Renormalization constants for $N_f = 2 + 1 + 1$

twisted mass QCD

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To my collaborators: Benoit Blossier, Mariane Brinet, Pierre Guichon, Vincent Morenas, Olivier Péne, Pepe Rodríguez-Quintero and to Tassos Vladikas for fruitful discussions
Motivation
RI’-MOM scheme-generalities
RCs
Conclusions and Outlook
Lattice formalism is bare QFT

One computes bare matrix elements of operators at fixed cutoff

Must renormalize to obtain continuum Physics

\[ O_R = Z_O O_b \]

Renormalization can be done perturbatively or non-perturbatively
Bare vs Renormalized

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$$O_R = Z_O O_b$$

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Lattice PT-notorious for its bad convergence

- MILC collaboration found that $m_s$ was raised by 14% once its renormalization constant (RC) known in 1-loop PT was calculated @ 2-loops.
- Göckeler et al found that $m_s$ was raised by 24% once its RC known in 1-loop PT was calculated non-perturbatively.
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Non-Perturbative Renormalization

- **RI-MOM scheme**  Martinelli et al (1995)

- Work on the calculation of the RCs by many groups many of them belonging to the ETMC
  

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focus on local fermion bilinears $O_\Gamma = \bar{\psi}(x)\Gamma\psi(x)$

Vladikas Les Houches lectures

$\Gamma$ can be any Dirac structure and can even potentially contain covariant derivatives

- inserting $O_\Gamma$ in the fermion 2-pt function
- $G_O = \langle u(x_1)O_\Gamma \bar{d}(x_2) \rangle$
- the amputated Green’s function
- $\Lambda_O(p_1,p_2) = S_u^{-1}(P_1)G_O(p_1,p_2)S_d^{-1}(p_2)$
- $\Gamma_O(p) = \frac{1}{12}\text{tr}[P_O\Lambda_O(p,p)]$
- $\Gamma_O(p)_R = \lim_{a \to 0} Z_q^{-1}Z_O\Gamma_O(p)$
- $Z_q(\mu^2 = p^2) = -\frac{i}{12p^2}\text{tr}[S_{bare}^{-1}(p)\slashed{p}]$
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impose that the amputated Green’s function in the chiral limit @ a large Euclidean scale $p^2 = \mu^2$ is equal to its tree level value

$$\Gamma_O(p)_R(\mu, g_R, m_R = 0) = \lim_{a \to 0} [Z_q^{-1}(a\mu, g_0)Z_O(a\mu, g_0)\Gamma_O(p, g_0, m)]_{p=\mu^2, m \to 0}$$
Window of applicability of RI-MOM

- $\Lambda_{QCD} \ll \mu \ll \frac{\pi}{a}$
- The first inequality ensures the possibility of matching with some perturbative scheme $\overline{\text{MS}}$ and protects from Goldstone pole contaminations.
- The second inequality ensures small cutoff effects.
- Whatever regularization and renormalization is necessary and sufficient at $T = 0$ and $\mu = 0$ is also necessary and sufficient at $T \neq 0$ and $\mu \neq 0$.
- $UV$ structure of the theory is unchanged by the intro of macroscopic parameters such as $T, \mu$.
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Conversion to $\overline{\text{MS}}$

- make connection with phenomenological calculations and experiments
- the decay width for the dominant channel $H \rightarrow \bar{b}b \propto m_b^2$
- one needs the RC for $m_b$ - in the TM framework it is given $1/Z_P$
- need to convert to $\overline{\text{MS}}$ with factors $Z_{q}^{\overline{\text{MS}}} = C_q^{-1} Z_{q}^{RI'-\text{MOM}}$
- and $Z_{\mathcal{O}}^{\overline{\text{MS}}} = C_{\mathcal{O}}^{-1} Z_{\mathcal{O}}^{RI'-\text{MOM}}$
- experiments usually provide results in $\overline{\text{MS}}$ at a reference scale $\mu = 2$ GeV
- evolve $\overline{\text{MS}}$ RCs $Z_{\mathcal{O}}^{\overline{\text{MS}}}$ using the scale dependence predicted by the RG equation
  \[ R_{\mathcal{O}(\mu, \mu_0)} := \frac{Z_{\mathcal{O}(\mu)}}{Z_{\mathcal{O}(\mu_0)}} = \exp \left\{ - \int_{\bar{g}(\mu_0^2)}^{\bar{g}(\mu^2)} dg \frac{\gamma(g)}{\beta(g)} \right\} \]
- $\beta$ is the usual QCD-beta function, $\gamma$ the anomalous dimension of operator $\mathcal{O}$ and $\bar{g}(\mu^2)$ the running coupling

