Thermal Schwinger model
with tensor network approach
- temperature dependence of
the chiral condensate in the Schwinger model -

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with M. C. Banuls, K. Cichy, I. Cirac and K. Jansen
Schwinger model for $N_f = 1$

- 1+1 dimensional QED model
  - not QCD, but similar to QCD:
    - confinement, chiral symmetry breaking (via anomaly for $N_f=1$)
    - exactly solvable in massless case $\Rightarrow$ a good test case
- Hamiltonian form.
  - To solve eigen equation with Hamiltonian $H$:
    $$ H |\psi\rangle = \lambda |\psi\rangle $$
    $\lambda$ : eigenvalue, $|\psi\rangle$ : eigenstate
- Hamiltonian of Schwinger model
  $$ H = x \sum_{n=0}^{N-2} [\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+] + \frac{\mu}{2} \sum_{n=0}^{N-1} [1 + (-1)^n \sigma_n^z] $$
  $$ + \sum_{n=0}^{N-2} \left[ l + \frac{1}{2} \sum_{k=0}^{n} ((-1)^k + \sigma_k^z) \right]^2 $$
  $H_{\text{hop}} + H_{\text{mass}} + H_{\text{g}}$
  - Gauss law
Tensor network (TN)

• Efficient approximation of quantum many-body state from quantum information

• **Matrix product state (MPS):** tensor network for 1d

\[ |\psi\rangle \approx \sum_{i_1, i_2, \ldots i_N} \text{Tr} \left[ M^{i_1} M^{i_2} \ldots M^{i_N} \right] |i_1 i_2 \ldots i_{N-1}\rangle \]

  - \( i_k \): physical indices at site \( k \), \( M^{i_k} \): tensor,
  - \( m, n (=1, \ldots, D) \): indices from this approximation, \( D : \text{bond dimension} \)

**Ex. 1/2-spin 2 particle system**

\[ |\psi\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right) = \sum_{i_1, i_2 = \uparrow, \downarrow} \text{Tr} \left[ M^{i_1} M^{i_2} \right] |i_1 i_2\rangle \]

- \( M^{i_1=\uparrow} = \begin{pmatrix} 0 & 1/\sqrt{2} \\ 0 & 0 \end{pmatrix} \), \( M^{i_1=\downarrow} = \begin{pmatrix} 0 & 0 \\ 1/\sqrt{2} & 0 \end{pmatrix} \), \( M^{i_2=\uparrow} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), \( M^{i_2=\downarrow} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \)

One choice
Note: This example is exact description, not approximation because of large value of bond dimension.
Advantages of TN and Variational method

**Bond dimension** \( D \)

* dominant in computational effort
* way to express sub-space of Hilbert space
* smaller than \( D \sim d^{N/2} \) can be enough

Ex.) In our studies, \( D \sim 100 \) is enough (\( \ll d^{N/2} \sim 10^{15} \))

* Hilbert space growing exponentially as increasing system size,
  \( \Rightarrow \) With TN, one can investigate sub-space growing **polynomially**

**Variational method**

* For computing ground state, some excited states
* Updating each element of one tensor with keeping the others fixed by searching for the minimum of \( \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} \) with linear eq. of the derivative, then sweeping through the chain.

Ex.) 1d spin system

By using MPS with \( D \),

\[
d^N \Leftrightarrow N d D^2
\]

\( d \) : d.o.f of physical index at each site, \( N \) : chain length

\( \Rightarrow \) If \( D \sim d^{N/2} \), no advantage of TN

F. Verstraete et al. PRL 93, 227204
Lattice gauge theory (LGT) with TN approach

- Earlier Study: critical behavior of Schwinger model with Density Matrix Renormalization Group
- Nowadays: various branches
  * Strong coupling exp.
  * TN rep. of LGT with continuous group
  * LGT with TN on higher dimension
  * Real time evolution
  * (as Hamiltonian) Quantum link model
  * (as different app.) Tensor Renormalization Group
  * Our previous studies

References:
- Y. Shimizu, Y. Kuramashi arXiv:1403.0642 (With Lagrangian)
- M. C. Banuls et al JHEP 1311, 158, LAT2013, 332 (2013)
Our previous study

