Wilson loops with fractional charges

P. Korcyl, M. Koren, J. Wosiek
Jagellonian University, Krakow
• Screening with dynamical quarks
• Nontrivial spectra of two dimensional gauge theories
• Lattice: partition function and its continuum limit
• Adding external charges:
  Wilson loops and Polyakov lines
  continuum limit
  interpretation
  theta states
  screening and effective fractional charge
• Fractional charges on a lattice and the new/classical continuum limit
• Nonabelian case
I. Screening *with* dynamical fermions

Massive Schwinger model

\[
\sigma_q = m \, e \left( 1 - \cos \left( \frac{2\pi q}{e} \right) \right) \quad m/e << 1, \quad [Coleman \ et \ al., \ '75]
\]

Figure 1:
II. Nontrivial spectra of trivial gauge theories

- Two dimensional gauge theories are trivial - no transverse degrees of freedom.
- True only if we neglect boundary conditions.

Quantum Maxwell Dynamics in 1+1 dimensions ($QMD_2$) on a circle

$$E_n = \frac{e^2}{2} L n^2, \quad n = 0, \pm 1, \pm 2, ...$$  \quad [Manton,'84]

An effective 1DOF hamiltonian

$$H = -\frac{e^2}{2L} \frac{d^2}{dA^2}, \quad 0 \leq A < L_A = \frac{2\pi}{L}$$  \quad (1)

The spectrum

$$\psi_n(A) = e^{inAL} = e^{ipnA}, \quad p_n = n\frac{2\pi}{L_A} = nL, \quad E_n = \frac{e^2}{2} Ln^2$$  \quad (2)

What is A ?

$$A_x(x, t) = A(x, t), \quad \frac{\partial_x A(x, t)}{A(x, t)} = 0 \quad A(x, t) = A(t) \neq 0$$
In a periodic (in $x$) world one cannot set a constant $A$ to 0 by a gauge transformation – 1 DOF left

Why periodicity in $A$?

If space is periodic, gauge transformations also have to be periodic (up to $2\pi n$)

$$g(x) = e^{i\Lambda(x)} = g(x + L), \quad \rightarrow \quad \Lambda(x + L) = \Lambda(x) + 2\pi n$$

Take $\Lambda(x) = 2\pi \frac{x}{L}$, then

$$A \rightarrow A + \partial_x \Lambda(x) = A + \frac{2\pi}{L}, \quad \text{are gauge equivalent} \quad \Rightarrow \quad A \in \left(0, \frac{2\pi}{L}\right]$$

Interpretation

• a string with $n$ units of electric flux winding around a circle

• Gauss’s law satisfied thanks to the nontrivial topology - topological strings

• electric charge even without electrons/sources!
A generalization: $\Theta$ parameter

a)

\[ H = -\frac{e^2}{2L} \left( \frac{d}{dA} + i\Theta L \right)^2, \]

\[ E_n = \frac{e^2}{2} L(n + \Theta)^2, \quad \psi_n(A) = e^{inAL} \]

b)

\[ \tilde{H} = -\frac{e^2}{2L} \frac{d^2}{dA^2}, \]

\[ \tilde{E}_n = \frac{e^2}{2} L(n + \Theta)^2, \quad \tilde{\psi}_n(A) = e^{i(n+\Theta)AL}, \]

\[ \tilde{\psi}_n(A) = e^{i\Theta AL} \psi_n(A) \]

Interpretation: $e\Theta$ – classic, constant electric field
II. QMD$_2$ on a lattice

![2x2 lattice](image)

Figure 2:

Partition function on a 2x2 lattice

\[
Z = \int_{0}^{2\pi} B(\theta_{12} + \vartheta_{22} - \theta_{11} - \vartheta_{12}) B(\theta_{22} + \vartheta_{12} - \theta_{21} - \vartheta_{22}) B(\theta_{11} + \vartheta_{21} - \theta_{12} - \vartheta_{11}) B(\theta_{21} + \vartheta_{11} - \theta_{22} - \vartheta_{21}) \\
\text{d(links)}
\]

\[
B(\phi_P) = e^{\beta \cos(\phi_P)}, \quad \text{d(links)} = \Pi_l \frac{d\alpha_l}{2\pi}
\]

A character expansion (Fourier analysis on a group)

\[
B(\phi) = \sum_{n=-\infty}^{\infty} I_n(\beta) \exp(in\phi),
\]

