \gamma_2 q \quad \eta : \bar{q} \gamma_5 q \quad \chi_{5,\text{con}} - \chi_{5,\text{disc}} \times 2 \]

\[ \begin{align*} 
\chi_{\pi} &= \chi_{\delta}, \quad \chi_{\sigma} = \chi_{\eta} \\
\chi_{\pi} &= \chi_{\sigma} \quad \Rightarrow \quad \chi_{\pi} - \chi_{\delta} = 2 \chi_{\text{disc}} \\
\chi_{\delta} &= \chi_{\eta} \quad \Rightarrow \quad \chi_{\pi} - \chi_{\delta} = 2 \chi_{5,\text{disc}} \\
\end{align*} \]

topological susceptibility: \[ Q_{\text{top}} = m \langle \bar{\psi} \gamma_5 \psi \rangle \quad \Rightarrow \quad \chi_{\text{top}} = m^2 \chi_{5,\text{disc}} \]

measure of axial symmetry breaking: (in the chiral symmetric phase)

\[ \chi_{\pi} - \chi_{\delta} = 2 \chi_{\text{disc}} = \frac{2 \chi_{\text{top}}}{m^2} \]
Measure of axial symmetry breaking from 2-pt functions

\[ \chi_{\pi} - \chi_{\delta} = 2 \chi_{\text{disc}} = \frac{2 \chi_{\text{top}}}{m^2} \]

chiral crossover:

\[ T_c = 154 (9) \text{ MeV} \]

non-zero at least till \[ T \sim 1.2T_c \]
Measure of axial symmetry breaking from 2-pt functions

\[ \chi_{\pi} - \chi_{\delta} = 2 \chi_{\text{disc}} = \frac{2 \chi_{\text{top}}}{m^2} \]

non-zero at least till \( T \sim 1.2T_c \)

very little volume & quark mass dependence
Axial symmetry breaking and the Dirac spectra

**χ SM**: \[ \langle \bar{\psi} \psi \rangle = \int_0^\infty d\lambda \frac{2m\rho(\lambda)}{m^2 + \lambda^2} \]

**U_A(1)**: \[ \chi_\pi - \chi_\delta = \int_0^\infty d\lambda \frac{4m^2\rho(\lambda)}{(m^2 + \lambda^2)^2} \]

Banks-Casher:

\[ \lim_{m \to 0} \langle \bar{\psi} \psi \rangle = \pi \text{sgn}(m) \rho(0) \]

QCD partition function is non-analytic at \( m=0 \)

Chiral symmetry restoration: \[ \langle \bar{\psi} \psi \rangle = 0 \Rightarrow \rho(0) = 0 \]

Axial symmetry breaking: \[ \chi_\pi - \chi_\delta \neq 0 \]

What is the form of \( \rho(\lambda) \), such that \( \rho(\lambda) = 0 \) but \( \chi_\pi - \chi_\delta \neq 0 \)?

Nature of the configurations that give rise to such \( \rho(\lambda) \)?
Axial symmetry breaking and the Dirac spectra

$\chi_{\text{SM}}$: $\langle \bar{\psi} \psi \rangle = \int_0^\infty d\lambda \frac{2m \rho(\lambda)}{m^2 + \lambda^2}$

$U_A(1)$: $\chi_\pi - \chi_\delta = \int_0^\infty d\lambda \frac{4m^2 \rho(\lambda)}{(m^2 + \lambda^2)^2}$

A gap in $\rho(\lambda)$ around $\lambda = 0$:

$\langle \bar{\psi} \psi \rangle = 0$, $\chi_\pi - \chi_\delta = 0$
A gap in Dirac spectra?

For the interacting theory there is no visible gap in the Dirac spectra till $T \approx 2T_c$. 

$T \approx 1.3T_c$

$T \approx 1.5T_c$

$T \approx 2.1T_c$
Possible forms of Dirac spectra

requirements, $m \to 0$:

$$\langle \bar{\psi} \psi \rangle = \int_0^\infty d\lambda \frac{2m\rho(\lambda)}{m^2 + \lambda^2} = 0$$

$$\chi_\pi = \langle \bar{\psi} \psi \rangle / m \sim \text{finite}$$

$$\chi_\delta = \int_0^\infty d\lambda \ 2\rho(\lambda) \frac{d}{dm} \left[ \frac{2m}{m^2 + \lambda^2} \right] \sim \text{finite}$$

$$\chi_{\text{disc}} = \int_0^\infty d\lambda \ \frac{2m(d\rho(\lambda)/dm)}{m^2 + \lambda^2} = \chi_\pi - \chi_\delta = \int_0^\infty d\lambda \ \frac{4m^2\rho(\lambda)}{(m^2 + \lambda^2)^2} \neq 0$$

quark mass dependence of $\rho(\lambda \to 0)$ is the key

$\rho(\lambda \to 0)$ independent of $m$ ruled out
Possible forms of Dirac spectra

chiral: \( SU_L(N_f) \times SU_R(N_f) \)

axial: \( U_A(1) \)

chiral symmetric phase:
also symmetric
under this \( Z(N_f) \)