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$\beta$ is the usual QCD-beta function, $\gamma$ the anomalous dimension of operator $O$ and $\bar{g}(\mu^2)$ the running coupling
Simulation setup for $N_f = 2 + 1 + 1$

- $S = S_{Iwa}^Y + S_f^Y + S_h^f$

$$
S = S_{Iwa}^Y + a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_{0l} + i \mu_I \gamma_5 \tau_3 \right) \chi_f \\
+ a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_{0h} + i \mu_h \gamma_5 \tau_1 + \mu_\delta \tau_3 \right) \chi_f
$$

Baron et al (2010)

- polar mass $M = \sqrt{m^2 + \mu^2}$ and twist angle $\omega = \arctan(\mu/m)$

- the quark doublet in the twisted basis is related to the one in the physical basis by the trafo

- $\psi_l = e^{i 2 \omega_l \gamma_5 \tau_3} \chi_l$ and $\bar{\psi}_l = \bar{\chi}_l e^{i 2 \omega_l \gamma_5 \tau_3}$

- $S_{ph} = a^4 \sum_{x,f} \bar{\psi}_f (D_{tW} + M) \psi_f$

- to achieve the benefits of the TM formulation one needs to work at maximal twist $\omega = \pi/2$ Frezzotti and Rossi (2003-2004)

- automatic $O(a)$ improvement
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  $+$ $a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_0_h + i\mu_h \gamma_5 \tau_1 + \mu_3 \tau_3 \right) \chi_f$

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- Polar mass $M = \sqrt{m^2 + \mu^2}$ and twist angle $\omega = \arctan(\mu/m)$

- The quark doublet in the twisted basis is related to the one in the physical basis by the transformation $\psi_l = e^{i \frac{\omega}{2} \gamma_5 \tau_3} \chi_l$ and $\bar{\psi}_l = \bar{\chi}_l e^{i \frac{\omega}{2} \gamma_5 \tau_3}$

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Simulation setup for $N_f = 4$

$$S = S_{Iwa}^{YM} + a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_0 + i r_f \mu_f \gamma_5 \right) \chi_f$$

- dedicated simulations with $N_f = 4$ light degenerate quarks to renormalize NP in a mass independent scheme (where RCs are defined in the chiral limit) the $N_f = 2 + 1 + 1$ ensembles - allow for a reliable chiral extrapolation.
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- dedicated simulations with $N_f = 4$ light degenerate quarks to renormalize NP in a mass independent scheme (where RCs are defined in the chiral limit) the $N_f = 2 + 1 + 1$ ensembles - allow for a reliable chiral extrapolation
The lattice spacing values are respectively $a = 0.062 \text{ fm for } \beta = 2.10$, $a = 0.078 \text{ fm for } \beta = 1.95$ and $a = 0.086 \text{ fm for } \beta = 1.90$.
Correlation functions of the pseudoscalar operator have pion pole contamination need to be addressed carefully.

An ansatz for the amputated pseudoscalar vertex is:

$$\Gamma_P = a_P + b_P m^2_\pi + \frac{c_P}{m^2_\pi}$$

$$\Gamma_{P}^{sub} = \Gamma_P - \frac{c_P}{m^2_\pi}$$
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\( u \) scalar (LHS) and pseudo-scalar (RHS) vertex functions versus pion mass squared (in lattice unit) for ensemble \( 3p \) for several values of \( a^2 \vec{p}^2 \). (Full-) empty circles correspond to (un-)subtracted values while * to the chiral extrapolation, \( (a.p^0 = \frac{\pi}{T} \) for all curves except the magenta one, for which \( a.p^0 = \frac{21\pi}{T} \)).
$Z_P/Z_S$ for ensemble 3mp ($\beta = 2.10$, $\mu = 0.0046$, volume $32^3.64$).

Lattice artifacts have been removed separately from $Z_S$ and $Z_P$. The ratio of these two RCs is compatible with a constant over the whole $a^2 p^2$ interval and $Z_P/Z_S = 0.717(3)$. 