The partition function "almost" factorizes

\[
Z = \sum_n I_n(\beta)^4 \rightarrow \sum_n I_n(\beta)^{N_V}, \quad N_V = N_t \ast N_x.
\]
The continuum limit

\[ Z = \# \sum_n \left( \frac{I_n(\beta)}{I_0(\beta)} \right)^{N_x N_t}, \]

\[ aN_t = T, \quad aN_x = L, \quad \beta = \frac{1}{e^2 a^2}, \quad a \to 0. \]

Asymptotic expansion of modified Bessel function

\[ I_n(\beta) \to \frac{e^\beta}{\sqrt{2\pi \beta}} \left( 1 - \frac{4n^2 - 1}{8\beta} + \ldots \right) \]

gives

\[ Z_{LQMD_2} \to \# \sum_n \left( 1 - \frac{e^2}{2} n^2 a^2 \right)^{N_x N_t} = \sum_n e^{-E_n T}, \quad E_n = \frac{1}{2} e^2 n^2 L, \]

\[ \longrightarrow \text{Manton fluxes result in the continuum limit of lattice } QMD_2 \]
Emergence of a constant mode - Coulomb gauge on a lattice

A single row of $N_x = 3$ horizontal links $\theta_1, \theta_2, \theta_3$

A local gauge transformation specified by $\alpha_1, \alpha_2, \alpha_3$

\[
\begin{align*}
\theta_1 & \rightarrow g\theta_1 = \theta_1 + \alpha_1 - \alpha_2 \\
\theta_2 & \rightarrow g\theta_2 = \theta_2 + \alpha_2 - \alpha_3 \\
\theta_3 & \rightarrow g\theta_3 = \theta_3 + \alpha_3 - \alpha_1
\end{align*}
\]

or

\[
g\theta_i = \theta_i + \beta_i, \quad \sum_{i=1}^{3} \beta_i = 0
\]

If we choose

\[
\begin{align*}
\beta_1 &= \frac{1}{3} (\theta_1 + \theta_2 + \theta_3) - \theta_1 \\
\beta_2 &= \frac{1}{3} (\theta_1 + \theta_2 + \theta_3) - \theta_2 \\
\beta_3 &= \frac{1}{3} (\theta_1 + \theta_2 + \theta_3) - \theta_3
\end{align*}
\]
then all new link angles are equal
\[ g\theta_1 = g\theta_2 = g\theta_3 = \frac{1}{3}(\theta_1 + \theta_2 + \theta_3) \equiv \theta_{\text{row}}. \]

⇒ Only one degree of freedom remains

• Volume reduction

III. Adding external charges

Wilson loops

Figure 3:
\[ W[\Gamma] = \Pi_{l \in \Gamma} e^{i \theta_l} \]  
\[ Z\langle W \rangle = \sum_n I_n(\beta)^{N_t \times N_t - n_x \times n_t} I_{n+1}(\beta)^{n_x \times n_t}. \]  

Time like Polyakov loops

\[ Z < P^\dagger(1)P(1 + n_x) > = \sum_n I_n(\beta)^{N_t \times (N_x - n_x)} I_{n+1}(\beta)^{N_t \times n_x}, \]  

Continuum limit

\[ aN_t = T, \quad aN_x = L, \quad \beta = \frac{1}{e^2 a^2}, \quad a \rightarrow 0. \]

\[ Z < P(0)^\dagger P(R) > = \sum_n e^{-E_{PP} n}, \]  
with

\[ E_{PP}^n = \frac{e^2}{2} \left( n^2(L - R) + (n + 1)^2 R \right), \quad n = 0, \pm 1, \pm 2, \ldots \]
A straightforward interpretation:

\[ E_n^{PP} = \frac{e^2}{2} \left( n^2(L - R) + (n + 1)^2R \right), \quad n = 0, \pm 1, \pm 2, \ldots \quad (8) \]

- Time like Polyakov lines modify Gauss’s law at spatial points 0 and R - they introduce external unit charges at these positions.

- Such charges cause additional unit of flux extending over distance R.

- Hence the two contributions to the eigenenergies: an ”old” flux over the distance \( L - R \) and the new one, bigger by one unit (fluxes are additive !) , over \( R \).