- Schwinger model with MPS method
- With variational method, computing:
  - spectrum
  - (subtracted) chiral condensate: \( \bar{\psi}\psi = \frac{\sqrt{x}}{L} \sum_{n} (-1)^{n} \left[ \frac{1 + \sigma_{n}^{z}}{2} \right] \) in spin language
- Continuum limit: \( \frac{1}{\sqrt{x}} \to 0 \)
  with inverse coupling \( x = \frac{1}{g^{2}a^{2}} \)

\[
\begin{align*}
\text{Fit function:} \\
f(x) &= A + F \frac{\log(x)}{\sqrt{x}} + B \frac{1}{\sqrt{x}} + C \frac{1}{x} \\
&\text{Logarithmic correction from analytic form of free theory}
\end{align*}
\]

\[
\begin{align*}
\text{M. C. Banuls et al JHEP 1311, 158, LAT2013, 332} \\
\text{H. Saito (NIC, DESY Zeuthen, Humboldt-Universitaet zu Berlin)}
\end{align*}
\]
This study
Chiral symmetry restoration of Schwinger model for $N_f = 1$

- Chiral symmetry breaking at $T = 0$ (via anomaly) ⇔ At finite $T$, the symmetry restoration
- Order parameter: chiral condensate $\langle \bar{\psi} \psi \rangle$
- Analytic formula with fermionic zero mode & instanton for gauge

$$
\langle \bar{\psi} \psi \rangle = \frac{m_\gamma}{2\pi} e^\gamma e^{2I(\beta m_\gamma)} = \begin{cases}
\frac{m_\gamma}{2\pi} e^\gamma & \text{for } T \to 0 \\
\frac{2\pi}{2Te^{-\pi T/m_\gamma}} & \text{for } T \to \infty
\end{cases}
$$

where $I(x) = \int_0^\infty \frac{1}{1 - e^x \cosh(t)} dt$, $\gamma = 0.57721 \ldots$ (Euler constant), $m_\gamma = e/\sqrt{\pi}$

- Expectation value at finite $T$: $\langle \mathcal{O} \rangle_\beta = \frac{\text{tr } [\mathcal{O} \rho(\beta)]}{\text{tr } [\rho(\beta)]}$

- $\rho(\beta/2)$ to ensure positivity: $\rho(\beta) = \rho(\beta/2) \rho(\beta/2)^\dagger$
- $T$-dep. by evolution of $T$ with: $\rho(\beta/2) = e^{-\frac{\delta}{2} H} \cdots e^{-\frac{\delta}{2} H}$ for $N_{\text{step}} = \beta/\delta$

thermal density operator: $\rho(\beta) \equiv e^{-\beta H}$ where $\beta = 1/T$


Ex.) For fixed $\delta$, larger $N_{\text{step}}$ corresponds to lower $T$

For each step, performing variational method with MPS approx. to $e^{-\frac{\delta}{2} H}$ (vectorized one)
Simulation setup

- Open Boundary condition
- Four simulation parameters
  1. inverse coupling $x$
  2. chain length $N$
  3. bond dimension $D$
  4. step size $\delta$ of $T$ evol.

- Two setups
  (i) For test: small $N$, single value of $x$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>20</td>
</tr>
</tbody>
</table>

  $D = 20\text{-}160$, $\delta = 0.0001\text{-}0.01$

(ii) For Continuum Limit: several $N$, $x$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>80\text{-}140</td>
</tr>
<tr>
<td>36</td>
<td>80\text{-}200</td>
</tr>
<tr>
<td>49</td>
<td>80\text{-}240</td>
</tr>
<tr>
<td>64</td>
<td>80\text{-}240</td>
</tr>
<tr>
<td>100</td>
<td>140\text{-}240</td>
</tr>
<tr>
<td>121</td>
<td>100\text{-}240</td>
</tr>
</tbody>
</table>

  moderate values $D = 80$, $\delta = 0.00001\text{-}0.00005$
Chiral condensate at finite $T$ with small system $\mathcal{N}$

- Result of (i) test case: small $\mathcal{N}$ and single $x$
- Dependence of bond dimension/step size

![Diagram showing analytic form and variational method with different colors and line types for varying $D$ and $\delta$.]