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RCs for $N_f = 2 + 1 + 1$ twisted mass QCD
Quark renormalisation constant (LHS) and scalar renormalisation constant (RHS) as a function of $a^2 p^{[2]}$. Both exhibit the typical "fishbone" structure induced by the breaking of the $O(4)$ rotational symmetry of the Euclidian space-time by the lattice discretization, into the hypercubic group $H(4)$. 
LHS: Effect of hypercubic corrections on quark renormalization constant, as a function of $a^2p^{[2]}$. RHS: renormalization constants as a function of $a^2p^{[2]}$, after removing $H(4)$ artifacts.
Correcting for artifacts

- hypercubic artifacts that respect $H(4)$ but not $O(4)$
- artifacts that respect $O(4)$ will be treated NP by introducing corrections to the running
  - egalitarian method (does not rely on the selection of diagonal momenta which have small $H(4)$ artifacts like the method of democratic cuts Boucaud et al (2003), de Soto et al (2007))
  - keeps maximum amount of info- allows for the testing of the running of RCs
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Correcting for artifacts

- perform an average over the orbits of $H^4$—several orbits correspond to the same value of $p^2$ e.g. $(1, 1, 1, 1)$ and $(2, 0, 0, 0)$
- we define the $H^4$ invariants

$$p [4] = \sum_{\mu=1}^{4} p^4_{\mu}, \quad p [6] = \sum_{\mu=1}^{4} p^6_{\mu}, \quad p [8] = \sum_{\mu=1}^{4} p^8_{\mu}$$

- any $H^4$ invariant polynomial can be written in terms of the four invariants $p^2, p [4], p [6], p [8]$

- Expand the RC already averaged over the cubic orbits around $p [4] = 0$

$$Z_{latt}(a^2 p^2, a^4 p [4], a^6 p [6], a p_4, a^2 \Lambda^2_{QCD}) = Z_{hypcorrected}(a^2 p^2, a p_4, a^2 \Lambda^2_{QCD}) + R(a^2 p^2, a^2 \Lambda^2_{QCD}) \frac{a^2 p [4]}{p^2} + \ldots$$

$$R(a^2 p^2, a^2 \Lambda^2_{QCD}) = \frac{dZ_{latt}(a^2 p^2, 0,0,0,a^2 \Lambda^2_{QCD})}{d\epsilon}|_{\epsilon=0} = c_{a_2 p_4} + c_{a_4 p_4} a^2 p^2$$
Correcting for artifacts

- perform an average over the orbits of $H(4)$-several orbits correspond to the same value of $p^2$ e.g. $(1, 1, 1, 1)$ and $(2, 0, 0, 0)$
- we define the $H(4)$ invariants
  $$p^{[4]} = \sum_{\mu=1}^{4} p^4_\mu, \quad p^{[6]} = \sum_{\mu=1}^{4} p^6_\mu, \quad p^{[8]} = \sum_{\mu=1}^{4} p^8_\mu$$
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- Expand the RC already averaged over the cubic orbits around $p^{[4]} = 0$
  $$Z_{latt}(a^2p^2, a^4p^{[4]}, a^6p^{[6]}, ap_4, a^2\Lambda_{QCD}^2) =$$
  $$Z_{hypcorrected}(a^2p^2, ap_4, a^2\Lambda_{QCD}^2) + R(a^2p^2, a^2\Lambda_{QCD}^2) \frac{a^2p^{[4]}}{p^2} + \ldots$$
- $$R(a^2p^2, a^2\Lambda_{QCD}^2) = \frac{dZ_{latt}(a^2p^2, 0, 0, 0, a^2\Lambda_{QCD}^2)}{d\epsilon} \bigg|_{\epsilon=0} =$$
  $$c_{a2p4} + c_{a4p4}a^2p^2$$
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- we define the $H(4)$ invariants
  $p^{[4]} = \sum_{\mu=1}^{4} p^{4}_\mu, \quad p^{[6]} = \sum_{\mu=1}^{4} p^{6}_\mu, \quad p^{[8]} = \sum_{\mu=1}^{4} p^{8}_\mu$
- any $H(4)$ invariant polynomial can be written in terms of the four invariants $p^2, p^{[4]}, p^{[6]}, p^{[8]}$
- Expand the RC already averaged over the cubic orbits around $p^{[4]} = 0$
  $Z_{latt}(a^2p^2, a^4p^{[4]}, a^6p^{[6]}, ap_4, a^2\Lambda^2_{QCD}) = Z_{hypcorrected}(a^2p^2, ap_4, a^2\Lambda^2_{QCD}) + R(a^2p^2, a^2\Lambda^2_{QCD}) \frac{a^2p^{[4]}}{p^2} + \ldots$
- $R(a^2p^2, a^2\Lambda^2_{QCD}) = \frac{dZ_{latt}(a^2p^2,0,0,0,a^2\Lambda^2_{QCD})}{d\epsilon}|_{\epsilon=0} = c_{a2p4} + c_{a4p4}a^2p^2$
consider for the running of $Z_q$ Blossier et al (2010)