- Interesting special cases:
  \( \rightarrow \) at large T the lowest, \( n = 0 \) and \( n = -1 \), states dominate. Then we just have standard (unit flux) strings of length R and L-R ,
  \( \rightarrow R = 0 \) – old topological flux with charge n.
  \( \rightarrow R = L \) – when external charges meet at the ”end point” of a circle, they annihilate \( (e^+\delta_P(0) + e^-\delta_P(L) = 0 ) \) and leave behind a topological string with length \( L \) and charge bigger by one unit.

- Varying R interpolates between integer valued topological fluxes.
Equivalent form

\[ E_n^{PP} = \frac{e^2}{2} L(n + \rho)^2 + \text{const.}(L, R), \quad \rho = \frac{R}{L}, \quad \text{const.} = \frac{e^2}{2} L \rho (1 - \rho) \]  

(9)

- Indeed \( e^R_L \) is the electric field, generated by two sources, \textit{averaged} over the whole volume.

- The system does not see any distances, \( A_x(x) = \text{const.} \), hence averaging over the volume.

- Changing \( R \) allows to mimic arbitrary real charge \( q = e(n + \rho) \).

- Only \( [\rho] \) is relevant.
• Θ parameter acquires now a straightforward interpretation

\[ \Theta_{Manton} = \rho = \frac{R}{L}, \]

• A new constant term.
**Θ-vacua**

- The transformation $A \rightarrow A + \frac{2\pi}{L}$ is a large gauge transformation, $\Lambda(x) = \frac{2\pi x}{L}$, $\Lambda(x + L) = \Lambda(x) + 2\pi$

- Full analogy 4D YM and/or a crystal: many classical configurations around which we can quantize

- $|\Theta\rangle = \sum_m e^{i\Theta m} |m\rangle$

- The wave function of a $\Theta$-state $\psi_\Theta(x) = \langle x|\Theta \rangle$ satisfies $\psi_\Theta(x - d) = e^{i\Theta} \psi_\Theta(x)$

- The solution (Bloch theorem): $\psi_\Theta(x) = e^{i\Theta x/d} u_\Theta(x)$, with periodic $u_\Theta(x)$

- Our case: $\psi_n(A) = e^{i(n+\rho)AL} = e^{i\rho AL} e^{i n AL}$ is exactly of Bloch type upon identification $x \rightarrow A$, $d \rightarrow 2\pi/L$, $\Theta \rightarrow 2\pi\rho$

- Introducing external charges fixes the $\Theta$-vacuum in $QMD_2$.

- D=4: in a $\Theta$-vacuum some field configurations acquire electric charge [Witten ’76].
More, different charges

$R_2$ - distance between doubly charged sources
$R_1$ - distance between singly charged ones

\[ Z < P(i)^\dagger P(j)^{2\dagger} P^2(j + n_2) P(i + n_1) > = \]

\[
\sum_n I_n(\beta)^{N_t(N_x-n_1)} I_{n+1}(\beta)^{N_t(n_1-n_2)} I_{n+3}(\beta)^{N_t n_2},
\]

- eigenenergies in the continuum limit

\[
E_{n}^{PPPP} = \frac{e^2}{2} \left( n^2(L - R_1) + (n + 1)^2(R_1 - R_2) + (n + 3)^2R_2 \right)
\]

\[
= \frac{e^2}{2} L \left( (n + \rho_1 + 2\rho_2)^2 + \rho_1(1 - \rho_1) + 4\rho_2(2 - \rho_1 - \rho_2) \right)
\]

etc. 1 DOF quantum mechanical systems can be also readily constructed.

- This time $\Theta = (R_1 + 2R_2)/L$, i.e. it is again equal to the external field averaged over the whole volume.
IV. Arbitrary charges on a lattice

Why? To learn about screening

Massive Schwinger model

\[ \sigma_q = m e \left(1 - \cos \left(\frac{2\pi q}{e}\right)\right) \quad m/e << 1, \quad [Coleman \ et \ al., \ '75] \]

⇒ generalizations for large N \( QCD_2 \).

⇒ How to put arbitrary (noncongruent with \( e \)) charges on a lattice?