QCD Lagrangian: \( m \to -m \)
(for even number of flavors)

\[ \psi_L \to e^{2i\pi/N_f} \psi_L \]

\[ \rho(m, \lambda \to 0) = \rho(-m, \lambda \to 0) \]

\[ \lim_{m \to 0} \langle \bar{\psi} \psi \rangle = \pi \text{sgn}(m) \rho(0) \]
Possible forms of Dirac spectra

\[
\begin{array}{cccccc}
\rho(m, \lambda \to 0) & \langle \bar{\psi} \psi \rangle & \chi_\pi & \chi_\delta & \chi_\pi - \chi_\delta & 2 \chi_{\text{disc}} \\
\hline
m^2 \delta(\lambda) & m & 1 & -1 & 2 & 2 \\
|m| & \pi m & \pi & 0 & \pi & \pi \\
m^2 & \pi m^2 & \pi m & 0 & \pi m & \pi m \\
\end{array}
\]

\[\rho(\lambda \to 0) \sim m^2 \delta(\lambda) \quad \text{a possible choice}\]
\[\rho(\lambda \to 0) \sim |m| \quad \text{a possible choice}\]

\[\rho(\lambda) \sim m^2: \text{no axial symmetry breaking in 2-pt functions for } m \to 0\]
not even in 4-pt functions;
rigorous proof: S. Aoki et. al, arXiv:1209.2061
LQCD Dirac spectra: DWF

zero or near-zero $\rho(\lambda \rightarrow 0) \sim m^2 \delta(\lambda)$ modes?
LQCD Dirac spectra: DWF

\[ \langle \bar{\psi} \psi \rangle = \int_0^\infty d\lambda \frac{2m \rho(\lambda)}{m^2 + \lambda^2} + \frac{\langle |Q_{top}| \rangle}{mV} \]

\[ \chi_\pi - \chi_\delta = \int_0^\infty d\lambda \frac{4m^2 \rho(\lambda)}{(m^2 + \lambda^2)^2} + \frac{2\langle |Q_{top}| \rangle}{m^2 V} \]

\[ \chi_{top} \sim V \Rightarrow Q_{top} \sim \sqrt{V} \]

contributions of exact zero modes vanishes in the thermodynamic limit

zero modes: \( \rho(0) \sim 1/\sqrt{V} \)

near-zero \( \rho(\lambda \rightarrow 0) \sim m^2 \delta(\lambda) \) modes ?

\[ V = 16^3 \rightarrow V = 32^3 \Rightarrow \rho(0) \sim 1/\sqrt{8} \]

no evidence !!
Axial symmetry breaking from Dirac spectra: DWF

\[
\chi_\pi - \chi_\delta = \int_0^\infty d\lambda \frac{4m^2 \rho(\lambda)}{(m^2 + \lambda^2)^2}
\]

\[
\rho(\lambda \to 0) = a_0 + a_1 \lambda + a_2 m^2 \delta(\lambda)
\]

\[
\chi_\pi - \chi_\delta = a_0 \pi/m + 2a_1 + 2a_2
\]

almost the entire contribution to the axial symmetry breaking measure \(\chi_\pi - \chi_\delta\) comes from near-zero modes \(m^2 \delta(\lambda)\) for \(T \geq 1.2T_c\)
A cleaner version: overlap

\[ \langle \gamma_5 \rangle \]

zero

near-zero

bulk

\[ |\lambda|^2 \]

\[ T \simeq 1.5T_c \]

T=1.5T_c  
\[ N_T=8 \]
A cleaner version: overlap

- **zero**
- **near-zero**
- **bulk**

\[ T \approx 1.5T_c \]

\[ |\lambda|^2 \]

\[ 1e^{-12} 1e^{-11} 1e^{-10} 1e^{-9} 1e^{-8} 1e^{-7} 1e^{-6} 1e^{-5} 0.0001 0.001 0.01 \]

\[ p(\lambda a)^3 \]

\[ \text{Min}(\lambda_{\text{max}}) \]

\[ \text{all} \]

\[ \text{no zero modes} \]

\[ T = 1.5T_c \]

\[ N_T = 8 \]

\[ \psi^* (x) \psi (x) \]

\[ y \]

\[ 0 \]

\[ 5 \]

\[ 10 \]

\[ 15 \]

\[ 20 \]

\[ 25 \]

\[ 30 \]

\[ 35 \]

\[ 40 \]

\[ 0 \]

\[ 0.02 \]

\[ 0.04 \]

\[ 0.06 \]

\[ 0.08 \]

\[ 0.1 \]

\[ 0.12 \]

\[ \psi^* (x) \psi (x) \]

\[ t \]

\[ 0 \]

\[ 5 \]

\[ 10 \]

\[ 15 \]

\[ 20 \]

\[ 25 \]

\[ 30 \]

\[ 35 \]

\[ 40 \]

\[ 0 \]

\[ 0.02 \]

\[ 0.04 \]

\[ 0.06 \]

\[ 0.08 \]

\[ 0.1 \]
A cleaner version: overlap

cleaner version

dilute instanton gas type picture?

ρ(λ → 0) ~ m^2 δ(λ)

chirality

wave function
Closer to the chiral crossover

\[ T \approx 1.04 T_c \]

overlap

no clear \( m^2 \delta(\lambda) \)
like structure

close enough to the chiral crossover O(4) universality driven behavior should dominate

\[ T \approx 1.02 T_c : \rho(\lambda \to 0) \sim |m| ? \]

inconclusive !!
Summary

LQCD answers to the fate of axial symmetry in the chiral symmetric phase of QCD

signature of axial symmetry clearly visible in 2-pt correlation function at least for $T_c \lesssim T \lesssim 1.2T_c$

dominant mechanism of axial symmetry appears to be similar to dilute instanton gas type picture for $T \approx 1.2T_c$

more interesting scenario for $T \sim T_c$? no conclusive answer yet