- Different color
  $\leftrightarrow$ different $D$,
  different line types
  $\leftrightarrow$ different $\delta$

- Smooth curve and similar to the analytic curve

- Some differences between various colors and line types
Chiral condensate at finite $T$ with small system 2

- Focusing on high/low temperature region

**Low $T$**

(i) convergence of $D$ for $D \gtrsim 80$

(ii) approaching to convergence of $\delta$ for $\delta \lesssim 0.001$

(iii) less dependence on $\delta$ for larger $D$

**High $T$**

Little dependence on $D, \delta$
Continuum extrapolation

- Naive estimate
  - Data of $N = 20-180$ fixed $N/\sqrt{x} = 20$
    - $N/\sqrt{x} \geq 20$ needed to see linear behavior of infinite volume extrapolation, from our results at $T = 0$
  - When taking cont. limit, result getting closer to analytic curve

- Four extrapolations
  - zero step size $\delta$ limit $\rightarrow$ sufficiently small $\delta$
  - large bond dimension $D$ limit $\rightarrow$ sufficiently large $D$
  - infinite volume limit
  - continuum limit

Changing color from red $\rightarrow$ blue $\rightarrow$ orange $\rightarrow$ pink $\rightarrow$... decreasing cutoff

Less dependence on $D, \delta$ at high $T$, let me support

Preliminary result cont. limit at high $T$
Continuum Limit - High $T$

$g\beta = 0.2$ (dimensionless inverse of temperature)

(i) Infinite volume extrapolation

- Linear behavior
- cond. in infinite vol. limit for each $x$

(ii) Continuum limit extrapolation

- Three fit functions: linear, quadratic, logarithmic $ay + by \log(y) + c$ where $y = 1/\sqrt{x}$
- different color $\Leftrightarrow$ different fit func.
- Logarithmic corr. in our previous study, the most reliable fit is logarithmic again
Summary

- Computing chiral condensate at finite \( T \) in Hamiltonian formalism with tensor network methods
- Evaluating dependence of bond dimension/step size
- As a preliminary result, by taking continuum limit at high temperature, we obtain results with consistent with an analytic formula.  I. Sachs and A. Wipf, arXiv:1005.1822
- Future plans
  i) Continuum limit in low \( T \) region
  ii) Many flavor Schwinger model
  iii) Schwinger model at finite \( \mu \)
  iv) Non-Abelian gauge theory
  v) Real time evolution
  vi) Higher dimension of TN
Backup slides
Thermal state calculation in detail

- Expectation value at finite $T$: $\langle \mathcal{O} \rangle_\beta = \frac{\text{tr} [\mathcal{O} \rho(\beta)]}{\text{tr} [\rho(\beta)]}$
- How to calculate the $\rho(\beta)$
  - $\rho(\beta/2)$ to ensure positivity: $\rho(\beta) = \rho(\beta/2) \rho(\beta/2)^\dagger$
  - Evolution of temperature: $\rho(\beta/2) = e^{-\frac{\delta}{2} H} \cdots e^{-\frac{\delta}{2} H}$ for high $T \rightarrow$ low $T$
- Our thermal density operator
  $e^{-\frac{\delta}{2} H} \approx e^{-\frac{\delta}{4} H_g} e^{-\frac{\delta}{2} (H_{\text{hop}} + H_{\text{mass}})} e^{-\frac{\delta}{4} H_g} \approx e^{-\frac{\delta}{4} H_e} e^{-\frac{\delta}{2} H_o} e^{-\frac{\delta}{4} H_e}$
- Multiplication of five $e^{-\delta H/4}$'s for each step of $\rho(\beta/2)$
- In each one of the five, updating tensor by searching the minimum of $\text{Tr} \left[ \Pi_i e^{-\delta H_{a(i)}/4} \Pi_j \left( e^{-\delta H_{a(j)}/4} \right)^\dagger \right]$ and sweeping (variational method)

$H_{a(i)} = H_g$ for $i=5l$, $5l+4$, $H_o$ for $i=5l+1$, $5l+3$, $H_o$ for $i=5l+2$


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Continuum Limit - High T

(i) Infinite volume limit

(ii) Continuum limit

Three fit functions: linear, quadratic, logarithmic

\[ ay + by \log(y) + c \]

where \( y = \frac{1}{\sqrt{x}} \)