• One way: as above \( q = e(n + R/L) \)
• Another way: new observables
Wilson loops with arbitrary charge

$$Z \langle W_Q \rangle = \int (W[\Gamma])^Q e^{-S}, \quad Q = q/e$$

Contras:
- gauge invariance – not if you carefully/consistently deal with multivaluedness
- dependence on the boundaries in angular variables – not if you do loops

Pros:
- Results are consistent ($MC \leftrightarrow TH$)
- New structure appears $QMD_2$
- Why not!
Q-loops theoretically

\[ Z\langle W_Q \rangle = \sum_{m,n} I_n^{N_xN_t-n_xn_t} I_m^{n_xn_t} S(Q - m + n)^{n_x+n_t}, \]

\[ S(x) = \left( \frac{\sin \pi x}{\pi x} \right)^2 \]
and ”experimentally” [P. Korcyl, M. Koren]

Figure 4:

- $Q$-loops can be defined on a lattice - MC agrees with TH
- They do not create states with arbitrary charge
  - they excite the only existing quantum states with integer charges
Continuum limit

\[ Z\langle W_Q \rangle \longrightarrow \Sigma_{m,n} \exp \left( -\frac{e^2}{2}n^2L(T - t) \right) \exp \left( -\frac{e^2}{2}(n^2(L - R) + m^2R)t \right) \exp S(Q - (n - m))(t + R)/a \]

does not exist at fixed, not integer \( Q \).

\[ \Rightarrow \text{However the } \textit{classical} \text{ limit:} \]
\[ Q \rightarrow \infty, \text{ with } q = Qe - \text{fixed}, \text{ on a fixed lattice } (a, N's, \text{const.}) \]

does exist!
Then \( \beta \equiv b^2 = 1/e^2a^2 \rightarrow \infty \), but not because \( a \rightarrow 0 \), but because \( e \rightarrow 0 \).

The spectrum of fluxes becomes continuous: \( n \rightarrow u = n/b, m \rightarrow v = n/b \)

Therefore \( (Q = q/e = \sqrt{\beta/\kappa} = b/g, g = 1/qa) \)

\[
ZK_{\Pi QQ} = \beta \int dudv \exp \left( -\frac{1}{2}(u^2(N_x - n_x) + v^2n_x) \right)
\]

\[
S(b(g^{-1} - (u - v))^2 e^{ibu(\Theta_{L-R} - \Theta'_{L-R})} e^{ibv(\Theta_{R} - \Theta'_{R})}
\]

using

\[
S(b\Delta) \xrightarrow{b \rightarrow \infty} \frac{1}{b} \delta(\Delta)
\]

gives

\[
ZK_{\Pi QQ} = \sqrt{\beta} \int du \exp \left( -\frac{1}{2}(u^2(N_x - n_x) + (u - g^{-1})^2n_x) \right)
\]

\[
e^{ibu(\Theta_{L-R} - \Theta'_{L-R})} e^{ib(u - g^{-1})(\Theta_{R} - \Theta'_{R})}
\]
Now, do the gaussian integral, take the continuum limit to obtain

$$Z K_{\Pi \Phi \Phi} = \sqrt{\beta} \sqrt{\frac{2\pi a}{L}} \exp \left( -\frac{L (A - A')^2}{2a} \right) \exp \left( -\frac{q^2}{2} \rho (1 - \rho) La \right)$$

\implies a free particle propagating over a time $a$, but in a constant background potential

$$V = \frac{q^2}{2} \rho (1 - \rho) L$$

with arbitrary, real value of a classical charge $q$.

- The classical energy with a continuous charge $q$ results from the contribution of many microscopic states with discrete charges.
- The structure (zeroes of the string tension)
V. Nonabelian case: $YM_2$ on a circle

- Continuum: problem reduces to $N$ constant in space, but constrained, angles $\theta_i, \Sigma_i \theta_i = 0$.
  Hamiltonian is again quadratic and the spectrum is known explicitly [Hetrick and Hosotani '89]

$$E_{\{n\}} = \frac{g^2 L}{4} \left( \Sigma_i n_i^2 - \frac{1}{N} (\Sigma_i n_i)^2 \right), \quad i = 1, ..., N - 1$$

- Continuum: different spectrum was obtained by Rajeev: $E_R = \frac{g^2 L C_2(R)}{2}$
- Discrepancy comes from the Casimir energy due to the curvature of the group manifold [Hetrick '93, Witten '91,'92]
- External charges in $YM_2$ – studied by many [Semenoff et al. '97] but above interpretation in terms of screening was not.
EU grant (via Foundation for Polish Science)

**Jagellonian University International PhD Studies on Physics of Complex Systems**

- 1 M Euro
- 4 years
- 14 PhD students (1/2 - 2 years abroad)
- 9 Local Supervisors
- 17 Foreign Partners: J. Ambjorn, J.P. Blaizot, H. Nicolai, S. Sharpe, ...