\[
Z_q^{hyp-corr}(a^2p^2) = Z_q^{pert\ RI'}(\mu^2) c_{0 Z_q}^{RI'}(\frac{p^2}{\mu^2}, \alpha(\mu)) \\
\times \left(1 + \frac{\langle A^2 \rangle_{\mu^2}}{32p^2} \frac{\overline{MS}_{2 Z_q}(\frac{p^2}{\mu^2}, \alpha(\mu))}{c_{0 Z_q}^{RI'}(\frac{p^2}{\mu^2}, \alpha(\mu))} \frac{c_{2 Z_q}^{RI'}(\frac{p^2}{\mu^2}, \alpha(\mu))}{\overline{MS}_{2 Z_q}(\frac{p^2}{\mu^2}, \alpha(\mu))}\right) \\
+ c_{a2p2} a^2 p^2 + c_{a4p4} (a^2 p^2)^2
\]

coefficients $c_{0 Z_q}^{RI'}$, $c_{0 Z_q}^{RI'}$ and $\overline{MS}_{2 Z_q}$ known from PT Chetyrkin et al (1999), Chetyrkin (2004), Chetyrkin et al (2009)

the running formula contains lattice artifact terms $\propto a^2 p^2$ and $\propto (a^2 p^2)^2$, not yet removed.

need to determine, $Z_q^{pert\ RI'}(\mu^2)$, $\langle A^2 \rangle_{\mu^2}$, $c_{a2p2}$ and $c_{a4p4}$
consider for the running of $Z_q$ \cite{Blossier:2010}

$$Z_q^{\text{hyp-corr}}(a^2p^2) = Z_q^{\text{pert RI'}}(\mu^2) c_{0Zq}^{RI'}(\frac{p^2}{\mu^2}, \alpha(\mu))$$

$$\times \left(1 + \frac{\langle A^2 \rangle_{\mu^2}}{32p^2} \frac{\overline{\text{MS}}_{2Zq}(\frac{p^2}{\mu^2}, \alpha(\mu))}{c_{2Zq}(\frac{p^2}{\mu^2}, \alpha(\mu))} \right)$$

$$+ c_{a2p2} a^2 p^2 + c_{a4p4} (a^2 p^2)^2$$

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$$\times \left(1 + \frac{\langle A^2 \rangle_{\mu^2}}{32p^2} \frac{\overline{MS}}{c_2^{Zq}(\frac{p^2}{\mu^2}, \alpha(\mu))} \frac{c_2^{RI'}(\frac{p^2}{\mu^2}, \alpha(\mu))}{c_2^{RI'}(\frac{p^2}{\mu^2}, \alpha(\mu))} \right)$$

$$+ c_{a2p2} a^2p^2 + c_{a4p4} (a^2p^2)^2$$

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consider for the running of $Z_q$ (Blossier et al, 2010)

$$Z_q^{\text{hyp}-\text{corr}} (a^2 p^2) = Z_q^{\text{pert \ RI'}} (\mu^2) \ c^{\text{RI'}}_0 Z_q (\frac{p^2}{\mu^2}, \alpha(\mu))$$

$$\times \left( 1 + \frac{\langle A^2 \rangle_{\mu^2}}{32 p^2} \ \frac{\overline{\text{MS}}_{2Z_q} (\frac{p^2}{\mu^2}, \alpha(\mu))}{c_{2Z_q}^{\text{RI'}} (\frac{p^2}{\mu^2}, \alpha(\mu))} \right)$$

$$+ \ c_{a2p2} \ a^2 p^2 \ + \ c_{a4p4} \ (a^2 p^2)^2$$

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- need to determine $Z_q^{\text{pert \ RI'}} (\mu^2)$, $\langle A^2 \rangle_{\mu^2}$, $c_{a2p2}$ and $c_{a4p4}$
Running of $Z_q$ for ensemble $3mp$ ($\beta = 2.10$, $\mu = 0.0046$, volume $32^3.64$) using different fitting formulae.
The same study is performed for scalar and pseudo-scalar RCs. $Z_S$ and $Z_P$ have the same running formula, namely:

$$Z_{P/S}(\mu) = Z_{P/S}(\mu_0) \frac{c^{RI'MOM}(\mu)}{c^{RI'MOM}(\mu_0)}$$

$$c^{RI'MOM}(\mu) = x^{\bar{\gamma}_0} \left\{ 1 + (\bar{\gamma}_1 - \bar{\beta}_1 \bar{\gamma}_0) x + \frac{1}{2} \left[ (\bar{\gamma}_1 - \bar{\beta}_1 \bar{\gamma}_0)^2 + \bar{\gamma}_2 + \bar{\beta}_1^2 \bar{\gamma}_0 - \bar{\beta}_1 \bar{\gamma}_1 - \bar{\beta}_2 \bar{\gamma}_0 \right] x^2 \right. $$

$$+ \left. \left[ \frac{1}{6} (\bar{\gamma}_1 - \bar{\beta}_1 \bar{\gamma}_0)^3 + \frac{1}{2} (\bar{\gamma}_1 - \bar{\beta}_1 \bar{\gamma}_0)(\bar{\gamma}_2 + \bar{\beta}_1^2 \bar{\gamma}_0 - \bar{\beta}_1 \bar{\gamma}_1 - \bar{\beta}_2 \bar{\gamma}_0) \right) + \frac{1}{3} (\bar{\gamma}_3 - \bar{\beta}_1^3 \bar{\gamma}_0 + 2 \bar{\beta}_1 \bar{\beta}_2 \bar{\gamma}_0 - \bar{\beta}_3 \bar{\gamma}_0 + \bar{\beta}_1^2 \bar{\gamma}_1 - \bar{\beta}_2 \bar{\gamma}_1 - \bar{\beta}_1 \bar{\gamma}_2) \right\} x^3 + O(x^4) \right\}$$

where $x = \alpha/\pi$, $\bar{\gamma}_i = \gamma_i/\beta_0$ and $\bar{\beta}_i = \beta_i/\beta_0$. $\beta_i$ are the coefficients of the QCD beta-function and they are given at four-loop in Chetyrkin et al (1999).
**Running of \( Z_S \) and \( Z_P \)**

**LHS:** running of \( Z_S \) for ensemble \( 3mP \) (\( \beta = 2.10, \mu = 0.0046, \) volume \( 32^3 \times 64 \)). The standard running formula is represented in solid blue line, the dashed cyan curve includes an \( 1/a^2p^2 \) and an \( a^2p^2 \) term. This latter fit leads to \( Z_S(10 \text{ GeV}) = 0.869(4) \).

**RHS:** Running of \( Z_P \) with the standard running expression *Chetyrkin et al (1999)* (solid blue curve), and adding an \( 1/a^2p^2 \) and an \( a^2p^2 \) terms (dashed cyan curve). The modified running gives \( Z_P(10 \text{ GeV}) = 0.623(2) \).
Fits of the residual $a^2p^2$ dependence of $Z_V$ and $Z_A$ for ensemble $3mp$ ($\beta = 2.10$, $\mu = 0.0046$, volume $32^3.64$)
LHS: \( N_f = 4 \) local RCs dependence with the pion mass. The straight dashed lines are constant fits for each \( \beta \) values. The red points correspond to \( \beta = 2.10 \), the black ones to \( \beta = 1.95 \), and the blue ones to \( \beta = 1.90 \).

RHS: RCs after chiral extrapolation, vs \( \log a^2 \). All RCs follow a linear dependence with \( \log a^2 \) to a very high accuracy.
converted our RI’-MOM results at 10 GeV to $\overline{\text{MS}}$ values at a reference scale of 2 GeV leads to the final RCs
- Provided NP results for the RCs of $N_f = 2 + 1 + 1$ Twisted Mass QCD
- Hypercubic artifacts were taken correctly into account by the ”egalitarian” method
- Complete the analysis of twist-2 operators
- Extend the analysis to fermion quadrilinears
- Extend our work to the new ensembles of ETMC with the large volumes $48^3 \times 96$ and masses @ the physical point
- Check the effect of Gribov copies
- Perform the analysis using the RI-SMOM scheme
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Stay Tuned!

for upcoming results . . .
Thank you for your attention!